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Director libration in nematoacoustics

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We extended the analysis of a variational theory for nematoacoustics recently proposed by Virga [Phys. Rev. E **80**, 031705 (2009)] by allowing the nematic director to vibrate about an average orientation at the frequency of a propagating wave, a periodic motion that we call the director *libration*. The acoustic susceptibilities, two phenomenological parameters that, in this theory, express the coupling between director and acoustic fields, are estimated along with an extra viscosity coefficient by using available experimental data.

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I. INTRODUCTION

It has long been recognized that sound waves interact with the orientational order of nematic liquid crystals, realigning liquid-crystal molecules [1–3]. Relevant experimental observations include the anisotropy and frequency dependence of both attenuation and dispersion of ultrasonic waves [4–8] and the reorienting acoustic action exerted on a uniformly aligned nematic liquid crystal [9–11]. To interpret these experimental results, it is essential to understand the interaction mechanism between the acoustic field with its wave vector $\boldsymbol{k}$ and the nematic director $\boldsymbol{n}$ representing the average molecular orientation. It was already anticipated by Helfrich [12], within a purely hydrodynamic theory, that an acoustic field can affect the orientation of $\boldsymbol{n}$. In 1972, the first experimental results were reported, which suggested introducing an elastic interaction energy between the acoustic wave and the nematic director field to account for a direct nematoacoustic coupling capable of inducing distortions in the director texture [13]. On the theoretical side, the nature of this interaction was later elucidated in Refs. [14] and [15] by postulating that the nematoacoustic interaction results from the coupling between the density modulation induced by the acoustic wave and the nematic director: Such a coupling occurs at time scales much larger than those of acoustic vibrations and, as a result, it is actually the time-averaged interaction energy that enters the nematic free energy. The proposed averaged interaction energy $V_{\rm avg}$ had the form $V_{\rm avg} = \frac{1}{2}u(\boldsymbol{k}\cdot\boldsymbol{n})^2$, where u is a coupling constant, which could have either sign, introduced as a phenomenological nematoacoustic susceptibility.

More recently, a variational theory for nematoacoustics has been based on a variant of this assumption [16]. According to this theory, liquid crystals are to be regarded as anisotropic *Korteweg fluids* [17] at the time and length scale at which the sound field produces density modulations. In general, the elastic stress tensor of a Korteweg fluid depends on both first and second gradients of the density field ϱ [18,19]. For anisotropic fluids, such as nematic liquid crystals, Korteweg theory was adapted in Ref. [16] to nematoacoustics by positing an additional elastic energy quadratic in the mass density gradient. On the other hand, in Ref. [16], at the time scale of the acoustic vibrations, the director texture is still regarded as immobile in the same spirit of previous work and in line with experimental studies where the director is kept fixed by external magnetic fields. Here, within the theory proposed in Ref. [16], we will relax such an assumption: The director is set free to vibrate in time and possibly to be distorted in space around a constant and uniform orientation. This latter is held fixed by an external (magnetic) field: We refer to such a motion as the director *libration*, and we imagine it to arise as a consequence of the sound wave that propagates through the nematic liquid crystal. The theoretical outcomes of our analysis will allow us to interpret the experimental results published long ago in the literature and to estimate some phenomenological parameters involved in the theory.

This paper is organized as follows. In Sec. II, we will briefly recall the general balance equations of the nematoacoustic theory put forward in Ref. [16]. In Sec. III, we seek plane wave solutions for these equations, including the director libration. In Sec. IV, we estimate some relevant phenomenological parameters of the theory from the available experimental data on anisotropic dispersion and wave attenuation. Finally, in the closing Sec. V, we summarize our conclusions and sketch future research directions suggested by this paper.

II. DYNAMICAL BALANCE EQUATIONS

In this section, to make our account self-contained, we review the fundamentals of the variational nematoacoustic theory proposed in Ref. [16]. This theory elaborates upon the idea that, on the time and length scales characteristic of acoustic modulations, nematic liquid crystals behave like anisotropic Korteweg fluids. In Ref. [16], the relevant balance equations have been derived from a principle of virtual power. At the same time, the theory presented in Ref. [16] generalizes the Ericksen-Leslie-Parodi theory for nematic liquid crystals [20,21] by removing the incompressibility constraint and by taking the density gradient $\nabla\varrho$ into appropriate account.

We consider a nematic liquid crystal occupying a region $\mathscr{B}$ and let the free energy stored in $\mathscr{B}$ be

$$\mathscr{F} := \int_{\mathscr{B}} F\, dV, \tag{1}$$

where V is the volume measure and

$$F := \tfrac{1}{2}\varrho \boldsymbol{v}^2 + W_{\rm e}(\boldsymbol{n},\nabla\boldsymbol{n}) + \varrho\sigma_{\rm K}(\varrho,\nabla\varrho,\boldsymbol{n}). \tag{2}$$

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Here, $\boldsymbol{v}$ is the flow field of the fluid and

$$W_{\mathrm{e}}(\boldsymbol{n},\nabla\boldsymbol{n}) := \tfrac{1}{2}K_1(\operatorname{div}\boldsymbol{n})^2 + \tfrac{1}{2}K_2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})^2 + \tfrac{1}{2}K_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2, \tag{3}$$

$$\sigma_{\mathrm{K}}(\varrho,\nabla\varrho,\boldsymbol{n}) := \sigma_0(\varrho) + \tfrac{1}{2}[u_1|\nabla\varrho|^2 + u_2(\nabla\varrho\cdot\boldsymbol{n})^2], \tag{4}$$

the former being the Frank distortion energy per unit volume and the latter being the Korteweg elastic energy per unit mass of an acoustic origin. While the Frank distortion energy is a classical contribution to the total free energy embodying the texture curvature elasticity, the Korteweg elastic energy is partly new in this context, and it is worth an additional comment. Physically, when an acoustic wave travels in a nematic medium, the condensation wave front, propagating along the density gradient $\nabla\varrho$, is affected by the local orientation of the nematogenic molecules. Hence, in addition to volume compressibility, reflected by the density ϱ, $\nabla\varrho$ has to be included in the energy and coupled to the director $\boldsymbol{n}$; they mutually interact leading to a distortion of the texture [22]. Intuitively, making elongated molecules denser along their common orientation must have a different energetic cost than making them denser in the orthogonal direction. Actually, the way $\boldsymbol{n}$ couples in time to $\nabla\varrho$ will become clearer from the balance equations we will derive later.

