Description of the suppression of the soliton self-frequency shift by bandwidth-limited amplification

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A perturbation study of the suppression of the soliton self-frequency shift by the bandwidth-limited optical amplification is proposed. The stability of the equilibrium point for the soliton amplitude and velocity identified by the adiabatic approximation of the soliton perturbation theory (SPT) is analyzed by a numerical solution of a linearized system in the neighborhood of the equilibrium point. The obtained analytical expressions for the eigenvalues of the linearized system allow the determination of the values of pulse and material parameters for which the equilibrium point is stable. A perturbation approach that leads to the research of the equation of strongly nonlinear Duffing–Van der Pol oscillator is suggested. The last equation is explored by two different methods. First, the recently obtained results for this equation by the hyperbolic perturbation method are used. Next, the hyperbolic Lindstedt-Poincare perturbation method is applied to the exploration of this equation. The equilibrium velocity of the perturbed stationary solution was calculated as a critical value of the control parameter in both methods. It turned out that the coupling of the equilibrium velocity and the amplitude of the perturbed stationary solution in both methods is similar to the relation between the soliton amplitude and velocity derived by the adiabatic approximation of SPT. The change in the form of the perturbed stationary solution has also been identified by means of the hyperbolic Lindstedt-Poincare perturbation method.

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I. INTRODUCTION

The intrapulse Raman scattering (IRS) is higher-order nonlinear effect that plays an important role in the propagation of femtosecond optical pulses in single mode optical fibers. It is related to the delayed nature of the Raman response in optical fibers. When the pulse spectrum becomes very broad its high-frequency components can transfer energy to its low-frequency components [1], which results in a continuous downshift of the soliton carrier frequency, a phenomenon known as the soliton self-frequency shift (SSFS) [1–4].

Because of their large bandwidth, erbium-doped fiber amplifiers (EDFAs) can be used to amplify optical pulses, overcoming transmission losses in long-haul fiber-optic communication systems. Moreover, it was shown that such bandwidth-limited amplification (BLA) can be used to reduce the noise-induced temporal jitter (the Gordon-Haus jitter) and the soliton-soliton interaction (see, for review, [2,3]). When relatively short (about 1 ps) pulses have to be amplified, IRS should also be taken into account. It was established [5,6] that the BLA can reduce the amount of spectral shift due to the SSFS and stabilize the soliton carrier frequency close to the gain peak. The observed physical phenomenon was called the suppression of the SSFS by BLA [5] or the trapping of an optical soliton by BLA [6].

The theoretical and experimental investigations on the IRS and BLA are reviewed in [1–3]. The important analytical tool of analysis is the soliton perturbation theory (SPT) [3,7,8]. SSFS of bright solitons has been described by means of adiabatic approximation of SPT [9]. Nonadiabatic description that includes change in the form of the bright soliton in the presence of SSFS has been suggested [10]. The properties of SSFS of bright solitons in birefringent fibers have been explored [11,12]. The approach for the analysis of perturbation dynamics of dark solitons and a general formula describing the SSFS of dark solitons with arbitrary amplitudes have been proposed [13] (see also [14]). The influence of BLA on the bright solitons has also been described by the adiabatic approximation of SPT [15]. It was pointed out that by proper choice of the parameters of BLA, solitons are asymptotically forced to acquire a fixed (singular) value of amplitude and velocity, an effect that has been interpreted as a soliton cooling [15,16]. The adiabatic approximation of SPT has been employed in the study of the IRS and BLA and the equilibrium point for the soliton amplitude and velocity has been identified in [5]. (As a matter of fact, I have found this equilibrium point before discovering the results in [5].)

The aim of the paper is to analyze the suppression of the SSFS by BLA [5] or the trapping of an optical soliton by BLA. First, I intend to study the stability of the equilibrium point for the soliton amplitude and velocity identified in [5] through the adiabatic approximation of SPT. I will pay special attention to the relation that couples the steady soliton amplitude, which will be numerically verified. The reason for this is that later I am going to use this relation as a argument for the validity of the proposed perturbation approach. Furthermore, I suggest a perturbation approach. The key idea in it is the usage of the equation of strongly nonlinear Duffing–Van der Pol oscillator, which will be obtained after searching for the stationary solution of the basic equation. This equation has been studied by the hyperbolic perturbation method of Chen and Chen [17] and the hyperbolic Lindstedt-Poincare perturbation method of Chen *et al.* [18]. The results in [17] are used directly, while the method [18] is applied to the equation of strongly nonlinear Duffing–Van der Pol oscillator. By means of these methods the equilibrium velocity

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of the perturbed stationary solution will be calculated. Results are compared with the ones achieved by the adiabatic approximation of SPT. The correction to the form of the perturbed stationary solution will also be explored through [18]. The partially reported results here have been first presented in [19].

The paper is organized as follows: After a mathematical formulation of the problem in Sec. II, the results are obtained by means of the adiabatic approximation of SPT are presented in Sec. III. In Sec. IV A the equation of strongly nonlinear Duffing–Van der Pol oscillator is introduced. Next in Sec. IV B, I use the results in [17] and apply the method in [18] to the analysis of the equation of strongly nonlinear Duffing–Van der Pol oscillator. Finally, in Sec. V, the results achieved by the adiabatic approximation of SPT and those earned with the help of the proposed approach are compared and discussed.

