Solvable quantum nonequilibrium model exhibiting a phase transition and a matrix product representation

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(Received 3 November 2010; published 11 January 2011)

We study a one-dimensional *XX* chain under nonequilibrium driving and local dephasing described by the Lindblad master equation. The analytical solution for the nonequilibrium steady state found for particular parameters in a previous study [M. Žnidarič, J. Stat. Mech. (**2010**) L05002] is extended to arbitrary coupling constants, driving, and a homogeneous magnetic field. All one-, two-, and three-point correlation functions are explicitly evaluated. It is shown that the nonequilibrium stationary state is not Gaussian. Nevertheless, in the thermodynamic and weak-driving limit it is only weakly correlated and can be described by a matrix product operator ansatz with matrices of fixed dimension 4. A nonequilibrium phase transition at zero dephasing is also discussed. It is suggested that the scaling of the relaxation time with the system size can serve as a signature of a nonequilibrium phase transition.

DOI: 10.1103/PhysRevE.83.011108 PACS number(s): 05.30.−d, 03.65.Yz, 05.70.Ln, 75.10.Pq

I. INTRODUCTION

Interesting processes in physics are frequently associated with nonequilibrium situations. For instance, for a device to do some work, an energy current must flow. The state of the system is therefore a nonequilibrium one. It is clearly desirable to understand nonequilibrium physics; unfortunately, as opposed to equilibrium physics, no general theory of nonequilibrium processes exists, for instance, for their stationary distribution. For equilibrium systems, in contrast, the invariant equilibrium distribution is well known. The main difference between equilibrium and nonequilibrium systems is that in an equilibrium system, a principle of detailed balance holds, stating that the net flow of probability between any two states x_1 and x_2 is 0. That is, in equilibrium the probability of going from x_1 to x_2 is equal to the probability of going from x_2 to x_1. This simple rule greatly simplifies the analysis. In a stationary nonequilibrium situation, in contrast, only the total probability flow out of a state x_1 has to be 0, and not along each connection individually. In view of the lack of general theory it would be desirable to find some exact solutions for nonequilibrium situations, from which we could perhaps draw some generic rules. Particularly simple are nonequilibrium states that do not change in time, also called nonequilibrium stationary states (NESSs).

In classical physics there are a number of exactly solvable nonequilibrium models, most notably, various stochastic lattice gasses described by exclusion processes [1–3]. The picture is quite different in quantum physics. There are hardly any analytically solvable models known; however, see, for instance, an example of a single spin coupled to a bath in a star configuration [4]. An exception of solvable models is those that are quadratic in the fermionic variables. One possibility for studying NESSs is to take an infinite system in which an infinite subpart serves as a bath. Such was the case in the doubly infinite *XY* chain studied in [5]. The other approach is to write an effective master equation that describes the evolution of only the central system without baths. The simplest master equation is the Lindblad equation, which can be diagonalized if the superoperator (a generator of a Markovian flow) is a quadratic function of fermionic operators [6], giving the exact solution for the NESS [7,8].

In the present work we are going to find a solution for quantum NESS in a system whose Lindblad superoperator $\mathcal{L}$ [Eq. (2)] is not a quadratic function of fermions but is, nevertheless, simple enough to enable an explicit solution. Some parts of the solution have been presented recently, in Refs. [9] and [10], while the same model has been numerically studied in [11]. The model we study here is the *XX* chain with a local dephasing at each site and coupled to nonequilibrium baths. A nonequilibrium *XX* model without dephasing was solved in [12]; for a compact solution see also [10]. Besides providing an explicit solution, we also show that it is fundamentally different from those in quadratic systems. Namely, the Wick theorem does not hold and the NESS is not Gaussian. Another interesting aspect is that in a certain thermodynamic or weak driving limit, the solution can be written as a matrix product operator (MPO) with matrices of small fixed dimension, 4. This extends the MPO solution obtained for the NESS in a model without dephasing [10]. The applicability of a matrix product ansatz has important implications for the properties of the system as well as for numerical methods that can be used to calculate NESS. It has been known for some time that the matrix product ansatz can describe ground states of certain one-dimensional systems [13], NESS of classical exclusion models [3,14], and is a rather useful concept in quantum information and numerical methods used to simulate quantum systems [15]. A MPO ansatz of small fixed dimension can describe the time evolution of certain operators in integrable systems [16–18]. The MPO solution presented here extends the applicability of a MPO ansatz to NESSs in nonquadratic systems.

The final section deals with phase transitions in NESSs. It is shown that the model exhibits a nonequilibrium phase transition at zero dephasing, going from the NESS that is Gaussian and displays a ballistic transport to the NESS that is non-Gaussian, shows diffusive spin transport, and exhibits long-range correlations for nonzero dephasing. We show that the characteristic feature of the phase transition point is a

faster closing of the gap of the superoperator with the system size than away from the phase transition. This seems to be a general property of nonequilibrium phase transitions; see recent studies in Refs. [7] and [19].

II. THE MODEL

The Hamiltonian of the *XX* spin chain in a homogeneous magnetic field is given by

$$H = \sum_{j=1}^{n-1} \left(\sigma_j^{\mathrm{x}}\sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}}\sigma_{j+1}^{\mathrm{y}}\right) + B\sum_{i=1}^{n}\sigma_i^{\mathrm{z}}, \tag{1}$$

with standard Pauli matrices. An exact evolution of a central system, like the *XX* chain in our case, which is coupled to the environment is, in general, complicated. However, if the environment correlation time is sufficiently small, that is, so that it has no "memory," and the evolution has the property of a semigroup, then the reduced evolution of the central system can be described by the Lindblad equation [20]:

$$\frac{d}{dt}\rho = \mathrm{i}[\rho, H] + \mathcal{L}^{\mathrm{dis}}(\rho) = \mathcal{L}(\rho). \tag{2}$$

The dissipative linear operator $\mathcal{L}^{\mathrm{dis}}$ can be written in terms of Lindblad operators L_k:

$$\mathcal{L}^{\mathrm{dis}}(\rho) = \sum_k ([L_k\rho, L_k^\dagger] + [L_k, \rho L_k^\dagger]). \tag{3}$$

The Lindblad master equation (2), can describe the most general completely positive trace preserving map that is a dynamical semigroup, that is, a map that is a semigroup for a continuous time parameter. While the complete positivity must arguably be satisfied by any evolution, the semigroup property can be violated when the environment has a "memory" that causes a back-action on the central system.

The model we study here has two different dissipative parts. One describes the action of two baths, inducing a nonequilibrium situation if they are different, while the other describes a local dephasing at each site, for instance, being due to the coupling of each site to unobservable degrees of freedom. The Lindblad dissipator is therefore a sum of two terms:

$$\mathcal{L}^{\mathrm{dis}} = \mathcal{L}^{\mathrm{bath}} + \mathcal{L}^{\mathrm{deph}}. \tag{4}$$

The dephasing part, $\mathcal{L}^{\mathrm{deph}} = \sum_{j=1}^{n}\mathcal{L}_j^{\mathrm{deph}}$, is a sum of $\mathcal{L}_j^{\mathrm{deph}}$, each of which acts only on the jth site and is described by a single Lindblad operator:

$$L_j^{\mathrm{deph}} = \sqrt{\frac{\gamma}{2}}\sigma_j^{\mathrm{z}}. \tag{5}$$

Sometimes it is useful to write a matrix representation of the dissipative superoperator $\mathcal{L}_j^{\mathrm{deph}}$. If we use a basis of Pauli matrices and we order them $\{\sigma_j^{\mathrm{x}}, \sigma_j^{\mathrm{y}}, \sigma_j^{\mathrm{z}}, \mathbb{1}_j\}$, we have a matrix representation:

$$\mathcal{L}_j^{\mathrm{deph}} = \begin{pmatrix} -2\gamma & 0 & 0 & 0 \\ 0 & -2\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Dephasing with strength γ causes an exponential decay of the off-diagonal elements of a density matrix, if we write it in the diagonal basis of σ^{z}. Using Jordan-Wigner transformation, this basis corresponds to the number basis of spinless fermions. Note that it is precisely the dephasing term that makes the superoperator nonquadratic, in fact quartic, in fermionic operators. Namely, the superoperator $\mathcal{L}^{\mathrm{deph}}$ is quadratic in the Lindblad operator $\propto \sigma^{\mathrm{z}}$, which is itself quadratic in fermionic operators.