Neither a kinetic energy for the rotational motion of $\boldsymbol{n}$ nor a potential energy for the external actions exerted on $\boldsymbol{n}$ are accounted for in Eq. (2). The former, which could be written as

$$\kappa := \tfrac{1}{2}\varrho\delta^2\dot{\boldsymbol{n}}^2, \tag{5}$$

with δ representing a molecular *radius of gyration*, is neglected in this theory,[1] while the latter is only omitted for simplicity. In the following, we will consider the one-constant approximation to W_{e}, so that Eq. (3) reduces to

$$W_{\mathrm{e}} = \tfrac{1}{2}K|\nabla\boldsymbol{n}|^2. \tag{6}$$

The acoustic susceptibilities u_1 and u_2 in Eq. (4) are assumed to be independent of the mass density ϱ. Moreover, they must obey the inequalities

$$u_1 \geqslant 0 \quad \text{and} \quad u_1 + u_2 \geqslant 0$$

for the energy density added to σ_0 in Eq. (4) to be positive semidefinite.

Nematic liquid crystals are dissipative fluids. Accordingly, we write the Rayleigh dissipation functional as

$$\mathscr{R} := \int_{\mathscr{B}} R_{\mathrm{a}}\,dV, \tag{7}$$

where R_{a} is the Rayleigh *dissipation function* adapted to the acoustic case: It is a frame-indifferent function of $\boldsymbol{n}$, $\mathbf{D}$, and $\mathring{\boldsymbol{n}}$, quadratic in the pair $(\mathring{\boldsymbol{n}},\mathbf{D})$. We denote, by $\mathring{\boldsymbol{n}}$, the *corotational* time derivative,

$$\mathring{\boldsymbol{n}} := \dot{\boldsymbol{n}} - \mathbf{W}\boldsymbol{n},$$

where $\dot{\boldsymbol{n}} := \frac{\partial\boldsymbol{n}}{\partial t} + (\nabla\boldsymbol{n})\boldsymbol{v}$ is the material time derivative of $\boldsymbol{n}$ and

$$\mathbf{W} := \tfrac{1}{2}[(\nabla\boldsymbol{v}) - (\nabla\boldsymbol{v})^{\mathsf{T}}]$$

is the *vorticity* tensor; we denote, by $\mathbf{D}$, the *stretching* tensor,

$$\mathbf{D} := \tfrac{1}{2}[(\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^{\mathsf{T}}].$$

Since a nematic liquid crystal is regarded as a compressible fluid here, and so the velocity field $\boldsymbol{v}$ is no longer solenoidal, the most general expression for R_{a} is the following:

$$R_{\mathrm{a}} := \tfrac{1}{2}\gamma_1\mathring{\boldsymbol{n}}\cdot\mathring{\boldsymbol{n}} + \gamma_2\mathring{\boldsymbol{n}}\cdot\mathbf{D}\boldsymbol{n} + \tfrac{1}{2}\gamma_3\mathbf{D}\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n} + \tfrac{1}{2}\gamma_4\mathbf{D}\cdot\mathbf{D} + \tfrac{1}{2}\gamma_5(\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n})^2 + \tfrac{1}{2}\gamma_6(\operatorname{tr}\mathbf{D})^2 + \gamma_7(\operatorname{tr}\mathbf{D})\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n}, \tag{8}$$

where $\gamma_1,\ldots,\gamma_7$ are viscosities, considered as functions of the mass density ϱ. As shown in Ref. [16], R_{a} in Eq. (8) is positive semidefinite provided that the following inequalities are satisfied:

$$\gamma_1 \geqslant 0, \tag{9a}$$
$$\gamma_3 + 2\gamma_4 \geqslant 0, \tag{9b}$$
$$\gamma_4 \geqslant 0, \tag{9c}$$
$$\gamma_4 + 2\gamma_6 \geqslant 0, \tag{9d}$$
$$\gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + 2\gamma_7 \geqslant 0, \tag{9e}$$
$$\gamma_1\gamma_3 + 2\gamma_1\gamma_4 - \gamma_2^2 \geqslant 0, \tag{9f}$$
$$\gamma_4(\gamma_3+\gamma_4+\gamma_5+\gamma_6+2\gamma_7) + 2\big[\gamma_6(\gamma_3+\gamma_4+\gamma_5) - \gamma_7^2\big] \geqslant 0. \tag{9g}$$

They will play a role in proving that sound waves are attenuated in a nematic liquid crystal.

A. Balance equations

Here, we finally recall the balance equations for nematoacoustics arrived at in Ref. [16] from the free-energy functional $\mathscr{F}$ in Eq. (1) and the Rayleigh dissipation functional $\mathscr{R}$ in Eq. (7) following the general variational principle posited in Ref. [23]. The balance of mass has the classical form of the continuity equation,

$$\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0. \tag{10}$$

In the absence of body forces, the balance of linear momentum is expressed by the equation,

$$\varrho\dot{\boldsymbol{v}} = \operatorname{div}(\mathbf{T}_{\mathrm{E}} + \mathbf{T}_{\mathrm{K}} + \mathbf{T}_{\mathrm{dis}}), \tag{11}$$

where $\dot{\boldsymbol{v}}$ is the acceleration field and the second-rank tensors $\mathbf{T}_{\mathrm{E}}$, $\mathbf{T}_{\mathrm{K}}$, and $\mathbf{T}_{\mathrm{dis}}$ represent the Ericksen elastic stress, the Korteweg elastic stress, and the dissipative stress. With the aid of Eqs. (4), (6), and (8), these tensors were given the following explicit expressions in Ref. [16]:

$$\mathbf{T}_{\mathrm{E}} = -K(\nabla\boldsymbol{n})^{\mathsf{T}}(\nabla\boldsymbol{n}), \tag{12}$$

$$\mathbf{T}_{\mathrm{K}} = -p_{\mathrm{K}}\mathbf{I} - \varrho[u_1\nabla\varrho\otimes\nabla\varrho + u_2(\nabla\varrho\cdot\boldsymbol{n})\nabla\varrho\otimes\boldsymbol{n}], \tag{13}$$

where

$$p_{\mathrm{K}} := p_0(\varrho) - \varrho\operatorname{div}\{\varrho[u_1\nabla\varrho + u_2(\nabla\varrho\cdot\boldsymbol{n})\boldsymbol{n}]\}, \tag{14}$$

with

$$p_0(\varrho) := \varrho^2\sigma_0'(\varrho)$$

[1] In Sec. III D, we will estimate the acoustic frequency that cannot be exceeded for this energy to be safely neglected in Eq. (2).