For completeness, the derivation of the important results of the hyperbolic perturbation method in [17] is shortly described in Appendix A. The main steps in my application of the hyperbolic Lindstedt-Poincare method in [18] to the equation of strongly nonlinear Duffing–Van der Pol oscillator are given in Appendix B.

II. BASIC EQUATION

As it is well known because of their large bandwidth EDFA can be used to amplify optical pulses. Erbium ions in doped fiber can be modeled as a two-level system. The Maxwell- Bloch equations for the slowly varying part of the polarization, responsible for the contribution of dopant and the population inversion density, together with the modified nonlinear Schrödinger equation (NLSE) for the slowly varying envelope of the electric field should be solved together. Considering optical pulses with width larger than that of the dipole relaxation time, the rate-equation approximation in which the polarization follows the optical field adiabatically can be used. The dispersive effects connected with the erbium ions can be included through the dopant susceptibility into the refractive index change and then into the modified NLSE. The important sequence of this procedure is that the dispersion parameters of the fiber become dependent on the dopant content. It turned out, however, that the dopant-induced change in the group velocity is negligible in practice. This is not the case, however, for the additional term to the group-velocity dispersion. This additional term represents the finite bandwidth of the fiber amplifier and is referred to as gain dispersion (see below). If the mode density and the dopant density are nearly uniform over the doped region and zero outside it, the relationship between the small-signal gain and the population inversion density transforms to linear one. In general, the dynamics of the gain depends on the small-signal gain, the fluorescence time, the saturation energy, and the pumping configuration. For EDFA, however, the typical fluorescence time is on order of 10 ms. As a result, it can be assumed that for the short optical pulses, the dependence on the pumping configuration may be neglected. The saturation energy for EDFA is on order of 1 μJ. As the typical pulse energy is much smaller, the gain saturation over the duration of single pulse can be neglected. All these circumstances lead to the following modified NLSE that describes pulse propagation in optical amplifiers [1,2]:

$$i\frac{\partial U}{\partial x}+\frac{1}{2}\frac{\partial^2 U}{\partial t^2}+|U|^2U=i\delta U+i\beta\frac{\partial^2 U}{\partial t^2}+\gamma U\frac{\partial}{\partial t}(|U|^2), \quad (1)$$

where the dimensionless variables (soliton units) are introduced as follows [1,2]:

$$x=z/L_D, \quad t=T/T_0, \quad U=(\gamma L_D)^{1/2}A.$$

Here, z and t' are real longitudinal coordinate in the fiber and time, $T=t'-z/v_g=t'-\beta_1 z$, v_g is the group velocity, $A(z,T)$ is the slowly varying envelope, $L_D=T_0^2/|\beta_2|$ is the dispersion length, T_0 is the width of the pulse, and β_2 represents the dispersion of the group velocity. $\gamma=n_2\omega_0/cA_{eff}$ is the nonlinear parameter, n_2 is the nonlinear-index coefficient, ω_0 is the carrier wavelength of the optical pulse, A_{eff} is the effective core area, and c is the speed of light in vacuum. The last term on the left-hand side of Eq. (1) is proportional to $\gamma=T_R/T_0$, where T_R is the first moment of the nonlinear response function (the slope of the Raman gain spectrum). This term is related to the delayed Raman response, describes the IRS, and is consequently responsible for the SSFS. Next,

$$\delta=(g_0-\alpha)L_D/2, \quad \beta=g_0L_D(T_2/T_0)^2/2,$$

where g_0 is the gain, α is the fiber losses, and T_2 is the dipole relaxation time. The term proportional to β represents the finite bandwidth of the fiber amplifier (gain dispersion). Here it was also taken that the pulse spectrum is narrower than the gain bandwidth, which allows approximating the gain spectrum by a parabola. I will assume that $0<\beta<1$, $0<\delta<1$, and $0<\gamma<1$, which reflects the discussed physical situation.

III. ADIABATIC PERTURBATION METHOD

For a small perturbation, in the adiabatic approximation of SPT, the soliton solution may be written as follows [3,5]:

$$U(x,t)=\eta(x)\text{sech}\{\eta(x)[t-\tau(x)]\}\exp\{i[-k(x)t+\sigma(x)]\}, \quad (2)$$

where $\eta(x)$ and $k(x)$ are the soliton amplitude and velocity (frequency), respectively. The soliton position $\tau(x)$ and phase $\sigma(x)$ are defined by the equations $d\tau(x)/dx=-k$ and $d\sigma(x)/dx=(\eta^2-k^2)/2$, respectively. Applying adiabatic perturbation method, the following system of ordinary differential equations that describe evolution of amplitude and velocity can be derived [3,5]:

$$\frac{d}{dx}\eta=2\delta\eta-2\beta\left(k^2+\frac{1}{3}\eta^2\right)\eta,$$

$$\frac{d}{dx}k=-\frac{4}{3}\beta k\eta^2-\frac{8}{15}\gamma\eta^4. \quad (3)$$

In case of BLA (γ=0), the system given by Eq. (3) has a singular point given by [15,3] $\eta_*=\sqrt{3\delta/\beta}$ and $k_*=0$. The eigenvalues of the linearized problem in the vicinity of this equilibrium point are negative, so it is stable, and the process of emerging of soliton with such parameters during the process of distributed amplification has been interpreted as a soliton cooling [3].