For the dissipative bath part $\mathcal{L}^{\mathrm{bath}}$ we take the simplest possible Lindblad operators that are still able to describe a nonequilibrium situation. First, they are going to act locally only on the first and the last spin, and second, there will be only two operators at each end. Writing the dissipative part, $\mathcal{L}^{\mathrm{bath}} = \mathcal{L}_{\mathrm{L}}^{\mathrm{bath}} + \mathcal{L}_{\mathrm{R}}^{\mathrm{bath}}$, as a sum of a part acting only at the left end (site index $j = 1$ and label "L") and a part acting only at the right end (site index $j = n$ and label "R"), we have $\mathcal{L}_{\mathrm{L,R}}^{\mathrm{bath}}(\rho) = \sum_{k=1,2}([L_k^{\mathrm{L,R}}\rho, L_k^{\mathrm{L,R}\dagger}] + [L_k^{\mathrm{L,R}}, \rho L_k^{\mathrm{L,R}\dagger}])$. The two Lindblad operators at the left end are

$$L_1^{\mathrm{L}} = \sqrt{\Gamma_{\mathrm{L}}(1-\mu+\bar{\mu})}\,\sigma_1^+, \quad L_2^{\mathrm{L}} = \sqrt{\Gamma_{\mathrm{L}}(1+\mu-\bar{\mu})}\,\sigma_1^-, \tag{7}$$

while on the right end we have

$$L_1^{\mathrm{R}} = \sqrt{\Gamma_{\mathrm{R}}(1+\mu+\bar{\mu})}\,\sigma_n^+, \quad L_2^{\mathrm{R}} = \sqrt{\Gamma_{\mathrm{R}}(1-\mu-\bar{\mu})}\,\sigma_n^-, \tag{8}$$

with $\sigma_j^{\pm} = (\sigma_j^{\mathrm{x}} \pm \mathrm{i}\,\sigma_j^{\mathrm{y}})/2$. The matrix representation of both superoperators is

$$\mathcal{L}_{\mathrm{L}}^{\mathrm{bath}} = \Gamma_{\mathrm{L}}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & -4(\mu-\bar{\mu}) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$\mathcal{L}_{\mathrm{R}}^{\mathrm{bath}} = \Gamma_{\mathrm{R}}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & -4(-\mu-\bar{\mu}) \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

Such simple local Lindbald operators involving σ^+ and σ^- are often used when studying transport in spin chains; see, for instance [21–25]. Two parameters, Γ_{L} and Γ_{R}, play the role of a coupling strength, while μ and $\bar{\mu}$ determine the magnetization that the bath tries to impose on the chain ends. To see that this is indeed the case, one can look for a stationary state of the bath $\mathcal{L}_{\mathrm{L}}^{\mathrm{bath}}$ dissipator only. That is, we look for a single-spin state $\tilde{\rho}$ that satisfies $\mathcal{L}_{\mathrm{L}}^{\mathrm{bath}}(\tilde{\rho}) = 0$. One easily finds that $\tilde{\rho} \sim \mathbb{1}_1 - (\mu - \bar{\mu})\sigma_1^{\mathrm{z}}$, and therefore $\mathrm{tr}\,(\tilde{\rho}\sigma_1^{\mathrm{z}}) = -\mu + \bar{\mu}$. The chosen bath is therefore such that it tries to induce a local magnetization of size $-\mu + \bar{\mu}$ at the left end and $\mu + \bar{\mu}$ at the right end (these two parameters can be thought of as magnetizations of an infinite bath to which the central system is coupled). Of course, the nonequilibrium stationary state of the whole master equation (2), which, in addition, includes a unitary part and a dephasing, will have a slightly different magnetization at the ends. One can also invert the problem and ask: Is it possible to choose Lindblad operators that will target an arbitrary stationary state $\tilde{\rho}$ of $\mathcal{L}^{\mathrm{bath}}$, even on many qubits? The answer is affirmative, with the explicit procedure given in [24].

The goal of this paper is to analytically find the NESS of the master equation (2). This state, simply denoted ρ here, is an eigenstate of the Lindblad superoperator with an eigenvalue of 0, $\mathcal{L}(\rho) = 0$.

III. THE SOLUTION

As shown in Ref. [9], when $\bar{\mu} = 0$ the solution, that is, the NESS, can be sought in the form of a series in powers of the driving μ. While a perturbative expansion is always possible, our solution is different. It is nonperturbative due to the special algebraic structure. We discuss this point in more detail later. The ansatz for the NESS is

$$\rho = \tfrac{1}{2^n}(\mathbb{1} + \mu R^{(1)} + \mu^2 R^{(2)} + \cdots \mu^r R^{(r)} + \cdots). \tag{11}$$

As we shall show, the term $\mu^r R^{(r)}$ is of the order μ^r and there is a closed set of equations that give $\mu^r R^{(r)}$. The solution can therefore be obtained term by term, without having to deal with the whole set of exponentially many equations. We first discuss the case with the zero bath magnetization offset, $\bar{\mu} = 0$. The nonzero case is then obtained as a simple modification of the solution for $\bar{\mu} = 0$.

It was shown in Ref. [9] that the solution ρ is a sum of terms, where each is a product of operators σ_j^z and $j_k = 2(\sigma_k^x \sigma_{k+1}^y - \sigma_k^y \sigma_{k+1}^x)$ at different sites. j_k is a spin current operator. This simple structure is a consequence of the fact that all other operators that could result in j_k or σ_j^z when operated on by $\mathcal{L}$ are 0. The NESS can therefore be sought within the algebra of σ_j^z and j_k. Before actually going to the solution itself, let us discuss the role of a homogeneous magnetic field of strength B.

The action of a single $B\,\sigma_j^z$ term in the Hamiltonian is simple. The matrix of the superoperator $\mathcal{L}_j^{(\mathrm{B})}$ has only two nonzero elements—$\mathcal{L}_j^{(\mathrm{B})}(\sigma_j^x) = 2B\,\sigma_j^y$ and $\mathcal{L}_j^{(\mathrm{B})}(\sigma_j^y) = -2B\,\sigma_j^x$—and all others are 0, $\mathcal{L}_j^{(\mathrm{B})}(\sigma_j^{z,\mathbb{1}}) = 0$. Because the NESS is a sum of products of only σ_j^z and currents j_k, and because for a homogeneous field we have $\sum_j \mathcal{L}_j^{(\mathrm{B})}(j_k) = 0$ as well as $\sum_j \mathcal{L}_j^{(\mathrm{B})}(j_k j_{k+1}) = 0$, the NESS for zero field, $B = 0$, is also an exact solution for arbitrary nonzero field $B \neq 0$. In short, the homogeneous magnetic field has no influence on the NESS.