assumed to be an *increasing* function of ϱ, and

$$\begin{aligned}\mathbf{T}_{\text{dis}} &= \tfrac{1}{2}\gamma_1(\boldsymbol{n}\otimes\mathring{\boldsymbol{n}}-\mathring{\boldsymbol{n}}\otimes\boldsymbol{n})+\tfrac{1}{2}\gamma_2(\boldsymbol{n}\otimes\mathbf{D}\boldsymbol{n}-\mathbf{D}\boldsymbol{n}\otimes\boldsymbol{n})\\ &\quad+\tfrac{1}{2}\gamma_2(\mathring{\boldsymbol{n}}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\mathring{\boldsymbol{n}})+\tfrac{1}{2}\gamma_3(\boldsymbol{n}\otimes\mathbf{D}\boldsymbol{n}+\mathbf{D}\boldsymbol{n}\otimes\boldsymbol{n})\\ &\quad+\gamma_4\mathbf{D}+(\gamma_5\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n}+\gamma_7\,\mathrm{tr}\,\mathbf{D})\boldsymbol{n}\otimes\boldsymbol{n}\\ &\quad+(\gamma_6\,\mathrm{tr}\,\mathbf{D}+\gamma_7\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n})\mathbf{I}.\end{aligned}\tag{15}$$

Similarly, the balance of torques is expressed by the following equation:

$$\left[\frac{\partial W_{\text{e}}}{\partial\boldsymbol{n}}-\mathrm{div}\left(\frac{\partial W_{\text{e}}}{\partial\nabla\boldsymbol{n}}\right)\right]\times\boldsymbol{n}+\varrho\frac{\partial\sigma_{\text{K}}}{\partial\boldsymbol{n}}\times\boldsymbol{n}+\frac{\partial R_{\text{a}}}{\partial\mathring{\boldsymbol{n}}}\times\boldsymbol{n}=\mathbf{0},\tag{16}$$

where *three* different torques are balanced, namely, an *elastic* torque arising from the distortion energy density W_{e}, an *acoustic* torque representing the action exerted by the acoustic field on the nematic director, and a *viscous* torque arising from the dissipation function R_{a}. By using the constitutive equations (4), (6) and (8) for σ_{K}, W_{e}, and R_{a}, we give Eq. (16) the following form:

$$\begin{aligned}&-K\,\mathrm{div}\,(\nabla\boldsymbol{n})\times\boldsymbol{n}+u_2\varrho(\nabla\varrho\cdot\boldsymbol{n})\nabla\varrho\times\boldsymbol{n}\\ &+\gamma_1\dot{\boldsymbol{n}}\times\boldsymbol{n}+(\gamma_2\mathbf{D}\boldsymbol{n}-\gamma_1\mathbf{W}\boldsymbol{n})\times\boldsymbol{n}=\mathbf{0}.\end{aligned}\tag{17}$$

As can be seen from both Eqs. (13) and (17), the acoustic action on a nematic liquid crystal is twofold: On one hand, an acoustic field affects the flow through the stress tensor $\mathbf{T}_{\text{K}}$; on the other hand, it is able to act on the nematic director by transferring a torque to it. Equation (17) actually suggests that the effect of $\nabla\varrho$ in the acoustic torque is the same as that of an external electric field. In Sec. IV, we will estimate the acoustic torque in typical experimental situations of wave propagation along with the magnitude of the electric field that would be required to produce the same effect.

Equations (10), (11), and (17), where $\mathbf{T}_{\text{E}}$, $\mathbf{T}_{\text{K}}$, and $\mathbf{T}_{\text{dis}}$ are as in Eqs. (12), (13), and (15), represent the basic balances of the theory. They will be solved later in a special setting.

III. PLANE WAVE SOLUTIONS

Here, we study the propagation of forced plane waves of condensation.[2] This propagation has already been studied in Ref. [16] under the simplifying assumption that $\boldsymbol{n}$ is held fixed by a compliant external action, such as a magnetic field. In such an approach, the elastic torque in Eq. (17) vanishes as does the viscous torque opposing the tumbling of $\boldsymbol{n}$, while the viscous torque opposing the acoustic flow vanishes once averaged in time. The only nonvanishing torque is the time-averaged acoustic torque, that, in Ref. [16], was imagined to be balanced by a reactive torque exerted by the external constraint keeping $\boldsymbol{n}$ fixed. Here, allowing the director to vibrate, we need to solve the balance equation of torques (17): We can no longer be content with reading the average unbalanced acoustic torque from it. Thus, our attention will now turn to the director motion and to its consequences on the balances of linear momentum and torque.

The linearized balance laws resulting from Eqs. (10), (11), and (17) and the constitutive relations in Eqs. (12), (13), and (15) are solved and are used to find the anisotropic dispersion of waves and to study the relationship between energy dissipation and wave attenuation. Solutions are sought in the plane wave form

$$\varrho(\boldsymbol{x},t)=\varrho_0(1+s_0\,\mathrm{Re}\,E),\quad \boldsymbol{v}(\boldsymbol{x},t)=s_0\,\mathrm{Re}\,(E\boldsymbol{a}),\tag{18}$$

where $E:=e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, Re denotes the real part of a complex number, $\boldsymbol{x}$ is the position vector, ϱ_0 is the unperturbed mass density and s_0 is a small dimensionless parameter measuring the scale of perturbation, $\boldsymbol{k}$ is the *complex* wave vector to be determined in terms of the angular frequency ω, and $\boldsymbol{a}$ is an unknown complex amplitude vector. We also allow for a director *libration*[3] described by

$$\boldsymbol{n}=[\mathbf{I}+s_0\,\mathrm{Re}\,(E\mathbf{A})]\boldsymbol{n}_0,\tag{19}$$

where $\mathbf{A}$ is a complex skew-symmetric tensor and $\boldsymbol{n}_0$ is a uniform unperturbed director field.

The basic governing equations are solved within the preceding class of flows, in the limit where s_0 is a small perturbation parameter. In particular, it follows from Eq. (18) that

$$\mathbf{D}=\tfrac{1}{2}s_0\,i\,E(\boldsymbol{a}\otimes\boldsymbol{k}+\boldsymbol{k}\otimes\boldsymbol{a}),\tag{20a}$$

$$\mathbf{W}=\tfrac{1}{2}s_0\,i\,E(\boldsymbol{a}\otimes\boldsymbol{k}-\boldsymbol{k}\otimes\boldsymbol{a}).\tag{20b}$$

Here, and in what follows, we fail to take the real part of equations such as Eqs. (20), while keeping in mind that, as in Eqs. (18) and (19), only their real parts bear a physical meaning. Up to first order in s_0, Eq. (10) becomes

$$\omega=\boldsymbol{a}\cdot\boldsymbol{k}.\tag{21}$$

By using Eqs. (18) and (20a), one readily arrives at

$$\varrho\dot{\boldsymbol{v}}=-s_0\,i\,\varrho_0\omega E\boldsymbol{a}+o(s_0).\tag{22}$$