In the event of IRS ($\delta=\beta=0$), it was obtained that [9,3] $dk/dx=-8\gamma\eta^4/15$, i.e., the amplitude does not change, but the velocity changes with distance.

In case both BLA and IRS are present, the following equilibrium point with positive amplitude (there is a similar one with the negative amplitude) has been identified [5]:

$$\eta_* = \sqrt{\frac{(5\sqrt{25\beta^4 + 144\delta\beta\gamma^2} - 25\beta^2)}{24\gamma^2}},$$

$$k_* = -\frac{(\sqrt{25\beta^4 + 144\delta\beta\gamma^2} - 5\beta^2)}{12\beta\gamma}. \qquad (4)$$

The singular point given by Eq. (4) has been obtained by means of symbolic computation with a computer software system MATHEMATICA. In addition to [5], I present the eigenvalues $\lambda_{1,2}$ of the linearized problem in the vicinity of the equilibrium point given by Eq. (4). The eigenvalues $\lambda_{1,2}$ are the solutions of the following quadratic equation in λ: $\lambda^2+p\lambda+q=0$, with $p=(5\beta/9\gamma^2)(-5\beta^2+\sqrt{25\beta^4+144\delta\beta\gamma^2})$, $q=(5\beta/162\gamma^4)[-125\beta^5-720\gamma^2\beta^2\delta+(25\beta^3+72\gamma^2\delta)\sqrt{\Lambda}]$, and $\Lambda=25\beta^4+144\beta\gamma^2\delta$. Recalling the assumption regarding β, δ, and γ we see that p will always be positive $p>0$. The eigenvalues $\lambda_{1,2}$ are given as

$$\lambda_{1,2} = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2-4q}$$
$$= \frac{1}{18\gamma^4}\{5\beta\gamma^2(5\beta^2-\sqrt{\Lambda}) \pm 2\sqrt{5}$$
$$\times\sqrt{\beta\gamma^4[125\beta^5+540\beta^2\gamma^2\delta-(25\beta^3+36\gamma^2\delta)\sqrt{\Delta}]}\}.$$

The case $q=0$ corresponds to a higher order of singular points and it requires $\beta=0$ and/or $\delta=0$, a situation which is not discussed here. Let us first consider $q>0$ ($p>0$). For $p^2>4q$, the roots λ_1 and λ_2 are real and of the same sign; the stable nodal points appear for $p>0$. For $p^2=4q$, the roots $\lambda_1=\lambda_2=-p/2$ are degenerate and the nodal point occurs. This situation appears in the case of the following system parameters: $\delta=0,15$; $\beta=0,45$; and $\gamma=5\times10^{-4}$. Then $q=0.36$, $p=1.2\Rightarrow p^2=4q=1.44$, and degenerate roots are $\lambda_1=\lambda_2=-0.6$. The corresponding nodal point is shown in Fig. 1. For $p^2<4q$, the roots λ_1 and λ_2 are complex conjugate; the stable focal points appear for $p>0$. Such an example case is illustrated for the following parameters: $\delta=0,5$; $\beta=0,01$; and $\gamma=5\times10^{-4}$, where $q=3.93$, $p=3.46$, or $p^2=12<4q=15.7$. The complex-conjugate eigenvalues are $\lambda_{1,2}=-1.73\mp0.97i$. They define the stable focal point, presented in Fig. 1.

In case of′$q<0$, the roots λ_1 and λ_2 are real and with opposite signs. Then the unstable saddle points occur. In all cases the values of parameters δ and β describing BLA are larger than the value of γ that describes the IRS.

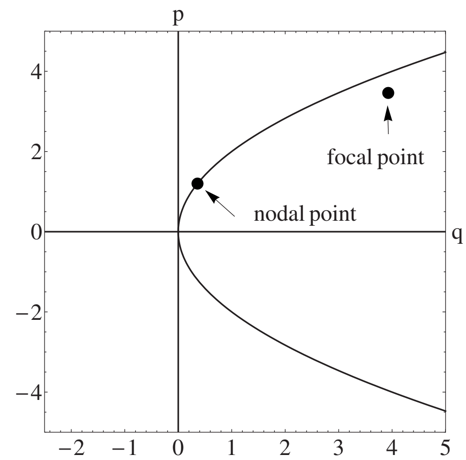

FIG. 1. The p-q plane for establishing types of the two equilibrium points: nodal point ($\delta=0,15$; $\beta=0,45$; $\gamma=5\times10^{-4}$) and focal point ($\delta=0,5$; $\beta=0,01$; $\gamma=5\times10^{-4}$). The solid curve represents the equation $p^2-4q=0$.