A. Hierarchy of connected correlations

Let us briefly argue why the NESS can be calculated term by term in the expansion over μ, Eq. (11), and why the set of equations for each term is closed. Without loss of generality we assume that $\bar{\mu} = 0$. A key is a simple algebra generated by various superoperators in the master equation. The dephasing term acts as $\mathcal{L}^{\mathrm{deph}}(j_k) = -4\gamma j_k$ and $\mathcal{L}^{\mathrm{deph}}(\sigma_j^z) = \emptyset$, as well as $\mathcal{L}^{\mathrm{deph}}(\mathbb{1}_j) = \emptyset$. The bath at the left end acts as $\mathcal{L}_{\mathrm{L}}^{\mathrm{bath}}(j_1) = -2j_1$, $\mathcal{L}_{\mathrm{L}}^{\mathrm{bath}}(\sigma_1^z) = -4\sigma_1^z$, $\mathcal{L}_{\mathrm{L}}^{\mathrm{bath}}(\mathbb{1}_1) = -4\mu\sigma_1^z$. Similar expressions hold for the bath at the right end. The unitary part due to H acts as $\mathcal{L}_{k,k+1}^{\mathrm{H}}(j_k) = 8(\sigma_{k+1}^z - \sigma_k^z)$, $\mathcal{L}^{\mathrm{H}}(\sigma_k^z) = (j_k - j_{k-1})$, and $\mathcal{L}^{\mathrm{H}}(\mathbb{1}_k) = \emptyset$. Note that there are no overlapping products of operators in ρ. The condition that there are no products of operators at the same site, that is, either $\sigma_j^z \sigma_j^z$ or $j_k j_k$, ensures that the explicit normalization of ρ by $1/2^n$ is not broken (11). Other terms not present due to hermiticity are $j_k \sigma_k^z + \sigma_k^z j_k \equiv \emptyset$ and $j_k \sigma_{k+1}^z + \sigma_{k+1}^z j_k \equiv \emptyset$; note, however, that $j_k j_{k+1} + j_{k+1} j_k \equiv -8(\sigma_k^x \mathbb{1}_{k+1} \sigma_{k+2}^x + \sigma_k^y \mathbb{1}_{k+1} \sigma_{k+2}^y)$ are allowed. From the action of superoperators we note two things: (i) if τ is a sum of operators, each of which is a product of nonoverlapping σ_j^z and j_k, then $\mathcal{L}(\tau)$ is again a sum of nonoverlapping σ_j^z and j_k; and (ii) the number of operators σ_j^z and j_k in each term is preserved by the action of $\mathcal{L}^{\mathrm{H}}$ and $\mathcal{L}^{\mathrm{deph}}$. The only superoperator that does not conserve the number of operators is that of the bath, which can create σ^z out of an identity. These two observations are crucial. Let us now write our solution ρ as a series in μ, Eq. (11), where the rth-order term is a sum of all possible nonoverlapping products, where each is a product of exactly r operators σ_j^z or j_k (here we always mean the number of nonidentity operators). We put all (unknown) coefficients appearing in the term $\mu^r R^{(r)}$ in a set $\mathcal{S}^{(r)}$. For instance, the set $S^{(0)}$ has only one coefficient, namely, a known normalization $1/2^n$ in front of $\mathbb{1}$, the set $S^{(1)}$ consists of n unknown coefficients in front of σ_j^z and $n-1$ unknown coefficients in front of j_k. The NESS must satisfy the equation $\mathcal{L}(\rho) = 0$, therefore, the coefficient in front of each operator appearing in $\mathcal{L}(\rho)$ must be 0. If we look at the coefficient in front of an operator that is a product of r nonidentity operators, then this coefficient is a linear function of coefficients in the set $S^{(r)}$ and coefficients in the set $S^{(r-1)}$ (these come from the action of the bath $\mathcal{L}^{\mathrm{bath}}$). This structure enables us to solve for $\mu^r R^{(r)}$, that is, coefficients in $S^{(r)}$, iteratively, starting with known $S^{(0)}$. Note that the only coefficient in $S^{(0)}$ is equal to $1/2^n \sim \mu^0$. Using this known $S^{(0)}$ we can now write a set of linear equations for coefficients in $S^{(1)}$, where the coefficient from $S^{(0)}$ will act as a source term, that is, an inhomogeneous part of equations. Because when $\mathcal{L}^{\mathrm{bath}}$ makes a term of type $S^{(r+1)}$ from $S^{(r)}$, it always multiplies it by μ, the source term will scale as μ^1. Therefore, all coefficients in $S^{(1)}$ are proportional to μ^1. Iteratively repeating the procedure we see that (i) coefficients in $S^{(r)}$ scale as μ^r, that is, the terms in $\mu^r R^{(r)}$ indeed scale as μ^r; and (ii) at each order we have to solve a closed set of equations for coefficients in $S^{(r)}$. These determine all r-point correlations in the NESS. Helping ourselves, for the moment, with the solution for the first three orders given in the following, we can say even more. If one writes equations for r-point connected correlations, instead of for nonconnected ones, we can also predict how the source term at the rth order scales with the number of spins n. Using equations from the previous order $S^{(r-1)}$, one can see that the source term scales as μ^r/n^r. In the following we see that the spin current, the coefficient of which is denoted b, scales in the same way. Therefore, the source terms scale as the rth power of the current, $\sim b^r$. A consequence of this is that the largest connected term in $\mu^r R^{(r)}$ scales as $\sim b^r n$ and comes from the connected correlation of r operators σ_j^z. The largest nonconnected term in $\mu^r R^{(r)}$, in contrast, scales as $\sim b^r n^r$ and comes from the product of r operators σ_j^z; see also the explicit ansatz, in the following.

We now find an explicit form of the first three orders. For special values of parameters, the solution for the first two orders was presented in [9].

B. No magnetization offset, $\bar{\mu} = 0$

Let us first find the NESS for a bath with a zero offset of magnetization, $\bar{\mu} = 0$. It is instructive to first find the

equilibrium stationary state in the absence of driving, when $\mu = 0$. In this case both baths (7) act with σ^+ and σ^- with equal probability, inducing no net magnetization. In fact, the equilibrium state is very simple and equal to the totally mixed state:

$$\rho_{\rm eq} = \frac{1}{2^n}\mathbb{1}. \tag{12}$$

This equilibrium state can therefore be thought of as an infinite temperature state.

1. First two orders

The ansatz for the first two orders is the following:

$$\mu R^{(1)} = \mu A + \mu B, \quad \mu A = \sum_{j=1}^{n} a_j \sigma_j^{\rm z}, \quad \mu B = \frac{b}{2}\sum_{k=1}^{n-1} j_k. \tag{13}$$

The spin current operator is $j_k = 2(\sigma_k^{\rm x}\sigma_{k+1}^{\rm y} - \sigma_k^{\rm y}\sigma_{k+1}^{\rm x})$. The second-order ansatz is

$$\mu^2 R^{(2)} = \frac{\mu^2}{2}(AB + BA) + \mu^2 C + \mu^2 D + \mu^2 F, \tag{14}$$

$$\mu^2 C = \sum_{j=1}^{n}\sum_{k=j+1}^{n}(C_{j,k} + a_j a_k)\sigma_j^{\rm z}\sigma_k^{\rm z},$$

$$\mu^2 D = \sum_{j=1}^{n-2}\frac{d_j}{2}\left(\sum_{l=j+1}^{n-1}\sigma_j^{\rm z} j_l - \sum_{l=1}^{n-1-j} j_l \sigma_{n+1-j}^{\rm z}\right), \tag{15}$$

$$\mu^2 F = \frac{f}{8}\sum_{\substack{k,l=1\\ k\neq l}}^{n-1} j_k j_l.$$

In the solution a specific factor will appear in all expressions for connected correlation functions. We denote it by the letter t,

$$t \equiv \frac{(\Gamma_{\rm L}+\Gamma_{\rm R})(1+\Gamma_{\rm L}\Gamma_{\rm R}) + 2(n-2)\gamma\Gamma_{\rm L}\Gamma_{\rm R}}{2\gamma\Gamma_{\rm L}\Gamma_{\rm R}}. \tag{16}$$

For large n it scales as $t \sim n$; in fact, for $\Gamma_{\rm L} = \Gamma_{\rm R} = 1$ it is equal to $n - 2 + 2/\gamma$. It therefore plays the role of an effective system size.