Since $\boldsymbol{n}_0$ is uniform in space, it follows from Eq. (12) that $\mathbf{T}_{\text{E}}$ is $o(s_0)$, and so, at the lowest approximation in s_0, the elastic stress does not contribute to the balance of linear momentum. On the other hand, by Eqs. (13) and (14), one obtains that

$$\mathrm{div}\,\mathbf{T}_{\text{K}}=-s_0\varrho_0 i\,E\left\{c_0^2+\varrho_0^2[u_1k^2+u_2(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2]\right\}\boldsymbol{k}+o(s_0),\tag{23}$$

where

$$c_0(\varrho_0):=\sqrt{p_0'(\varrho_0)}\tag{24}$$

is the velocity of sound in the *isotropic* limit where $u_1=u_2=0$. Finally, by Eqs. (20a), (20b), and (21), one also arrives at

$$\begin{aligned}\mathrm{div}\,\mathbf{T}_{\text{dis}}&=\tfrac{1}{2}(\gamma_1+\gamma_2)s_0\omega E(\boldsymbol{k}\cdot\mathbf{A}\boldsymbol{n}_0)\boldsymbol{n}_0\\ &\quad+\tfrac{1}{2}(\gamma_2-\gamma_1)s_0\omega E(\boldsymbol{k}\cdot\boldsymbol{n}_0)\mathbf{A}\boldsymbol{n}_0\\ &\quad-\tfrac{1}{2}s_0E\left\{\left[\tfrac{1}{2}(\gamma_1-2\gamma_2+\gamma_3)(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2+\gamma_4k^2\right]\boldsymbol{a}\right.\\ &\quad+\left[\tfrac{1}{2}(\gamma_3-\gamma_1+4\gamma_7)(\boldsymbol{a}\cdot\boldsymbol{n}_0)(\boldsymbol{k}\cdot\boldsymbol{n}_0)\right.\end{aligned}$$

[2]Not to be confused with the shear acoustic waves studied in Refs. [24] and [25].

[3]By *libration*, here, we mean a motion in which the director keeps a nearly uniform orientation $\boldsymbol{n}_0$ and vibrates about it as a consequence of the flow perturbation.

$$+(\gamma_4+2\gamma_6)(\boldsymbol{a}\cdot\boldsymbol{k})\big]\boldsymbol{k}$$
$$+\big[\tfrac{1}{2}(\gamma_3-\gamma_1+4\gamma_7)(\boldsymbol{k}\cdot\boldsymbol{n}_0)(\boldsymbol{k}\cdot\boldsymbol{a})$$
$$+\tfrac{1}{2}(\gamma_1+2\gamma_2+\gamma_3)(\boldsymbol{a}\cdot\boldsymbol{n}_0)k^2$$
$$+2\gamma_5(\boldsymbol{a}\cdot\boldsymbol{n}_0)(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2\big]\boldsymbol{n}_0\big\}, \tag{25}$$

where $k^2 := \boldsymbol{k}\cdot\boldsymbol{k}$. Equations (22), (23), and (25) will be used in Sec. III B to derive the propagation equation that applies in the presence of director libration. To this end, we also need to solve the balance equation of torques (17).

A. Libration equation

At the lowest order in s_0 and under the assumption that $\boldsymbol{n}_0$ is uniform in space and constant in time, Eq. (17) becomes

$$[Kk^2 - i\,\gamma_1\omega](\mathbf{A}\boldsymbol{n}_0)\times\boldsymbol{n}_0 + \tfrac{1}{2}i\,(\gamma_2-\gamma_1)(\boldsymbol{k}\cdot\boldsymbol{n}_0)\boldsymbol{a}\times\boldsymbol{n}_0 + \tfrac{1}{2}i\,(\gamma_2+\gamma_1)(\boldsymbol{a}\cdot\boldsymbol{n}_0)\boldsymbol{k}\times\boldsymbol{n}_0 = \mathbf{0}. \tag{26}$$

By the constraint set in Ref. [16] on the director motion, the balance of torque was there only satisfied on average since, as shown by Eq. (26), for a proper instantaneous balance, the viscous torque exerted by the acoustic flow entrains a director vibration. It is also worth noting that, at the lowest order of approximation in the acoustic condensation parameter s_0, the acoustic torque in Eq. (17) does not contribute to the libration equation (26). As already shown in Ref. [16], the acoustic torque is second order in s_0 and manifests itself at times longer than the acoustic period through a time-averaged action similar in character to an *acoustic streaming*. We will further elaborate on this in Secs. IV and V, where we will also estimate the acoustic torque at second order in s_0 and compare it with the first-order viscous torques appearing in Eq. (26); here, we only remark that, among the second-order effects that characterize the slow (streaming) dynamics taking place at time and length scales larger than the acoustic ones, we should also include Ericksen elastic stress in the balance equation of linear momentum.

Since $\mathbf{A}$ is skew symmetric, $\mathbf{A}\boldsymbol{n}_0$ is orthogonal to $\boldsymbol{n}_0$. Letting $\mathbf{A}\boldsymbol{n}_0 = \boldsymbol{d}\times\boldsymbol{n}_0$, with $\boldsymbol{d}\perp\boldsymbol{n}_0$, we can easily solve Eq. (26) for $\boldsymbol{d}$ and then arrive at

$$\mathbf{A}\boldsymbol{n}_0 = -\Sigma\big\{\gamma_2(\boldsymbol{k}\cdot\boldsymbol{n}_0)(\boldsymbol{a}\cdot\boldsymbol{n}_0)\boldsymbol{n}_0 + \tfrac{1}{2}(\gamma_1-\gamma_2)(\boldsymbol{k}\cdot\boldsymbol{n}_0)\boldsymbol{a} - \tfrac{1}{2}(\gamma_1+\gamma_2)(\boldsymbol{a}\cdot\boldsymbol{n}_0)\boldsymbol{k}\big\}, \tag{27}$$

where

$$\Sigma := \frac{1}{\gamma_1\omega + i\,Kk^2}.$$

A consequence of Eq. (27) is especially worth taking notice of. In the limit as both viscosities γ_1 and γ_2 vanish, so does $\mathbf{A}\boldsymbol{n}_0$; this implies, by Eq. (19), that no director libration occurs in that limit, and a wave can propagate while $\boldsymbol{n}$ remains immobile, even in the absence of any external restraining field. In brief, one could also say that the director libration is a motion fed by dissipation. Hereafter, we will assume that $\gamma_1 > 0$.

We now use the explicit solution in Eq. (27) for the director libration to derive the equation that governs the wave propagation from the balance of linear momentum Eq. (11).