Using the obtained singular point given by Eq. (4), the following relation between the velocity (frequency) k_* and the square of amplitude η_*^2 is obtained [5]:

$$k_* = -2\gamma\eta_*^2/(5\beta). \qquad (5)$$

(In truth, I have found this relation before discovering the results in [5].) According to this relation the soliton's velocity is directly proportional to the parameter describing the IRS and the square of the soliton's amplitude and inversely proportional to the parameter that describes the finite bandwidth of the fiber amplifier β. For fixed parameter describing the IRS γ, with the increase of the parameter that describes the finite bandwidth β of the fiber amplifier, the soliton's velocity decreases. In other words, the observed reduction of the amount of spectral shift due to the IRS and stabilization of the soliton's carrier frequency close to the gain peak in the case of BLA may be analytically explained. The relation given by Eq. (5), which is obtained by the adiabatic approximation of SPT, will be further used as a measure of the validity of our next perturbation considerations.

In order to confirm numerically the appearance of the singular point given by Eq. (4), as well as relation (5), I solved numerically the system of Eq. (3) for the case $\delta=0,5$; $\beta=0,01$; and $\gamma=5\times10^{-4}$, with the initial value for η, $\eta(0)=0.1$, and initial values for k varying from −6 to 6 in steps of 2.5. The $\eta(x)$ and $k(x)$ as a function of x are shown in Fig. 2.

As can be seen in Fig. 2, the stable focal point occurs with the asymptotic values of the amplitude $\eta_*=11.39$ and the frequency $k_*=2.59$, which satisfy the relation given by Eq. (5).

IV. PERTURBATION APPROACH

An alternative perturbation approach to the adiabatic perturbation of SPT is proposed in this section that comprises of two steps. In the first one, in Sec. IV A, analyzing the stationary solution of the basic equation [Eq. (1)], the equation of strongly nonlinear Duffing–Van der Pol oscillator [see Eq.

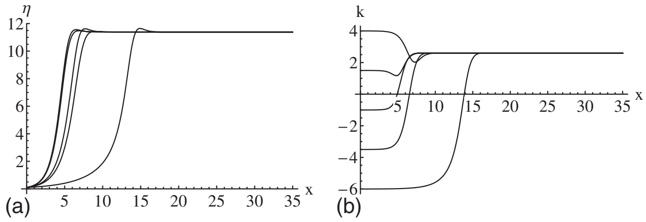

FIG. 2. Evolution of $\eta(x)$ (a) and $k(x)$ (b) as a function of x, demonstrating the appearance of the stable amplitude and the frequency of the soliton for the case $\delta=0,5$; $\beta=0,01$; and $\gamma=5\times10^{-4}$.

(11) below] is introduced. In the second step, in Sec. IV B, I directly use results in [17] and apply the method in [18] to Eq. (11). The derivation of the important results in [17] is shortly described in Appendix A. The main steps in my application of the hyperbolic Lindstedt-Poincare method in [18] to the equation of strongly nonlinear Duffing–Van der Pol oscillator are given in Appendix B.

A. Stationary solution

I look for the stationary pulse solution of Eq. (1) into the form

$$U(x,t)=u(\xi)\exp\{i[f(\xi)+Kx]\},\tag{6}$$

where $\xi=t-Mx$ and M and K are real numbers. M has a meaning of the unknown equilibrium velocity (more precisely the inverse velocity). Inserting Eq. (6) into Eq. (1), the following nonlinear system of ordinary differential equations for the functions $u(\xi)$ and $f(\xi)$ is obtained:

$$-\delta u-\beta u''+uf''/2+f'u'-Mu'+\beta u(f')^2=0,\tag{7a}$$

$$u''/2+\beta uf''+2\beta u'f'-2\gamma u^2u'-u(f')^2/2+Muf'-Ku+u^3=0.\tag{7b}$$

We should mention here that Eqs. (7a) and (7b) in the absence of distributed optical amplification ($\delta=\beta=0$) transform to Eqs. (2.8a) and (2.8b) in [20]. Let us assume that due to the smallness of the parameters describing distributed BLA, δ and β, Eq. (7a) can be approximated in the following way:

$$uf''+2f'u'-2Mu'=0\Rightarrow(u^2f'-Mu^2)'=0.\tag{8}$$

The applicability of this important approximation will be eventually justified by the validity of final results. The solution of Eq. (8) can be written as

$$f=M\xi+S_0\int d\xi/u^2,\tag{9}$$

where S_0 is arbitrary constant. The velocity M in Eqs. (9) and (6) can be related to the soliton velocity (frequency) k in the adiabatic approximation of the SPT. Using Eq. (8), Eq. (7b) can be transformed to

$$u''+(M^2-2K)u+2u^3-\frac{S_0^2}{u^3}+4(\beta M-\gamma u^2)u'=0.\tag{10}$$

In this equation, β and γ are related to distributed BLA and γ is connected to IRS. We consider the BLA as the primary perturbation, and IRS is considerably weaker than BLA. (See the values of δ and β for which stable equilibrium point is possible in accordance with SPT.) We expect that the velocity M (due to the effect of IRS) will eventually have certain smallness $M\sim\gamma\sim\varepsilon$. So, we neglect the term proportional to M^2, while the term proportional to βM will be kept in Eq. (10). It was established in [20] that Eq. (10) cannot be transformed into the Painleve-type equations for $M=0$, $\gamma_3\neq0$, and $S_0\neq0$. In the event of $M=0$, $\gamma=0$, and $S_0\neq0$, the exact solutions are available [21]. As a means to explore the influence of the IRS, M and γ should be different from zero. Implying that the arbitrary constant can be set equal to zero, $S_0=0$, the phase function becomes proportional to ξ: $f=M\xi$. Equation (10) can then be cast into the form