The solution for the first and second orders is found in exactly the same way as in Ref. [9] for specific parameters. One writes a closed set of equations and solves it. We do not repeat the details of the derivation here, but just present the solution for general values of parameters. The first-order terms are

$$b = -\mu\frac{2\Gamma_{\rm L}\Gamma_{\rm R}}{(\Gamma_{\rm L}+\Gamma_{\rm R})(1+\Gamma_{\rm L}\Gamma_{\rm R}) + 2(n-1)\gamma} = -\frac{\mu}{(t+1)\gamma} \tag{17}$$

and

$$\begin{aligned} a_1 &= -\mu - \frac{b}{\Gamma_{\rm L}},\\ a_2 &= a_1 - b(\Gamma_{\rm L} + 2\gamma),\\ a_3 &= a_2 - 2\gamma b,\\ &\vdots\\ a_{n-1} &= a_{n-2} - 2\gamma b,\\ a_n &= a_{n-1} - b(\Gamma_{\rm R} + 2\gamma) = \mu + \frac{b}{\Gamma_{\rm R}}. \end{aligned} \tag{18}$$

Alternatively, we can express local magnetizations a_j as

$$a_j = -\mu - b\, k_j^{(\rm L)} = \mu + b\, k_j^{(\rm R)}, \tag{19}$$

$$k_j^{(\rm L)} = \left\{\frac{1}{\Gamma_{\rm L}}, \frac{1+\Gamma_{\rm L}^2}{\Gamma_{\rm L}} + 2\gamma, \ldots, \frac{1+\Gamma_{\rm L}^2}{\Gamma_{\rm L}} + 2(n-1)\gamma + \Gamma_{\rm R}\right\},$$

$$k_j^{(\rm R)} = \left\{\frac{1+\Gamma_{\rm R}^2}{\Gamma_{\rm R}} + 2(n-1)\gamma + \Gamma_{\rm L}, \ldots, \frac{1+\Gamma_{\rm R}^2}{\Gamma_{\rm R}} + 2\gamma, \frac{1}{\Gamma_{\rm R}}\right\}. \tag{20}$$

Away from the boundary we have $k_{j+1}^{(\rm L)} - k_j^{(\rm L)} = 2\gamma$, while $k_{j-1}^{(\rm R)} - k_j^{(\rm R)} = 2\gamma$. The expectation value of the magnetization at site j is just $\langle\sigma_j^z\rangle = a_j$, and that of the spin current $\langle j_k\rangle = 2b$. Several interesting observations can be made. First, our one-spin Lindblad bath is constructed in such a way that it targets magnetization $\mp\mu$ on the first and the last sites. Because there are other terms besides the bath in the Lindblad equation, the actual magnetization in the NESS at the boundaries is slightly different. Namely, we have $a_1 = -\mu - b/\Gamma_{\rm L}$ and $a_n = \mu + b/\Gamma_{\rm R}$. Expectedly, one can see that if the coupling strength $\Gamma_{\rm L}$ (or $\Gamma_{\rm R}$) is very small, the difference from the targeted magnetization is large. Second, the magnetizaton difference between two neighboring sites is $2\gamma b$ in the bulk, while it is $b(2\gamma + \Gamma_{\rm L})$ at the left end and $b(2\gamma + \Gamma_{\rm R})$ at the right end. The magnetization profile is therefore linear, apart from two sites at both ends, where a contact resistance due to $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$ is felt. One could in fact generalize our solution to allow for an inhomogeneous dephasing γ_j at each site. The difference in the first-order solution, Eq. (18), would have the terms $\gamma_j + \gamma_{j+1}$ instead of 2γ. Using $\gamma_1 = \gamma_n = 0$ and with $\Gamma_{\rm L} = \Gamma_{\rm R} = \gamma$, one could therefore achieve a perfectly linear profile, without any contact resistance jumps at the boundaries. However, higher-order terms in μ are more complicated in this case (although of the same form), and we do not discuss inhomogeneous dephasing in the present work.

The expectation value of the current $2b$ is maximal for intermediate couplings $\Gamma_{\rm L,R}$. Keeping the system size and the dephasing strength γ fixed, we can see that b has a maximum at $\Gamma_{\rm L} = \Gamma_{\rm R} = 1$. It is equal to $|b_{\max}| = \mu/[2 + (n-1)\gamma]$. If the couplings are smaller, the current is lower, because the NESS is only weakly nonequilibrium due to the weak coupling. In contrast, if Γs are large, the current is again lower because the Hamiltonian part H, which transports magnetization, is less important compared to baths. The dependence of the maximal current on γ is trivial; the smaller the dephasing, the higher the current. Another observation is that the magnetization difference between the two ends is $a_n - a_1 = -b[\Gamma_{\rm L} + \Gamma_{\rm R} + 2\gamma(n-1)]$. The transport coefficient κ, defined via $j = -\kappa\frac{a_n - a_1}{n}$, is $\kappa = 2n/[\Gamma_{\rm L} + \Gamma_{\rm R} + 2(n-1)\gamma]$. As long as the dephasing is nonzero, it asymptotically behaves as $\kappa \sim 1/\gamma$ and is independent of the system size n. In contrast, if $n\gamma$ is smaller than $\Gamma_{\rm L}, \Gamma_{\rm R}$, for instance, if the dephasing is 0, then in the limit of weak coupling, $\Gamma_{\rm L}, \Gamma_{\rm R} \to 0$, κ diverges.

The expressions for the second-order terms are rather simple:

$$f = b^2(1 + 1/t), \tag{21}$$

$$d_{i,j} = \frac{b^2}{t}\begin{cases} -k_i^{(\rm L)}, & j > i,\\ k_i^{(\rm R)}, & j < i, \end{cases} \tag{22}$$

$$C_{i,j} = -\frac{b^2}{t}\left(k_i^{(\rm L)} k_j^{(\rm R)} + (1+t)\,\delta_{i+1,j}\right). \tag{23}$$

The connected correlation function of magnetization, $\langle\sigma_i^z\sigma_j^z\rangle_c = \langle\sigma_i^z\sigma_j^z\rangle - \langle\sigma_i^z\rangle\langle\sigma_j^z\rangle$, is equal to $1-a_i^2$ for $i=j$, while it is equal to $\langle\sigma_i^z\sigma_j^z\rangle_c = C_{i,j}$ for nondiagonal $i\neq j$. For large system size n the connected correlation function $C_{i,j}$ achieves its maximal value at $i\approx j\approx n/2$, when a product of $k_i^{(\mathrm{L})}k_j^{(\mathrm{R})}$ is the largest. We therefore have $\max(C_{i,j})\sim(\gamma nb)^2/t$. The maximum is again achieved at $\Gamma_\mathrm{L}=\Gamma_\mathrm{R}=1$. The dependence of the maximal z-z connected correlation on γ is, however, the opposite of that of the current. Here the correlations monotonously increase with increasing γ.

2. Third order

The third-order terms are obtained in an analogous way. The calculation is a little tedious; nevertheless, we managed to obtained closed expressions. Detailed results are given in the Appendix. At this point we just list the dominant term for large n. It is a three-point connected correlation of σ^z. Away from boundaries and for non-neighboring indexes, where the Kronecker δ terms are 0, the general expression, Eq. (A4), simplifies. For instance, for $\gamma=\Gamma_\mathrm{L}=\Gamma_\mathrm{R}=1$ and $i<j<k$, it is just

$$\left\langle\sigma_i^z\sigma_j^z\sigma_k^z\right\rangle_c = \frac{2(2\mu)^3}{n(n-1)}x(1-2y)\,(1-z), \tag{24}$$

where we have introduced rescaled position variables $x=\frac{i}{n+1}$, $y=\frac{j}{n+1}$, and $z=\frac{k}{n+1}$. For the same parameter values and $i<j$, the two-point function is

$$\left\langle\sigma_i^z\sigma_j^z\right\rangle_c = -\frac{(2\mu)^2}{n}x(1-y). \tag{25}$$

Scaling of the first two connected correlations on μ and n therefore follows a general rule: the r-point connected correlation function is $\sim\mu^r/n^{r-1}$, in accordance with the general discussion in Sec. III A. The form of these two correlation functions is the same as in some classical exclusion processes [2]; for a recent work on an analogous quantum master equation exhibiting some features similar to our model, see [27]. Whether this is only a consequence of the same hydrodynamic limit or whether there perhaps exists an exact mapping from the quantum model to a classical one is unknown at present.

C. Nonzero offset, $\bar{\mu}\neq 0$

Let us now go to the case where there is an offset of magnetization in the baths. For nonzero $\bar{\mu}$ the equilibrium state, when $\mu=0$, is again simple, although not trivial, $\sim\mathbb{1}$, as in the case of zero $\bar{\mu}$. It is a product state with nonzero magnetization:

$$\rho_\mathrm{eq} = \frac{1}{2^n}\prod_{j=1}^{n}\left(\mathbb{1}_j+\bar{\mu}\sigma_j^z\right). \tag{26}$$

It is therefore a state with all connected correlations being 0 (trivially Gaussian state) apart from nonzero magnetization. This equilibrium state can be equated to the grand canonical state $\rho_\mathrm{gc}\sim\exp\left[-\beta(H-\chi\,\Sigma^z)\right]$, with $\Sigma^z=\sum_j\sigma_j^z$, having an infinite temperature $\beta=0$ and a finite chemical potential times the inverse temperature, $\beta\,\chi=\frac{1}{2}\ln\left[(1+\bar{\mu})/(1-\bar{\mu})\right]$. One should note, however, that integrable systems, such as our XX chain, do not thermalize for generic local Lindblad baths [26].