B. Wave propagation

By employing Eqs. (23) and (25), with the aid of Eq. (27), up to first order in s_0, Eq. (11) is reduced to the purely kinematic form

$$2\,i\,\varrho_0\omega\boldsymbol{a} = 2\,i\,\varrho_0\big\{c_0^2 + \varrho_0^2[u_1k^2 + u_2(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2]\big\}\boldsymbol{k}$$
$$+\big[\tfrac{1}{2}(\gamma_1-2\gamma_2+\gamma_3)(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2+\gamma_4k^2 - \tfrac{1}{2}(\gamma_2-\gamma_1)^2\Sigma\omega(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2\big]\boldsymbol{a}$$
$$+\big[\tfrac{1}{2}(\gamma_3-\gamma_1+4\gamma_7)(\boldsymbol{a}\cdot\boldsymbol{n}_0)(\boldsymbol{k}\cdot\boldsymbol{n}_0)+(\gamma_4+2\gamma_6)\times(\boldsymbol{a}\cdot\boldsymbol{k})+\tfrac{1}{2}(\gamma_1^2-\gamma_2^2)\Sigma\omega(\boldsymbol{k}\cdot\boldsymbol{n}_0)(\boldsymbol{a}\cdot\boldsymbol{n}_0)\big]\boldsymbol{k}$$
$$+\big[\tfrac{1}{2}(\gamma_3-\gamma_1+4\gamma_7)(\boldsymbol{k}\cdot\boldsymbol{n}_0)(\boldsymbol{k}\cdot\boldsymbol{a}) + \tfrac{1}{2}(\gamma_1+2\gamma_2+\gamma_3)(\boldsymbol{a}\cdot\boldsymbol{n}_0)k^2$$
$$+2\gamma_5(\boldsymbol{a}\cdot\boldsymbol{n}_0)(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2 + 2\gamma_2^2\Sigma\omega(\boldsymbol{k}\cdot\boldsymbol{n}_0)^2(\boldsymbol{a}\cdot\boldsymbol{n}_0) + \tfrac{1}{2}(\gamma_1^2-\gamma_2^2)\Sigma\omega(\boldsymbol{k}\cdot\boldsymbol{n}_0)(\boldsymbol{a}\cdot\boldsymbol{k})$$
$$-\tfrac{1}{2}(\gamma_1+\gamma_2)^2\Sigma\omega(\boldsymbol{a}\cdot\boldsymbol{n}_0)k^2\big]\boldsymbol{n}_0, \tag{28}$$

where all γ_i's are evaluated at the unperturbed density ϱ_0. We let $\boldsymbol{k}$ and $\boldsymbol{a}$ be represented as

$$\boldsymbol{k} = k\boldsymbol{e} \quad \text{and} \quad \boldsymbol{a} = a_e\boldsymbol{e} + a_n\boldsymbol{n}_0, \quad a_n, a_e, k \in \mathbb{C}, \tag{29}$$

where the unit vector $\boldsymbol{e}$ designates the propagation direction and k, a_e, and a_n are all complex numbers to be determined. The imaginary part k_2 of k is associated with the attenuation of the wave: $1/k_2$ represents the attenuation length,[4] that is, the length over which the wave amplitude is reduced by the factor $1/e$. Equation (28) must be supplemented with the mass continuity equation (21), which, by Eq. (29), takes the form

$$ka_e + ka_n\cos\beta = \omega, \quad \text{with} \quad \cos\beta := \boldsymbol{e}\cdot\boldsymbol{n}_0. \tag{30}$$

It follows from Eqs. (29) and (30) that, whenever $\sin\beta = 0$, a_e and a_n are not uniquely defined; we resolve this ambiguity by setting $a_n = 0$ for $\sin\beta = 0$.

Were $\Sigma = 0$, Eq. (28) would reduce to the propagation equation found in Ref. [16] in the absence of director libration. What makes Eq. (28) more difficult to solve than that equation is the way Σ depends on the unknown k^2. Here, we assume that

$$\gamma_1\omega \gg K|k^2|, \tag{31}$$

so that, in Eq. (28), $\Sigma\omega$ can be approximated by $1/\gamma_1$. Physically, this approximation amounts to disregarding the elastic torque in the balance equation (17); thus, elastic effects disappear from the balances of both linear momentum and torque, although for different reasons. In Sec. III D, we will derive the *upper* bound to be imposed on ω to make (31) compatible with the solution for the propagation equation obtained here. As in Ref. [16], we also consider the limit of Eq. (28) where all viscosities are small. More precisely, we assume that there is a small dimensionless parameter ε_0 such that

$$\gamma_i = \varrho_0\frac{c_0^2}{\omega}\,O(\varepsilon_0), \quad i = 1,\ldots,7, \tag{32}$$

[4]In Sec. III C, we will show that inequalities (9) imply $k_2 \geqslant 0$.

and we further seek solutions of Eq. (28) such that

$$k_2 = \frac{\omega}{c_0}\, O(\varepsilon_0), \quad a_n = c_0\, O(\varepsilon_0), \tag{33}$$

where c_0 is as in Eq. (24). The validity of Eq. (32) requires that ω does not exceed an upper bound that will be discussed in Sec. III D along with the one that makes (31) acceptable.

Under assumptions (31)–(33), proceeding exactly as in Ref. [16], we finally arrive at the following solution of Eqs. (28) and (30) at the lowest order of approximation in ε_0:

$$k = \frac{\omega}{c} + i\,k_2, \tag{34a}$$

$$k_2 = \frac{\omega^2}{2\varrho_0 c_0^3}\frac{1}{\frac{c}{c_0}+\frac{1}{2}\omega^2\tau^2\frac{c_0}{c}}\left[\gamma_4+\gamma_6+\left(\gamma_3+2\gamma_7-\frac{\gamma_2^2}{\gamma_1}\right)\cos^2\beta + \left(\gamma_5+\frac{\gamma_2^2}{\gamma_1}\right)\cos^4\beta\right], \tag{34b}$$

$$a_n = -\frac{i}{2}\frac{\cos\beta}{\sin^2\beta}\frac{\omega}{c\varrho_0}\left[\gamma_3+2\gamma_7-\frac{\gamma_2^2}{\gamma_1} - \left(\gamma_3-2\gamma_5+2\gamma_7-3\frac{\gamma_2^2}{\gamma_1}\right)\cos^2\beta - 2\left(\gamma_5+\frac{\gamma_2^2}{\gamma_1}\right)\cos^4\beta\right] \quad \text{for } \sin\beta \neq 0, \tag{34c}$$

$$a_e = c - i\,\frac{c^2}{\omega}k_2 - a_n\cos\beta. \tag{34d}$$

Here, c is the velocity of sound along $\boldsymbol{e}$, which depends on both ω and β through the equation,

$$\frac{c}{c_0} = \frac{\omega\tau}{\sqrt{2(\sqrt{1+\omega^2\tau^2}-1)}}, \tag{35}$$

where τ is the *anisotropic* characteristic time defined by

$$\tau := 2\frac{\varrho_0}{c_0^2}\sqrt{u_1+u_2\cos^2\beta}. \tag{36}$$

The same expression (35) for c was obtained in Ref. [16] in the absence of director libration, while the expressions for k_2, a_n, and a_e would formally reduce to the corresponding ones found in Ref. [16] in the limit as $\gamma_1/\gamma_2 \to \infty$, where the director libration would be hampered by an arbitrarily large rotational viscosity γ_1. Here, we record, for future reference, the limiting expression of k_2 in the absence of libration:

$$k_2^\infty = \frac{\omega^2}{2\varrho_0 c_0^3}\frac{1}{\frac{c}{c_0}+\frac{1}{2}\omega^2\tau^2\frac{c_0}{c}} \times [\gamma_4+\gamma_6+(\gamma_3+2\gamma_7)\cos^2\beta+\gamma_5\cos^4\beta]. \tag{37}$$