$$u''+c_1u+c_3u^3=\varepsilon(\mu-\mu_1u^2)u'=\varepsilon g(\mu,u,u'),\tag{11}$$

where $c_1=-2K$, $c_3=2$, $\mu=-4\beta M/\gamma$, and $\mu_1=-4$. The coefficients c_1,c_3 on the left-hand side of Eq. (11) are generally not small. Neglecting the right-hand side of Eq. (11), we get the Duffing equation

$$u''+c_1u+c_3u^3=0,\tag{12}$$

which will be called here the generating (unperturbed) equation and represents the strongly nonlinear oscillator. The solution (homoclinic) of the generating equation in case of $c_1<0$ and $c_3>0$ can be expressed by

$$u_0=a_0\ \text{sech}\ \tau,\quad \tau=\omega_0\xi,\tag{13}$$

where the amplitude a_0 and the frequency ω_0 are given by $a_0^2=-2c_1/c_3=2K$ and $\omega_0^2=-c_1=2K$, respectively. The parameter K should be a positive number. In the hyperbolic methods employed below the parameter μ will be considered as a control parameter. The term on the right-hand side of Eq. (11) can be considered as a Van der Pol perturbation to generating equation. Equation (11) will be called here the equation of strongly nonlinear Duffing–Van der Pol oscillator.

B. Hyperbolic perturbation methods

The hyperbolic perturbation method proposed in [17] can be characterized as follows. First, contrary to the classical perturbation methods applicable for analysis of perturbations of weakly nonlinear systems [20], it is applicable to the perturbations of strongly nonlinear oscillators. Second, it assumes that at each order, the perturbed solution has the shape of the unperturbed solution of its generating (unperturbed) equation. The same assumption is made in the adiabatic approximation of SPT, where the shape of the soliton solution is also preserved, which makes it possible to compare the achieved results with both methods.

In accordance with [17], the solution of Eq. (11) can be written as

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n = \sum_{n=0}^{\infty} \varepsilon^n a_n h(\tau) = (a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots) h(\tau), \tag{14a}$$

where

$$d\tau/d\xi = \omega(\tau) = \sum_{n=0}^{\infty} \varepsilon^n \omega_n(\tau) = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots . \tag{14b}$$

The amplitudes a_n and τ depend on the small parameter ε. As it can be seen from Eq. (14a), at each order the approximate solutions $u_n(\tau)=a_n h(\tau)=a_n \operatorname{sech}(\tau)$ have the shape of the solution of the generating (unperturbed) equation, $\operatorname{sech}(\tau)$. The control parameter μ is also expanded in the power of ε,

$$\mu = \sum_{n=0}^{\infty} \varepsilon^n \mu_{Cn} = \mu_{C0} + \varepsilon\mu_{C1} + \varepsilon^2 \mu_{C2} + \cdots . \tag{15}$$

The critical control parameter μ_{C0} should then be identified under which a solution forms. It has been found [17] (see also Appendix A) that the value of μ_{C0} is given by

$$\mu_{C0} = 2\mu_1 a_0^2/5. \tag{16}$$

Using Eq. (16) and the definition of $\mu_{C0}=-4\beta M_{C0}/\gamma$, we obtain for the critical value of the unknown velocity M_{C0},

$$M_{C0} = 2\gamma a_0^2/(5\beta). \tag{17}$$

Comparison between Eq. (17), which relates the velocity M_{C0} with amplitude a_0^2 of the stationary solution, and Eq. (5), which couples the soliton amplitude η_* and velocity k_*, shows remarkable correspondence. (The comparison should be taken into account that $f=M\xi$.) It was further obtained [17] (see also Appendix A) that $\omega_1(\tau)=A_2 \tanh \tau=(-a_0^2\mu_1/5)\tanh \tau$. Finally, the perturbed solution of the equation of strongly nonlinear Duffing–Van der Pol oscillator [Eq. (11)] is given by

$$u = a_0 \operatorname{sech} \tau + O(\varepsilon^2), \tag{18a}$$

$$u' = -a_0[\omega_0 - (4\gamma a_0^2/5)\tanh \tau]\operatorname{sech} \tau \tanh \tau + O(\varepsilon^2), \tag{18b}$$

where a_0^2 and ω_0^2 are defined above. The correction to the frequency of the perturbed solution that describes the influence of the IRS on the stationary solution is $\varepsilon\omega_1(\tau)=(-a_0^2\mu_1\gamma/5)\tanh \tau$. Let us point out that the correction term in Eq. (18b) does not depend on the parameters of BLA.