Out of equilibrium, when $\mu\neq 0$, the solution is actually very similar to the one with zero $\bar{\mu}$. Namely, the only thing that changes is the expression for magnetization a_j, while all other n-point connected correlations are the same as in the case of no offset, $\bar{\mu}=0$. For nonzero offset the solution is therefore

$$a_j = \bar{\mu}-\mu-b\;k_j^{(\mathrm{L})}, \tag{27}$$

with the same $k_j^{(\mathrm{L})}$ [Eq. (20)] as before. All other terms (17), (23), (A4), (A5), and (A6), are the same.

IV. THE NESS IS NON-GAUSSIAN

In this section we show that the NESS obtained previously is not Gaussian, that is, that the Wick theorem does not hold. Because three-point connected correlation functions are nonzero [see, e.g., Eq. (A4)], it is obvious that the Wick theorem does not apply in spin variables. However, it is not clear that it does not apply in the spinless fermion picture.

A system of spin-1/2 particles can be mapped to spinless fermions using the Jordan-Wigner transformation. Denoting by c_j and $c_j^\dagger$ canonical fermionic annihilation and creation operators, satisfying anticommutators $\{c_j,c_k\}=0$, $\{c_j^\dagger,c_k^\dagger\}=0$, $\{c_j,c_k^\dagger\}=\delta_{j,k}\mathbb{1}$, the transformation is given by the mapping

$$\begin{aligned} c_j &= -\left(\sigma_1^z\cdots\sigma_{j-1}^z\right)\sigma_j^+,\\ c_j^\dagger &= -\left(\sigma_1^z\cdots\sigma_{j-1}^z\right)\sigma_j^-, \end{aligned} \tag{28}$$

or its inverse,

$$\begin{aligned} \sigma_j^x &= -\left(\sigma_1^z\cdots\sigma_{j-1}^z\right)(c_j+c_j^\dagger),\\ \sigma_j^y &= -\mathrm{i}\left(\sigma_1^z\cdots\sigma_{j-1}^z\right)(c_j-c_j^\dagger),\\ \sigma_j^z &= c_jc_j^\dagger-c_j^\dagger c_j = \mathbb{1}-2n_j, \end{aligned} \tag{29}$$

where we denote by $n_j=c_j^\dagger c_j$ a number (density) operator at site j. Denoting by $Z_j^{(r)}=\sigma_j^z\cdots\sigma_{j+r-1}^z$ a string of r consecutive σ^z, and introducing an energy-density-like operator $H_j^{(r+1)}\equiv\sigma_j^xZ_{j+1}^{(r-1)}\sigma_{j+r}^x+\sigma_j^yZ_{j+1}^{(r-1)}\sigma_{j+r}^y$ and a current-like operator $B_j^{(r+1)}\equiv\sigma_j^xZ_{j+1}^{(r-1)}\sigma_{j+r}^y-\sigma_j^yZ_{j+1}^{(r-1)}\sigma_{j+r}^x$, we have

$$\begin{aligned} H_j^{(r+1)} &= 2(c_j^\dagger c_{j+r}-c_jc_{j+r}^\dagger),\\ B_j^{(r+1)} &= 2\mathrm{i}(c_j^\dagger c_{j+r}+c_jc_{j+r}^\dagger),\\ H_j^{-(r+1)} &= 2(c_j^\dagger c_{j+r}^\dagger-c_jc_{j+r}),\\ B_j^{-(r+1)} &= 2\mathrm{i}(c_j^\dagger c_{j+r}^\dagger+c_jc_{j+r}), \end{aligned} \tag{30}$$

where $H_j^{-(r+1)}\equiv\sigma_j^xZ_{j+1}^{(r-1)}\sigma_{j+r}^x-\sigma_j^yZ_{j+1}^{(r-1)}\sigma_{j+r}^y$ and $B_j^{-(r+1)}\equiv\sigma_j^xZ_{j+1}^{(r-1)}\sigma_{j+r}^y+\sigma_j^yZ_{j+1}^{(r-1)}\sigma_{j+r}^x$. From our solution for the NESS (see also comments in Ref. [9]), we can see that expectations for all $H_j^{(r)}$, $B_j^{(r+1)}$, $H_j^{-(r)}$, and $B_j^{-(r)}$, apart from $B_j^{(2)}$, are 0. The only nonzero two-point

fermionic expectations are therefore $\langle c_j^\dagger c_j\rangle = (1-a_j)/2$ and $2\mathrm{i}\langle c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger\rangle = b$. An important point is that only on-site or nearest-neighbor two-point fermionic correlations are nonzero. For the NESS to be Gaussian and the Wick theorem to hold, all connected correlations beyond nearest neighbor would have to be 0. Because this is not the case, the NESS is clearly not Gaussian. For instance, rewriting the connected z-z correlation as $C_{i,j} = \langle\sigma_i^z\sigma_j^z\rangle - \langle\sigma_i^z\rangle\langle\sigma_j^z\rangle = 4(\langle c_i^\dagger c_i\, c_j^\dagger c_j\rangle - \langle c_i^\dagger c_i\rangle\langle c_j^\dagger c_j\rangle)$, and using the Wick theorem, we would have $C_{i,j} = -\langle c_i^\dagger c_j\rangle\langle c_i c_j^\dagger\rangle$. The last expression is nonzero only if $j = i+1$, whereas, in contrast, we have long-range correlations with all $C_{i,j}$ being nonzero if $\gamma \neq 0$. The Wick theorem therefore does not hold if $\gamma \neq 0$. The NESS is non-Gaussian and presents an interesting new solvable model. Whether it is, nevertheless, equivalent to some existing solvable model is at present unknown.

A. Matrix product operator ansatz

Although the NESS is non-Gaussian, it is still, in a sense, weakly correlated, meaning that it can be represented in terms of an MPO,

$$\rho = \frac{1}{2^n}\sum_{\alpha_j}\left\langle 1\middle|A_1^{(\alpha_1)}A_2^{(\alpha_2)}\cdots A_n^{(\alpha_n)}\middle|1\right\rangle \sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\cdots\sigma_n^{\alpha_n}, \qquad (31)$$

with matrices of small size. Note that in an MPO formulation a density matrix is treated as an element (a pure state) of a 4^n-dimensional Hilbert space of operators. For $\gamma = 0$, matrices A_j^α of finite size $D = 4$ that are independent of the system length suffice [10]. For nonzero dephasing an exact representation with matrices of fixed size is not possible. We have numerical indications, however, that the Schmidt rank, that is, the number of nonzero Schmidt coefficients, for a bipartite cut after the first m spins is $4m$. We have verified this conjecture by numerically computing an exact NESS for systems of up to $n = 10$ spins. Because the Schmidt rank is equal to the necessary dimension of matrices in the MPO ansatz, the exact representation of the NESS requires an MPO of dimension $D = 2n$.

Looking at the series solution for NESS (11), we can see that the largest connected term in the rth order $\mu^r R^{(r)}$ (i.e., r-point connected correlation function) scales, for large n, as $\sim b^r\, n = 1/n^{r-1}$ and comes from the connected correlation of r σ^z values. Therefore, in the thermodynamic limit of large n, one could neglect all higher-order connected correlations, keeping in the NESS only the leading terms that scale with n as ~ 1 and $\sim 1/n$. These are magnetization σ_j^z, spin current j_k, and z-z connected correlations, as well as their products decaying no faster than $\sim 1/n$ that appear in higher-order nonconnected correlations. Because all these terms are already present in an exact MPO solution for $\gamma = 0$ [10] having $D = 4$, we can expect that in the limit of a large system, one might be able to approximately describe the NESS with an MPO of dimension smaller than $2n$. An MPO with dimension $D = 4$ that correctly describes all terms larger than or equal to $\sim\mathcal{O}(1/n)$ in the NESS is a simple extension of the solution for $\gamma = 0$ [10]. Four matrices at each site have to have the following form:

$$A_i^{(z)} = \begin{pmatrix} a_i & c_i & 0 & 0\\ s_i & a_i & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad A_i^{(\mathbb{1})} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$A_j^{(\mathrm{x})} = \left(\cos\frac{\pi}{2}j - \sin\frac{\pi}{2}j\right)P, \quad A_j^{(\mathrm{y})} = -\left(\cos\frac{\pi}{2}j + \sin\frac{\pi}{2}j\right)R,$$

$$P = \begin{pmatrix} 0 & 0 & 0 & -b\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & -b & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \qquad (32)$$

where $c_i = \frac{b}{\sqrt{t}}k_i^{(\mathrm{L})}$ and $s_i = -\frac{b}{\sqrt{t}}k_i^{(\mathrm{R})}$. Parameters a_i and b are given in Eqs. (17) and (18). Matrices $A_i^{(\mathrm{x})}$ and $A_i^{(\mathrm{y})}$ are periodic with period 4, $A_{i+4}^{(\mathrm{x})} = A_i^{(\mathrm{x})}$, and can be concisely written as $A^{(x)} = (-P, -P, P, P, -P, \ldots)$ and $A^{(y)} = (-R, R, R, -R, -R, \ldots)$, if one writes them as a vector.