As already pointed out in Ref. [16] for k_2^∞, since c in Eq. (35) is a function of ω, k_2 in Eq. (34b) does not depend on ω in a purely quadratic fashion, as in earlier theoretical studies on wave propagation in nematic liquid crystals [26–29]. Such a nonquadratic dependence is a characteristic signature of our assumption on the Korteweg nature of the acoustic coupling; it will be quantitatively compared in Sec. IV with the available experimental data.

C. Wave attenuation

We now proceed to show that both k_2 and k_2^∞ are not negative whenever the dissipation inequalities (9) are satisfied. To this end, we set $z := \cos^2\beta$ and call $h(z)$ the function defined by the expression enclosed in brackets on the right side of Eq. (34b):

$$h(z) := \gamma_4+\gamma_6+(\gamma_3+2\gamma_7)z+\gamma_5 z^2 - \frac{\gamma_2^2}{\gamma_1}[-(z^2-z)].$$

Clearly, the sign of k_2 is the same as the sign of h. Since we assumed that $\gamma_1 > 0$, by Eq. (9f), we also have that

$$\frac{\gamma_2^2}{\gamma_1} \leqslant \gamma_3 + 2\gamma_4,$$

and so,

$$h(z) \geqslant \gamma_4+\gamma_6+2(\gamma_7-\gamma_4)z+(\gamma_5+\gamma_3+2\gamma_4)z^2 =: h_0(z) \quad \forall\, z \in [0,1]. \tag{38}$$

It readily follows from Eqs. (9c)–(9e) that both $h_0(0) \geqslant 0$ and $h_0(1) \geqslant 0$. Thus, if h_0 is either linear or concave, that is, if $\gamma_5+\gamma_3+2\gamma_4 \leqslant 0$, then $h_0(z) \geqslant 0\, \forall z \in [0,1]$, which, by (38), is the desired conclusion. If, on the other hand, h_0 is convex, that is, if $\gamma_5+\gamma_3+2\gamma_4 > 0$, the desired conclusion follows from the inequality $h_0(z_m) \geqslant 0$, where z_m is the minimizer of h_0 in $\mathbb{R}$. Indeed, an easy computation shows that

$$h_0(z_m) = \frac{(\gamma_4+\gamma_6)(\gamma_3+\gamma_5+2\gamma_4)-(\gamma_7-\gamma_4)^2}{\gamma_3+\gamma_5+2\gamma_4}.$$

While the denominator of this ratio is positive by assumption, the numerator can be shown to be non-negative by taking the product of the left sides of (9c) and (9e) and by adding the result to the left side of (9g).

Similarly, we show that $k_2^\infty \geqslant 0$. By Eq. (37), h is then replaced by

$$h^\infty(z) := \gamma_4+\gamma_6+(\gamma_3+2\gamma_7)z+\gamma_5 z^2,$$

which can also be written as

$$h^\infty(z) = h(z) + \frac{\gamma_2^2}{\gamma_1}[-(z^2-z)] \geqslant h(z) \geqslant 0, \quad \forall z \in [0,1],$$

since $\gamma_1 > 0$.

D. Admissible frequency ranges

Several simplifying assumptions have been made to arrive at Eqs. (34); here, we identify the ranges where the angular frequency ω of the propagating wave must be chosen to make these assumptions admissible.

First, we identify the values of ω that make ε_0 in Eq. (32) a small parameter. From Ref. [30], estimating the velocity of sound $c_0 = 1.3 \times 10^3$ ms^{-1}, from Ref. [21] (p. 231), estimating the average viscosity $\gamma = 10^{-1}$ Pa s, and from Ref. [31], estimating the mass density $\varrho_0 = 10^3$ kg m^{-3}, one easily sees, from Eq. (32), that $\varepsilon_0 \ll 1$ whenever $\omega \ll \omega_\gamma$, with

$$\omega_\gamma := \frac{c_0^2\varrho_0}{\gamma} = 0.8 \times 10^{10}\ \text{s}^{-1} \sim 10^4\ \text{MHz}.$$

Second, we identify the angular frequencies that make the solution (34) of the propagation equation (28) compatible with the assumption (31). To this end, we recall, from Ref. [16], that $c \approx c_0$ for $\omega\tau < 10$, and so, since by Eq. (34a), $k \sim \omega/c$, estimating γ_1 as 2γ and taking $K \sim 10^{-11}$ N from Ref. [21] (p. 103), we see that Eq. (31) is satisfied for the solution (34) whenever $\omega \ll \omega_K$, with

$$\omega_K := \frac{c_0^2\gamma_1}{K} = 3.38 \times 10^{16}\ \mathrm{s}^{-1} \sim 10^{10}\ \mathrm{MHz}.$$

Finally, as remarked in Sec. II, our theory has neglected the director inertia. Such an approximation is valid if κ in Eq. (5) is much smaller than the kinetic energy density associated with the acoustic flow. By Eqs. (18), (29), and (34d), this latter can be estimated as $\frac{1}{2}\varrho_0 s_0^2 c_0^2$. On the other hand, by Eq. (19), $|\dot{\boldsymbol{n}}| = s_0\omega|\mathbf{A}\boldsymbol{n}_0| + O(s_0^2)$, and by Eq. (27), for the solution (34), $|\mathbf{A}\boldsymbol{n}_0| = O(1)$, so that κ can be neglected whenever $\omega \ll \omega_\delta$, with[5]

$$\omega_\delta := \frac{c_0}{\delta} = 6.5 \times 10^{12}\ \mathrm{s}^{-1} \sim 10^6\ \mathrm{MHz}.$$

It is clear from the previous estimates that the largest upper bound on ω for the validity of our theory is ω_K: Such a bound of an elastic origin can still not be exceeded when the rotational kinetic energy κ comes into play. For $\omega \sim \omega_\delta$, however, our solution (34) ceases to be valid, as the upper bound ω_γ is violated, and so, Eq. (32) no longer applies. Thus, the most stringent bound on ω is ω_γ; it will follow from the estimate of τ in Sec. IV that this easily complies with the requirement that $\omega\tau < 10$.