The hyperbolic Lindstedt-Poincare perturbation method in [18] is also proposed for the analysis of perturbations of strongly nonlinear oscillators. It allows, however, the accounting for the change in the shape of perturbed solution. Following [18], solution of Eq. (11) may be written as

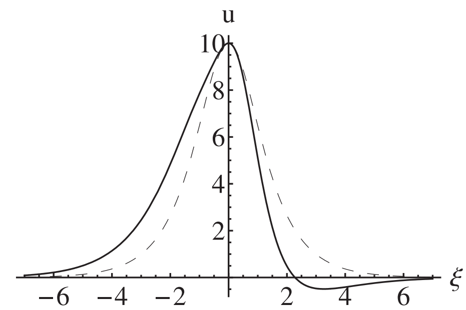

FIG. 3. Change in the shape of the soliton in the presence of IRS predicted by Eq. (20a) for the values of γ: $\gamma=0.0008$ (dashed line) and $\gamma=0.8$ (solid line).

$$u(\tau) = \sum_{n=0}^{\infty} \varepsilon^n u_n(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \cdots , \tag{19}$$

where $u_0(\tau)=a_0 \operatorname{sech}(\tau)$. The expansion of the control parameter μ in the power of ε coincides with Eq. (15) [18]. Applying the idea in [18] to Eq. (11), I have found the critical control parameter μ_{C0} under which a solution forms (see Appendix B), and it turned out that the values of μ_{C0} and M_{C0} are again given by Eqs. (16) and (17), respectively. So, the expressions for M_{C0} given by Eq. (17) obtained here in the framework of the proposed perturbation approach that includes the usage of the two hyperbolic perturbation methods [17,18] coincide. Moreover, Eqs. (17) and (5) obtained by the adiabatic approximation of the SPT agree well. This fact can be considered as justification of an approximation [see Eq. (8)], which has been done in order to obtain the equation of strongly nonlinear Duffing–Van der Pol oscillator. I have further obtained (see Appendix B) $u_1(\tau)=(a_0^3\mu_1/5\omega_0)\ln[\cosh(\tau)]\operatorname{sech}(\tau)\tanh(\tau)$. So, the perturbed solution of the equation of strongly nonlinear Duffing–Van der Pol oscillator is given by

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + O(\varepsilon^2) = a_0\{1 - (4\gamma a_0/5)\ln[\cosh(\tau)]\tanh(\tau)\}\operatorname{sech}(\tau) + O(\varepsilon^2), \tag{20a}$$

$$\mu = \mu_{C0} + \varepsilon\mu_{C1} + O(\varepsilon^2) = -8a_0^2/5 + O(\varepsilon^2). \tag{20b}$$

The obtained functional form of the change in pulse shape, given by Eq. (20a), is quite similar to the functional form of Eq. (14) in [10]. This is an interesting observation, having in mind the quite different approaches of investigation used in [10].

Next we should point out that the correction term in Eq. (20a) does not depend on the parameters of BLA. For the value of IRS considered until now, namely, $\gamma=0.0005$, it is clear that the amplitude in front of the correction term in Eq. (20a) should be very small and therefore difficult to observe numerically. Next, Fig. 3 shows graphics of Eq. (20a) for two different values of γ: $\gamma=0.0008$ and $\gamma=0.8$.

As can be seen from Fig. 3, the correction term in Eq. (20a) in the case of $\gamma=0.0008$ (dashed line) is very small compared to the soliton one, while in the case of $\gamma=0.8$ (solid line) it leads to clearly observed asymmetry in the shape of the pulse.

Comparing the applicability of both hyperbolic perturbation methods we can expect the following. In the case of BLA (and without IRS) an exact chirped "sech-like" solution of the basic equation is well known [2,3]. So in the case of larger values of the parameters describing BLA than that of IRS, one can expect that the changes in the form of the pulse will be weakly expressed, and therefore the first hyperbolic perturbation method will be more appropriate. [Note that in accordance with Eqs. (18b) and (20a), the corrections in the frequency and the shape of the solution are proportional to the first and second powers of the amplitude of solution, respectively.] With the increase of the influence of the IRS and the amplitude of the solution, however, the change in the shape of the soliton solution will increase its importance.

V. CONCLUSION

In this paper, I analyzed the reduction of the amount of spectral shift due to the SSFS in the presence of BLA or the suppression of the SSFS by BLA. Two different methods, namely, the adiabatic approximation of SPT and the proposed approach here, have been used.

The stability of the equilibrium point for the soliton amplitude and velocity identified earlier by the adiabatic approximation of SPT [5] is analyzed by numerical solution of linearized system in the neighborhood of equilibrium point. The obtained analytical expressions for the eigenvalues of the linearized system allow the determination of the values of pulse and material parameters for which the equilibrium point is stable. The relation between the stationary amplitude and the soliton's velocity [Eq. (5)] is numerically verified. According to the derived relation between the soliton's amplitude and velocity of the soliton [Eq. (5)], the soliton's velocity is directly proportional to the parameter describing the IRS and the square of the soliton's amplitude and inversely proportional to the parameter that describes the finite bandwidth of the fiber amplifier β. For fixed parameter describing the IRS γ, with the increase of the amplifier's bandwidth parameter β, the soliton's velocity decreases.