Whether the preceding MPO with $D = 4$ actually suffices to describe the NESS in the thermodynamic limit depends on the scaling of the next largest Schmidt coefficient, λ_5 (here we denote by $\lambda_j, j = 1, \ldots,$ the Schmidt coefficients for a symmetric bipartition at $n/2$ spins; $\sum_j \lambda_j^2 = 1$). If $D = 4$ is to suffice, λ_5 must decay more rapidly than λ_4. Because the scaling of Schmidt coefficients is not simply related to the scaling of expansion coefficients of the NESS, we used numerical simulation to study the scaling of λ_j. We have calculated the NESS using a time-dependent density matrix renormalization-group method (tDMRG) [11,28] for systems with up to $n = 128$ spins using an MPO ansatz with a fixed small matrix dimension $D = 10$. In all results that follow we have fixed $\Gamma_\mathrm{L} = \Gamma_\mathrm{R} = \gamma = 1$ and $\bar{\mu} = 0$. As an independent check of our asymptotic MPO solution, Eq. (32), we also compared the four largest Schmidt coefficients obtained from the tDMRG with those obtained from the $D = 4$ MPO (32). Having an explicit representation of matrices, Eq. (32), it is easy to calculate the Schmidt coefficients.

The results for $\lambda_{1,\ldots,6}(n)$ are shown in Fig. 1. Several interesting things can be observed. First, the largest four Schmidt coefficients obtained by the tDMRG agree with the analytical calculation from Eq. (32). Interestingly, around $n \approx 60$ for $\mu = 0.1$ and around $n \approx 30$ for $\mu = 0.2$, an avoided crossing occurs between $\lambda_{2,3}$ and λ_4. A consequence of this is that, for large n, we have scaling $\lambda_4 \sim 1/n$, while we have $\lambda_4 \approx n^0$ for small n. From the data for different μ we can infer that the crossing happens when $\mu n = \nu_\mathrm{c}$, with $\nu_\mathrm{c} \approx 6$. A similar avoided crossing at the same value of n also occurs between λ_5 and λ_6. For $\mu n < \nu_\mathrm{c}$, λ_5 decays faster than $1/n$, while it decays as $\lambda_5 \sim 1/n$ for $\mu n > \nu_\mathrm{c}$. We therefore see that if we want λ_5 to decay with n faster than λ_4—in other words, for $D = 4$ MPO [Eq. (32)] to really give the leading term solution—one must have $\mu n < \nu_\mathrm{c}$ when going to the thermodynamic limit $n \to \infty$.

Another possibility for the $D = 4$ solution to give the leading order would be a limit of small driving, $\mu \to 0$. If λ_5 decays with μ faster than λ_4, this would still be enough. We therefore also looked at the scaling of λ_i with μ. Looking at Fig. 2, and noting that the transition point of the

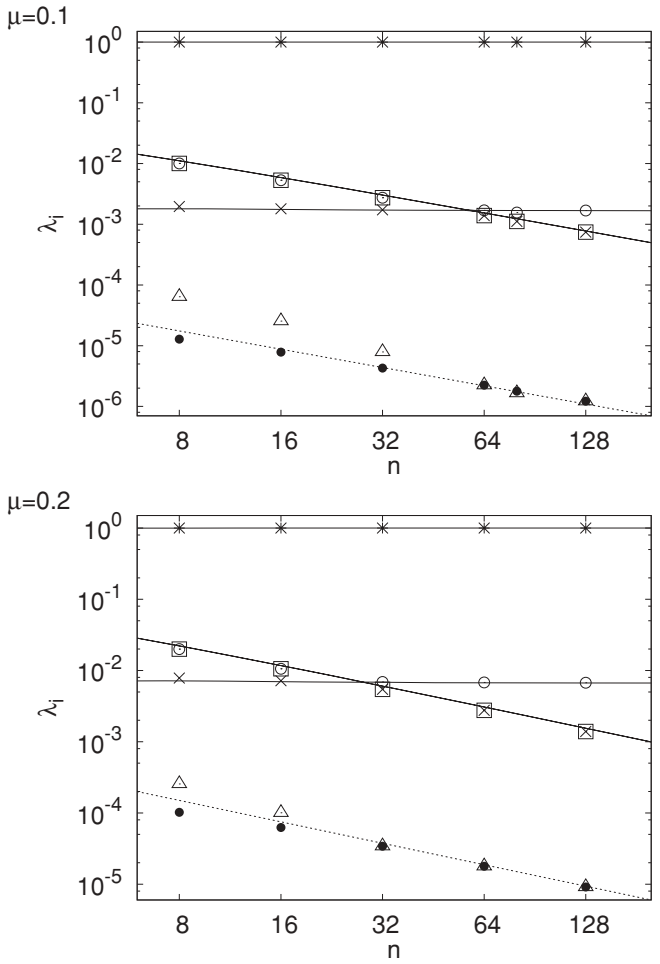

FIG. 1. Scaling of the six largest Schmidt coefficients λ_i of the NESS with the system size. Top: $\mu = 0.1$. Bottom: $\mu = 0.2$. Symbols are tDMRG results for λ_1, λ_2, λ_3, λ_4, λ_5, and λ_6 (stars, open circles, squares, crosses, triangles, and filled circles, respectively); solid lines are analytic Schmidt coefficients from an MPO with $D = 4$, Eq. (32), while the dotted line indicates asymptotic $\sim 1/n$ scaling of λ_5. We use a symmetric bipartite cut after $n/2$ spins, $\Gamma_L = \Gamma_R = \gamma = 1$, $\bar{\mu} = 0$.

avoided crossing $\mu n = \nu_c$ happens at $\mu \approx 0.05$ for $n = 128$ and $\mu \approx 0.1$ for $n = 64$, we can observe the following scaling for the largest coefficients: (i) $\lambda_2 \sim \mu$ for $\mu n < \nu_c$, while $\lambda_2 \sim \mu^2$ for $\mu n > \nu_c$; (ii) $\lambda_3 \sim \mu$; (iii) $\lambda_4 \sim \mu^2$ for $\mu n < \nu_c$, while $\lambda_4 \sim \mu$ for $\mu n > \nu_c$; and (iv) $\lambda_5 \sim \mu^2$ for $\mu n < \nu_c$, while $\lambda_5 \sim \mu^3$ for $\mu n > \nu_c$. We can see that for $\mu n > \nu_c$, λ_5 decays faster with μ than λ_4. The MPO with $D = 4$ [Eq. (32)] therefore also gives the leading-order solution in the limit of weak driving, $\mu \to 0$, provided we have $\mu n > \nu_c$. Presumably there are further avoided crossings in the spectrum of Schmidt coefficients, besides the one at $\mu n = \nu_c$. μn should therefore be smaller than the value at these higher crossings, however, these points need further investigation.