IV. PHENOMENOLOGICAL PARAMETERS

Using data published in the literature for *N*-(*p*-methoxybenzylidene)-*p*-butylaniline (MBBA), numerical evaluations for both dispersion and attenuation are made in this section and are compared to acoustic experiments. Our objective is to estimate the phenomenological parameters introduced by our theory, namely, u_1, u_2, γ_6, and γ_7.

To account for the experimental data available in Refs. [4,13], we introduce a measure of anisotropy for the speed of sound c in Eq. (35) as

$$\Delta c := \frac{c - c|_{\beta=\pi/2}}{c|_{\beta=0}}. \tag{39}$$

Δc is a function of both ω and β, which vanishes for $\beta = \frac{\pi}{2}$. By assuming that

$$\varepsilon := \frac{u_2}{u_1} \tag{40}$$

is a small parameter, as in Ref. [16], we arrive at[6]

$$\Delta c = \varepsilon f(\omega\tau_1)\cos^2\beta + O(\varepsilon^2), \tag{41}$$

where

$$\tau_1 := 2\frac{\varrho_0}{c_0^2}\sqrt{u_1}, \qquad f(x) := \frac{1}{4}\frac{x^2 - 2(\sqrt{1+x^2}-1)}{\sqrt{1+x^2}(\sqrt{1+x^2}-1)}. \tag{42}$$

By Eqs. (40) and (42), we can write Eq. (36) as

$$\tau = \tau_1\left(1 + \tfrac{1}{2}\varepsilon\cos^2\beta\right) + O(\varepsilon^2).$$

Since f is a positive function, the speed of propagation along the average orientation of the nematic director $\boldsymbol{n}_0$ is larger than the speed of propagation at right angles to it whenever $\varepsilon > 0$.

Similarly, we introduce the following measure of anisotropy for the attenuation k_2:

$$\Delta k_2' := \frac{\Delta k_2}{\Delta_\perp k_2}, \tag{43}$$

where $\Delta k_2 := k_2 - k_2|_{\beta=0}$ expresses the change in attenuation and $\Delta_\perp k_2 := \Delta k_2|_{\beta=\pi/2}$ is the value of Δk_2 when the wave propagates at right angles to $\boldsymbol{n}_0$. It follows from Eq. (34b) that, in the limit of small ε,

$$\Delta k_2' = G(\beta) + O(\varepsilon), \tag{44}$$

where

$$G(\beta) := \sin^2\beta\left(1 + \frac{\gamma_1\gamma_5 + \gamma_2^2}{\gamma_1(\gamma_3+\gamma_5+2\gamma_7)}\cos^2\beta\right).$$

To represent how $\Delta_\perp k_2$ depends on ω relative to a reference angular frequency ω_0, we introduce

$$\Delta k_2'' := \frac{\Delta_\perp k_2}{\Delta_\perp k_2|_{\omega=\omega_0}},$$

which, in the limit of small ε, becomes

$$\Delta k_2'' = \frac{\omega}{\omega_0}\sqrt{\frac{\sqrt{1+\tau_1^2\omega^2}-1}{\sqrt{1+\tau_1^2\omega_0^2}-1}}\sqrt{\frac{1+\tau_1^2\omega_0^2}{1+\tau_1^2\omega^2}} + O(\varepsilon). \tag{45}$$

In this approximation, $\Delta k_2''$, unlike $\Delta k_2'$, is independent of the viscosities: It depends on a single phenomenological parameter, that is, τ_1.

We used formulas (39), (41), (44), and (45) to fit the data measured for Δc in Ref. [13] for MBBA at the wave frequency $\omega/2\pi = 10$ MHz and the data measured for Δk_2 in Ref. [4] in the range of 2–6 MHz for $\omega/2\pi$. The former data were taken from Fig. 2 of Ref. [13], and the latter data were taken from Figs. 1 and 2 of Ref. [4].

We started from Fig. 2 of Ref. [4], which exhibits the dependence of Δk_2 on the wave frequency $\omega/2\pi$. We fitted these data with formula (45) using $\omega_0/2\pi = 6$ MHz as the reference frequency. By employing the built-in function FINDFIT in MATHEMATICA [32] for the least-squares fit, we obtained $\tau_1 = 3.47 \times 10^{-8}$ s for the only fitting parameter. From this value, we computed $\tau_1\omega_1 = 2.18$ at the frequency $\omega_1/2\pi = 10$ MHz used in Ref. [13]. We inserted this value of $\omega_1\tau_1$ in the exact formula (39) for Δc to validate our assumption about the smallness of ε, which was then found to be $\varepsilon = 7.74 \times 10^{-3}$ by fitting the data in Fig. 2 of Ref. [13] for $u_2 = \varepsilon u_1$. Thus, our using Eq. (45) to fit the data in Ref. [4] was fully justified. Moreover, the data in Fig. 2 of

[5]Here, we take the molecular radius of gyration δ as a typical molecular length, and following Ref. [21] (p. 98), we estimate $\delta \approx 2$ nm.

[6]In Eq. (41), we actually correct a typographical error that occurred in Eq. (87) of Ref. [16].

Ref. [13] for Δc could also be fitted directly with the function $A\cos^2\beta$, which gave $A = 11.25\times10^{-4}$, consistent with the value $A = 12\times10^{-4}$ found in Ref. [13] from the row data. On the other hand, by using the approximate formula (41) for Δc, we found instead $A = \varepsilon f(\tau_1\omega_1) = 11.28\times10^{-4}$, which is in very good agreement with the value found by the direct fit and, thus, further confirms the validity of our assumption on ε. Finally, we used Eq. (44) to find the value of the viscosity γ_7, taking, for $\gamma_1,\dots,\gamma_5$ in G, the standard values for MBBA (see p. 231 of Ref. [21]). Thus, we arrived at $\gamma_7 = 1.58\times10^{-1}$ Pa s. Since both $\Delta k_2'$ and $\Delta k_2''$ are independent of γ_6, this latter viscosity would only be determined with the aid of Eq. (34b) from direct measurements of k_2.