An alternative perturbation approach has been suggested based on the introduction of the equation of strongly nonlinear Duffing–Van der Pol oscillator. This equation is explored here by two different methods. First, the recently obtained results of Chen and Chen [17] for this equation are used. Next, the hyperbolic Lindstedt-Poincare perturbation method of Chen *et al.* [18] is applied to the exploration of the equation of strongly nonlinear Duffing–Van der Pol oscillator. It turned out that the coupling of the equilibrium velocity and amplitude of perturbed stationary solution in both methods coincides with the relation between the soliton amplitude and velocity derived by the adiabatic approximation of SPT.

It turned out, however, that the pertubation approach proposed here can bring more additional useful information. The application in [18] to the equation of the strongly nonlinear Duffing–Van der Pol oscillator, proposed in this work, allowed the description of the change in the functional form of the perturbed stationary solution. It turned out that the obtained change here in the form of the perturbed stationary solution is similar to the correction to the form of soliton solutions found in [10] [see Eq. (14) there]. The latest result is particularly important as the qualitative description of the change of the form of solution is a result that is beyond the abilities of the adiabatic approximation of SPT. In addition, it was also shown (by means of [17]) how the frequency of the perturbed stationary solution of the equation of the strongly nonlinear Duffing–Van der Pol oscillator changes due to the IRS.

The reported results intimate that the proposed perturbation approach can be used in the further exploration of properties of optical pulses in the presence of BLA and IRS. Our consideration points out the relation between two dissipative systems, one described by the basic nonlinear partial differential equation and the other characterized by strongly nonlinear ordinary differential equation of Duffing–Van der Pol oscillator.

A comment concerning the applicability of earned results is in order. Independently of the fact, that the correctness of the results obtained the adiabatic approximation of SPT that has been checked through the direct numerical solution of the basic equation [5], further numerical investigation is required in order to confirm the perturbation results obtained here. In the occurrence of strong influence of the BLA and higher-order effects, the phase modulation of the optical pulses should be accounted for. Regarding BLA, the well-known reason for this is the existence of exact chirped solutions of corresponding equation [3,2]. For example, due to the presence of BLA, the initial Schrödinger solitons can evolve in the course of propagation into the exact chirped solutions, a transformation that can significantly change the character of the soliton interaction [22].

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APPENDIX A: HYPERBOLIC PERTURBATION METHOD

In this appendix, the main steps of the hyperbolic perturbation method of Chen and Chen [17], which lead to Eqs. (18a) and (18b), are shortly repeated. After substituting Eqs. (14a) and (14b) into Eq. (11) equations in different orders of magnitude have been identified (see Eqs. (3.9)–(3.11) in [17]), solving which the values of $\mu_{C0},\mu_{C1},\ldots$ and each order solutions $u_0,u_1,\ldots$ have been obtained. Using quantity

$$I(\tau)=\int g(\mu,u_0,u_0')u_0'd\tau, \tag{A1}$$

the necessary condition for the existence of the perturbed homoclinic solutions of Eq. (11) can be formulated as the requirement that μ_{C0} should be resolved by the equation [17]

$$I(\tau)|_{-\infty}^{+\infty}=0. \tag{A2}$$

To have this equation fulfilled A_1 should be zero, i.e., $A_1=(5\mu_{C0}-2\mu_1 a_0^2)/15=0$, or the value of μ_{C0} is given by Eq.

(16). Applying the condition $I(\tau)|_0^{+\infty}=0$ it was proved that

$$a_1=0 \quad \text{and} \quad u_1=u_1'=0. \tag{A3}$$

For the first correction of frequency it was found that

$$\omega_1(\tau)=I(\tau)/(\omega_0 u_0'^2). \tag{A4}$$

It turned out that $I(\tau)=\omega_0 a_0^2 A_2 \operatorname{sech}^2(\tau)\tanh^3\tau$, where $A_2=-\mu_1 a_0^2/5$ [17]. Using Eq. (A4), it has been obtained that $\omega_1(\tau)=A_2\tanh\tau=(-a_0^2\mu_1/5)\tanh\tau$. For nonzero a_0 and ω_0, it was shown that $\mu_{C1}=0$. Then making use of Eqs. (A3) and (A4), the solution of Eq. (11) can be written in the form of Eqs. (18a) and (18b).