To summarize, an MPO ansatz of size $D = 4$ [Eq. (32)] gives the leading-order solution in two limits: either $n \to \infty$, while keeping $\mu n < \nu_c$, or $\mu \to 0$, while keeping $\mu n > \nu_c$. The two conditions can in fact be put under the same hood. Observing that to fulfill $n\mu < \nu_c$ in the thermodynamic limit, one must necessarily have $\mu \to 0$, and similarly, to have

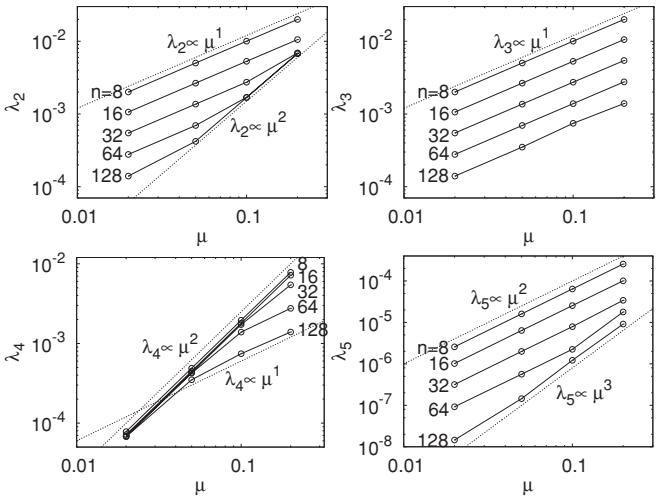

FIG. 2. Scaling of the Schmidt coefficients $\lambda_{2,3,4,5}$ with μ, obtained by the tDMRG (circles) for $n = 8$, 16, 32, 64, and 128. Dashed lines suggest the asymptotic scaling with μ. We use a symmetric bipartite cut after $n/2$ spins, $\Gamma_L = \Gamma_R = \gamma = 1$, $\bar{\mu} = 0$.

$n\mu > \nu_c$ in the weak-driving limit, one must necessarily have $n \to \infty$, one can reformulate both in a single statement, saying that the MPO of dimension $D = 4$ [Eq. (32)] gives the leading-order solution in the limit $n \to \infty$ having, at the same time, $\mu \to 0$.

Another interesting point is that for $\mu n > \nu_c$ the second largest Schmidt coefficient is independent of n and scales as $\lambda_2 \sim \mu^2/n^0$. This means that despite the fact that our system is at an infinite effective temperature, there is a nonzero bipartite entanglement present, even in the thermodynamic limit of large n (at a fixed μ). This entanglement at an infinite temperature is of purely nonequilibrium origin.

V. NONEQUILIBRIUM PHASE TRANSITION

From the exact solution we can see that the NESS undergoes a transition from a state without long-range correlations for $\gamma = 0$ to one with long-range correlations for $\gamma \neq 0$. A point in a parameter space where a system undergoes a sudden change in some expectation values is usually called a phase transition point. We use the same nomenclature here and call this transition a nonequilibrium phase transition, because the nature of the correlations changes. Transport properties also change suddenly at $\gamma = 0$, going from a ballistic (superconducting) state to a diffusive state for nonzero dephasing. In the phase with long-range correlations, two-point z-z correlations scale as $\sim \mu^2/n$ [9] and are therefore of purely nonequilibrium origin. They also go to 0 in the thermodynamic limit, making this transition different from equilibrium phase transitions. The correlation function has a plateau because in the thermodynamic limit the decay of $C_{i,j}$ with the distance between indices $|i - j|$ gets increasingly slower [Eq. (25)]. Recently, similar quantum nonequilibrium phase transitions have been discovered in the XY model [7,8], in which the long-range correlations scale as $\sim \mu^2/n$, and in the XXZ model with dephasing [11] (the same scaling of the correlation plateau as here), as well as in the XXZ model without dephasing [19], where, however, the correlation

plateau scales as $\sim\mu^2/n^0$ (the plateau is independent of n!) at an infinite temperature and as $\sim\mu^2/n^\alpha$ at a finite temperature. It has been conjectured that long-range correlations are a generic feature of quantum nonequilibrium steady states [19].

It would be nice to understand this phase transition in more detail. At equilibrium, phase transitions are connected with the nonanalyticity of the free energy or, equivalently, zeros of the partition function. The difficulty with the nonequilibrium situation is that there is no general theory; in particular, a free-energy-like quantity whose analytic properties could be studied is not known. Nevertheless, because observable properties of the system change suddenly at our nonequilibrium phase transition point, there must be some underlying nonanalytic property. We can mention that the Lee-Yang theory of equilibrium phase transitions has been used with some success in certain classical nonequilibrium systems [29,30,32] whose stationary state can be represented in a matrix product form. For some classical exclusion processes in the thermodynamic limit even a free-energy-like functional can be calculated [31]. Until recently [10] no quantum stationary state solvable by a matrix product ansatz has been known and the theory of quantum nonequilibrium phase transitions has lagged behind the classical. For classical nonequilibrium systems the product of all nonzero eigenvalues of a superoperator can play the role of a "partition" function, $Z=\prod_{\lambda_j\neq 0}(-\lambda_j)$, whose zeros then determine the location of nonequilibrium phase transitions. It has been shown [32] that the product of nonzero eigenvalues is equal to the sum of all expansion coefficients of the NESS. There are problems with such an approach, however. The principal difficulty is that Z is defined only up to to an arbitrary normalization factor and it is *a priori* not known which normalization one should take.

In the present work we look for nonanalytic signatures in the spectrum of the superoperator, trying to see if there is any characteristic change at the phase transition point. Because the NESS is an eigenstate of $\mathcal{L}$ with an eigenvalue of 0, a change in the properties of the NESS should be connected with the closing of the gap between the second largest eigenvalue and that of the NESS. If this gap goes to 0, there is the possibility for a scenario reminiscent of an avoided crossing between two eigenenergies happening at a zero-temperature quantum phase transition, where at the crossing the nature of the two levels involved is exchanged. Of interest for nonequilibrium phase transitions is, therefore, the second largest eigenvalue λ_2 of the Lindblad superoperator $\mathcal{L}$ or, in particular, the gap Δ, $\Delta=-\lambda_2$. We note that the size of the gap also determines the decay rate, that is, the relaxation rate, towards the NESS. Namely, the initial nonstationary state will, for large times, relax to the NESS as $\exp(-\Delta t)$. Because we, in general, expect the relaxation time, in the case of local coupling to baths at chain ends, to grow with the system size, at least as $\propto n$ (a larger system simply needs more time to relax), the gap is expected to decrease with n, even if we are not at the nonequilibrium phase transition point. The signature of the nonequilibrium phase transition therefore cannot be simply the closing of the gap Δ. What is generally observed, however, is that the closing of the gap Δ with n is faster at the phase transition point than away from it. Such behavior has been observed, for instance, in an open XY chain [7], where at a nonequilibrium phase transition point the gap scales as $\Delta\sim 1/n^5$, while it scales as only $\Delta\sim 1/n^3$ away from the transition point. We therefore studied the scaling of the gap in our model. The results are shown in Fig. 3. We numerically exactly calculated the gap Δ for systems of up to $n=11$ spins. For larger systems a relaxation time τ of some observable can be used as an estimate of the gap. Using the tDMRG we have solved the Lindblad equation in time, thereby obtaining time-dependent expectations of observables. We fitted an exponential function to the relaxation of the magnetization at site $n/2$, $z_{n/2}(t)-z_{n/2}(\infty)\sim\exp(-t/\tau)$, and determined τ for systems with up to $n=128$ spins. As the second largest eigenvalue λ_2 is nondegenerate for $\gamma=1$, the relaxation time can serve to estimate the gap through $\Delta\sim 1/\tau$. From Fig. 3 we can see that at the phase transition point the gap scales as $\Delta\sim 1/n^3$. This, of course, agrees with a previous analytic result for the isotropic XY chain [7]. Away from the nonequilibrium phase transition, at $\gamma=1$, the gap decays more slowly, as $\Delta\sim 1/n^2$. We therefore conjecture that the characteristic feature of nonequilibrium phase transitions is that the gap of the superoperator decays more rapidly at the nonequilibrium phase transition point than in its neighborhood. At a nonequilibrium phase transition point the relaxation slows down, similarly as at an equilibrium phase transition point. How much more rapidly the gap decays depends on a particular system. Here we have $1/n^3$ vs. $1/n^2$; for the XY model, in contrast, one has $1/n^5$ vs. $1/n^3$ [7]. Note that this criterion is different from those used in equilibrium physics, as it makes a reference also to the neighborhood of the transition point. Interesting, in our model the relaxation time of the spin current is shorter by a factor of ≈ 4 than that of $z_{n/2}$. The reason must lie in the fact that some of the eigenstates that are just below the NESS in the spectrum carry no current. Another observation about the spectrum of the Lindblad superoperator for the XX model with dephasing is that the eigenvalues are