Now, we use the preceding estimates of phenomenological parameters in typical situations of acoustic wave propagation to justify our linear approximation in the balance equations of the theory. In particular, as already anticipated in Sec. III A, we estimate the magnitude of the second-order acoustic torque and compare it with the viscous torques surviving in the linearized equation (26). To this end, we recall from Ref. [16] that the time average $\boldsymbol{K}_\mathrm{a}$ of the acoustic torque over a period of the acoustic wave is given by

$$\boldsymbol{K}_\mathrm{a} = u_2\langle\varrho(\nabla\varrho\cdot\boldsymbol{n})\nabla\varrho\rangle\times\boldsymbol{n} = -\mathrm{sgn}\,(u_2)K_0(\boldsymbol{n}_0\cdot\boldsymbol{e})\boldsymbol{n}_0\times\boldsymbol{e}, \tag{46}$$

where

$$K_0 = \frac{1}{4}\frac{I_0}{c_0}\varepsilon\omega^2\tau_1^2\left(\frac{c_0}{c}\right)^2. \tag{47}$$

Here,

$$I_0 = \tfrac{1}{2}\varrho_0 s_0^2 c_0^3 e^{-2k_2\boldsymbol{x}\cdot\boldsymbol{e}} \tag{48}$$

is related to the acoustic intensity I_a carried by the wave through the equation,

$$I_\mathrm{a} = \langle p_\mathrm{K}\boldsymbol{v}\cdot\boldsymbol{e}\rangle = I_0\left[\left(\frac{c}{c_0}\right) + \frac{1}{4}\omega^2\tau^2\left(\frac{c}{c_0}\right)\right]. \tag{49}$$

To estimate K_0, we need to estimate the perturbation parameter s_0 and to find I_0 accordingly. To this end, we identified I_a with the applied acoustic intensity in a typical experimental situation. More precisely, we considered the range 10–10^3 mW cm^{-2} for the applied acoustic intensity. We also assumed that $|k_2\boldsymbol{x}\cdot\boldsymbol{e}|\ll 1$ and that $c\sim c_0$, as already done in Sec. III D. With the foregoing estimates for $\tau_1\omega_1 = 2.18$ and $\varepsilon = 7.74\times10^{-3}$, valid at the wave frequency 10 MHz, by Eqs. (48) and (49), we finally arrived at

$$s_0 = \sqrt{\frac{I_\mathrm{a}}{1.1\varrho_0 c_0^3}}. \tag{50}$$

Thus, choosing $I_\mathrm{a} = 10$ mW cm^{-2} and taking $c_0 = 1.3\times10^3$ ms^{-1} and $\varrho_0 = 10^3$ kg m^{-3}, as in Sec. III D, the perturbation parameter turns out to be $s_0 = 6.43\times10^{-6}$. Correspondingly, the magnitude of the acoustic torque per unit volume was found to be $K_0 = 3.22\times10^{-4}$ N m^{-2}, comparatively small with respect to the viscous torque per unit volume in the libration equation (26), estimated by $\gamma\omega_1 s_0 = 4.04\times10$ N m^{-2}, with γ as in Sec. III D. Then, choosing $I_\mathrm{a} = 1$ W cm^{-2}, we found $s_0 = 6.43\times10^{-5}$, $K_0 = 3.22\times10^{-2}$ N m^{-2}, and $\gamma\omega_1 s_0 = 4.04\times10^2$ N m^{-2}.

Equation (46) makes the analogy with the effect of an electric field already mentioned in Sec. II even stricter. In this case, the interaction energy density W_E and the corresponding torque $\boldsymbol{K}_\mathrm{E}$ suffered by the director $\boldsymbol{n}_0$ would be

$$W_\mathrm{E} = -\tfrac{1}{2}\epsilon_0\epsilon_a(\boldsymbol{n}_0\cdot\boldsymbol{E})^2, \quad \boldsymbol{K}_\mathrm{E} = \epsilon_0\epsilon_a|\boldsymbol{E}|^2(\boldsymbol{n}_0\cdot\boldsymbol{e})\boldsymbol{n}_0\times\boldsymbol{e},$$

where $\boldsymbol{E} = E\boldsymbol{e}$ is the external electric field, $\epsilon_0 = 8.8544\times10^{-12}$ C^2 N^{-1} m^{-2} is the dielectric permittivity in vacuum, and ϵ_a is the relative dielectric anisotropy. The typical value of ϵ_a for MBBA at 25 °C is −0.7 [33, p. 27]. With the aid of Eqs. (47), (48), and (50), the simple identification,

$$K_0 = \epsilon_0|\epsilon_a|E^2$$

shows that I_a is proportional to E^2 and yields $E = 0.72$ V μm^{-1} for $I_\mathrm{a} = 1$ W cm^{-2}.

V. CONCLUSIONS

We extended a variational theory for nematoacoustics recently proposed in Ref. [16] to the case where the nematic director can freely librate around an average orientation. According to this theory, the acoustic field interacts with the nematic texture through an additional elastic energy of Korteweg type, characterized by the phenomenological susceptibilities u_1 and u_2 introduced in Eq. (4). We estimated these constitutive parameters with the aid of experimental data available in the literature for the anisotropy of both sound speed and wave attenuation. We also estimated one additional viscosity (γ_7) introduced in the Rayleigh dissipation function (8) by the relaxation of the incompressibility constraint.

We solved the balance equations of the theory in the linear approximation of acoustic propagation. In particular, we sought plane wave solutions also involving the director libration, and we proved that the wave attenuation is not negative as a consequence of the semipositive definiteness of the Rayleigh dissipation function.

Seen from an experimentalist's point of view, perhaps the most relevant outcomes of our theory are the anisotropies in both sound speed and attenuation. In particular, formula (34b) for the attenuation differs from the ones in earlier works [26–29] in two ways. On one hand, it exhibits a nonquadratic dependence on the frequency ω, as the speed of propagation c, which enters Eq. (34b) also depends on ω. On the other hand, it is affected by the librational motion of the director in the coefficients of the anisotropic factor, as seen by comparing Eqs. (34b) and (37). New experiments could confirm these distinctive features of theory and possibly determine the viscosity γ_6 by direct attenuation measurements.

We showed that, in the (fast) acoustical regime, elastic stresses and torques do not affect the motion: Ericksen stress tensor is second order in the acoustical perturbation parameter, and the elastic torque is negligible with respect to viscous torques as long as the angular frequency of the propagating wave does not exceed an upper bound much larger than the frequency at which the rotational inertia of the director—neglected here as usual—should also be taken into account.

Also, the torque transferred to the director by the acoustic wave is second order in the acoustical perturbation parameter and is negligible with respect to viscous torques, as confirmed by the direct estimate in Sec. IV. Thus, acoustic torques and elastic stresses act at time and length scales larger than the acoustic ones: They should be regarded as *streaming* sources that affect the flow through their time averages. Other steady motions of the director texture can then take place at time and length scales much larger than the acoustic characteristic times and lengths, including the director relaxation dynamics. Correspondingly, although no net hydrodynamic flow takes place at the time scale of the acoustic vibrations, at longer time scales, even an initially stagnant fluid may develop steady flows by *acoustic* streaming, as already proved for isotropic viscous fluids in a vast literature [34–37], originating from Rayleigh's work [38–40]. For nematic liquid crystals, elastic stresses and acoustic torques are new streaming sources: They both affect the slow director and flow dynamics, mutually interwoven, for which the appropriate balance equations will be derived and studied in Ref. [41].

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