APPENDIX B: HYPERBOLIC LINSTEDT-POINCARE PERTURBATION METHOD

In this appendix, I describe the main steps of the application of the hyperbolic Lindstedt-Poincare method of Chen *et al.* [18] to the equation of strongly nonlinear Duffing–Van der Pol oscillator proposed here. After substituting Eq. (19) and Eq. (15) into Eq. (11), the following equations in different orders of magnitude have been identified:

$$\varepsilon^0{:}\,\omega_0^2 u_0''+c_1u_0+c_3u_0^3=0, \tag{B1a}$$

$$\varepsilon^1{:}\,\omega_0^2 u_1''+(c_1+3c_3u_0^2)u_1=g(\mu_{C0},u_0,u_0'), \tag{B1b}$$

$$\varepsilon^2{:}\,\omega_0^2 u_2''+(c_1+3c_3u_0^2)u_2=\mu_{C1}g_\mu(\mu_{C0},u_0,u_0')+u_1g_u(\mu_{C0},u_0,u_0')+u_1'g_{u'}(\mu_{C0},u_0,u_0')-3c_3u_1^2u_0. \tag{B1c}$$

Differentiating Eq. (B1a) with respect to ξ leads to

$$\omega_0^3u_0'''+c_1\omega_0u_0'+3c_3\omega_0u_0^2u_0'=0. \tag{B2}$$

It can be seen from Eq. (B2) that u_0' is a solution of the homogeneous part of Eq. (B1b). The particular solution of Eq. (B1b) can be written as

$$u_1=u_0'\int\frac{1}{u_0'^2}\left[\int u_0'g(\mu_{C0},u_0,u_0')d\xi\right]d\xi. \tag{B3}$$

Similar to the classical Lindstedt-Poincare method [20], the homogeneous solution of u_1 is ignored. Multiplying both sides of Eq. (B1b) with u_0' and integrating the equation from $-\infty$ to $+\infty$, one can achieve

$$(u_0'u_1'-u_0''u_1)_{-\infty}^{+\infty}|+\int_{-\infty}^{+\infty}(u_0'''+c_1u_0'+3c_3u_0^2u_0')u_1d\xi=\int_{-\infty}^{+\infty}g(\mu_{C0},u_0,u_0')u_0'd\xi. \tag{B4}$$

Putting into action (B1a), Eq. (B4) is transformed into

$$(u_0'u_1'-u_0''u_1)|_{-\infty}^{+\infty}=\int_{-\infty}^{+\infty}g(\mu_{C0},u_0,u_0')u_0'd\xi. \tag{B5}$$

As we are looking for a homoclinic solution (u,u') that approaches a saddle point in the phase plane as $\xi\to\pm\infty$, (u_1,u_1') and (u_2,u_2') should be bounded as $\xi\to\pm\infty$ or

$$u_1(\pm\infty)\neq\pm\infty,\quad u_1'(\pm\infty)\neq\pm\infty, \tag{B6}$$

$$u_2(\pm\infty)\neq\pm\infty,\quad u_2'(\pm\infty)\neq\pm\infty. \tag{B7}$$

Then the necessary condition for the existence of the homoclinic solution of Eq. (11) is

$$I(\xi)|_{-\infty}^{+\infty}=\int_{-\infty}^{+\infty}g(\mu_{C0},u_0,u_0')u_0'd\xi=0. \tag{B8}$$

By solving Eq. (B8) one can determine the critical value of μ_{C0} and consequently avoid secular terms of u_1. Multiplying both sides of Eq. (B1c) with u_0' and integrating the equation from $-\infty$ to $+\infty$ we get

$$(u_0'u_2'-u_0''u_2)|_{-\infty}^{+\infty}=\int_{-\infty}^{+\infty}[\mu_{C1}g_\mu(\mu_{C0},u_0,u_0')+u_1g_u(\mu_{C0},u_0,u_0')+u_1'g_{u'}(\mu_{C0},u_0,u_0')-3c_3u_1^2u_0]u_0'd\xi. \tag{B9}$$

Taking into account conditions given by Eq. (B7) from Eq. (B8) may be obtained,

$$\int_{-\infty}^{+\infty}[\mu_{C1}g_\mu(\mu_{C0},u_0,u_0')+u_1g_u(\mu_{C0},u_0,u_0')+u_1'g_{u'}(\mu_{C0},u_0,u_0')-3c_3u_1^2u_0]u_0'd\xi=0. \tag{B10}$$

By solving Eq. (B10) the value of μ_{C1} under which there exists a homoclinic solution of Eq. (11) can be determined. One can eliminate the secular terms of the second-order solution u_2,

$$u_1=u_0'\int\frac{1}{u_0'^2}\left[\int u_0'[\mu_{C1}g_\mu(\mu_{C0},u_0,u_0')+u_1g_u(\mu_{C0},u_0,u_0')+u_1'g_{u'}(\mu_{C0},u_0,u_0')-3c_3u_1^2u_0]d\xi\right]d\xi. \tag{B11}$$

It turns out that $I(\tau)=\omega_0a_0^2[A_1+A_2\operatorname{sech}^2(\tau)]\tanh^3\tau$, where $A_1=(5\mu_{C0}-2\mu_1a_0^2)/15$ and $A_2=-\mu_1a_0^2/5$. Making use of Eq. (B8), the value of μ_{C0} can be calculated and it is given again by Eq. (16). It follows from Eq. (B3) that $u_1(\tau)=(a_0^3\mu_1/5\omega_0)\ln[\cosh(\tau)]\operatorname{sech}(\tau)\tanh(\tau)$. Taking advantage of Eq. (B11), one can obtain that $2\mu_{C1}a_0^2\omega_0/3=0$ and as a_0 and ω_0 are different from zero then $\mu_{C1}=0$. Finally, the homoclinic solution of Eq. (11) can be written in the form of Eqs. (20a) and (20b).

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