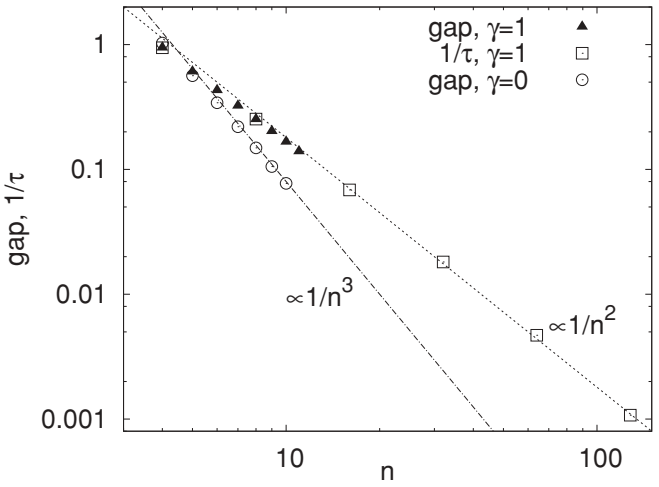

FIG. 3. Scaling of the gap of the Lindblad superoperator with system size for the model without long-range order at $\gamma=0$ and for the system with long-range order in the case of $\gamma=1$. At the nonequilibrium phase transition point at $\gamma=0$, the gap decays faster, as $\Delta\sim 1/n^3$, than away from the transition point ($\gamma=1$), where $\Delta\sim 1/n^2$. We also show the scaling of the inverse relaxation time of the magnetization at site $n/2$ (squares). $\Gamma_{\mathrm{L}}=\Gamma_{\mathrm{R}}=1$, $\bar{\mu}=0$, and $\mu=0.1$.

independent of the driving μ; only the eigenvectors depend on μ. This probably happens due to the hierarchical structure of the master equation.

VI. SUMMARY

We have provided exact expressions for all one-point, two-point, and three-point connected correlations in the NESS of the *XX* model with dephasing. A hierarchical structure of stationary equations is explained, a consequence of which is that equations determining all n-point correlations form a closed set. The NESS is non-Gaussian because the Wick theorem does not apply. In the thermodynamic and weak-driving limit the solution can be written in terms of a matrix product ansatz with matrices of fixed dimension 4. At zero dephasing the model exhibits a nonequilibrium phase transition from a state with only nearest-neighbor correlations to a state possessing long-range correlations. It is conjectured that at a quantum nonequilibrium phase transition point, the gap of the superoperator closes with the system size more rapidly than in the vicinity of the transition point.

ACKNOWLEDGMENTS

Support from Program No. P1-0044 and Grant No. J1-2208 of the Slovenian Research Agency, Project No. 57334 of CONACyT, Mexico, and Project No. IN114310 of the University of Mexico, as well as discussion with T. Prosen, is acknowledged.

APPENDIX: THIRD-ORDER TERMS

We are mainly interested in the connected three-point correlations, defined for arbitrary operators A, B, and C as $\langle ABC\rangle_{\rm c} = \langle ABC\rangle - \langle A\rangle_{\rm c}\langle B\rangle_{\rm c}\langle C\rangle_{\rm c} - \langle AB\rangle_{\rm c}\langle C\rangle_{\rm c} - \langle AC\rangle_{\rm c}\langle B\rangle_{\rm c} - \langle BC\rangle_{\rm c}\langle A\rangle_{\rm c}$. The third-order ansatz $\mu^3 R^{(3)}$ is the sum of five terms:

$$\mu^3 R^{(3)} = G_{\rm zzz} + G_{\rm zzj} + G_{\rm zjj} + G_{\rm jjj} + G'_{\rm jjj}. \tag{A1}$$

The terms are

$$\begin{aligned}
G_{\rm zzz} &= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}(Z_{i,j,k} + a_i a_j a_k + a_i C_{j,k} \\
&\quad + a_j C_{i,k} + a_k C_{i,j})\sigma_{\rm i}^{\rm z}\sigma_{\rm j}^{\rm z}\sigma_{\rm k}^{\rm z}, \\
G_{\rm zzj} &= \sum_{k=1}^{n-1}\sum_{\substack{i=1\\ i\notin\{k,k+1\}}}^{n}\sum_{\substack{j=i+1\\ j\notin\{k,k+1\}}}^{n}\frac{1}{2}\big(X_{k,i,j} + a_i a_j b + b C_{i,j} \\
&\quad + a_i\langle j_k\sigma_j^{\rm z}\rangle_{\rm c} + a_j\langle j_k\sigma_i^{\rm z}\rangle_{\rm c}\big)j_k\sigma_{\rm i}^{\rm z}\sigma_{\rm j}^{\rm z}, \\
G_{\rm zjj} &= \sum_{\substack{i,k=1\\ k>i}}^{n-1}\sum_{\substack{j=1\\ j\notin\{i,i+1,k,k+1\}}}^{n}\big(Y_{j,i,k} + a_j b^2 + a_j(f-b^2) \\
&\quad + b\langle\sigma_j^{\rm z} j_k\rangle_{\rm c} + b\langle\sigma_j^{\rm z} j_i\rangle_{\rm c}\big)\sigma_j^{\rm z}\frac{j_i j_k + j_k j_i}{8}, \\
G_{\rm jjj} &= \sum_{\substack{i,k,l=1\\ i\neq k, i\neq l, k\neq l;\ \text{at most one overlap}}}^{n-1}\frac{w}{8}j_i j_k j_l, \\
G'_{\rm jjj} &= \sum_{i=1}^{n-3} g\left(\sigma_i^{\rm x}\sigma_{i+3}^{\rm y} - \sigma_i^{\rm y}\sigma_{i+3}^{\rm x}\right).
\end{aligned} \tag{A2}$$

In Eqs. (A2) the summations are such that no two operators appear at the same site, and in addition, there are no overlapping terms like $j_k\sigma_{k+1}^{\rm z}$. In the $G_{\rm jjj}$ we have, at most, one overlapping current on neighboring sites, like $j_k j_{k+1}$. The solutions for unknown coefficients are

$$w = b^3\frac{(1+t)^2}{6t(t-1)}, \quad g = b^3\frac{(1+t)^2}{t(t-1)}, \tag{A3}$$

$$Z_{i,j,l} = -b^3\frac{2}{t(t-1)}\left\{k_i^{\rm (L)}\left(k_j^{\rm (R)} - k_j^{\rm (L)}\right)k_l^{\rm (R)} + (t+1)\left(k_l^{\rm (R)}\delta_{j,i+1} - k_i^{\rm (L)}\delta_{j,l-1}\right)\right\}, \tag{A4}$$

$$X_{k,i,j} = -b^3\frac{2}{t(t-1)}\cdot\begin{cases} k_i^{\rm (L)}\left(k_j^{\rm (R)} - k_j^{\rm (L)}\right) + (t+1)\delta_{j,i+1}, & k > j, \\ -k_j^{\rm (R)}\left(k_i^{\rm (R)} - k_i^{\rm (L)}\right) + (t+1)\delta_{j,i+1}, & k < i, \\ 2k_i^{\rm (L)}k_j^{\rm (R)}, & i < k < j-1, \end{cases} \tag{A5}$$

$$Y_{j,i,k} = -b^3\frac{2}{t(t-1)}\cdot\begin{cases} -2k_j^{\rm (R)}, & j > k+1, \\ 2k_j^{\rm (L)}, & j < i, \\ \left(k_j^{\rm (L)} - k_j^{\rm (R)}\right), & i+1 < j < k. \end{cases} \tag{A6}$$

The three-point connected correlation function is equal to $\langle\sigma_i^{\rm z}\sigma_j^{\rm z}\sigma_k^{\rm z}\rangle_{\rm c} = Z_{i,j,k}$ for nondiagonal indices.

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