

Universality in disordered systems: The case of the three-dimensional random-bond Ising model

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We study the critical behavior of the $d=3$ Ising model with bond randomness through extensive Monte Carlo simulations and finite-size scaling techniques. Our results indicate that the critical behavior of the random-bond model is governed by the same universality class as the site- and bond-diluted models, clearly distinct from that of the pure model, thus providing a complete set of universality in disordered systems.

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Understanding the role of impurities on the nature of phase transitions is of great importance, both from experimental and theoretical perspectives. First-order phase transitions are known to be significantly softened under the presence of quenched randomness [1–5], whereas continuous transitions may have their exponents altered under random fields or random bonds [6,7]. There are some very useful phenomenological arguments and some, perturbative in nature, theoretical results, pertaining to the occurrence and nature of phase transitions under the presence of quenched randomness [2,8,9]. Historically, the most celebrated criterion is that suggested by Harris [6]. This criterion relates directly the persistence, under random bonds, of the nonrandom behavior to the specific-heat exponent α_p of the pure system. According to this criterion, if $\alpha_p > 0$, then disorder will be relevant, i.e., under the effect of the disorder, the system will reach a new critical behavior. Otherwise, if $\alpha_p < 0$, disorder is irrelevant and the critical behavior will not change. Pure systems with a zero specific heat exponent ($\alpha_p = 0$) are marginal cases of the Harris criterion and their study, upon the introduction of disorder, has been of particular interest [10]. The paradigmatic model of the marginal case is, of course, the general random two-dimensional (2D) Ising model and this model has been extensively debated [11].

Respectively, the three-dimensional (3D) Ising model with quenched randomness—which is a clear case in terms of the Harris criterion having a positive specific heat exponent in its pure version—has also been extensively studied using Monte Carlo (MC) simulations [12–19] and field theoretical renormalization group approaches [20–22]. Especially, the diluted model can be treated in the low-dilution regime by analytical perturbative renormalization group methods [23–25], where a new fixed point independent of the dilution has been found, yet for the strong dilution regime only MC results remain valid. Historically, the first numerical studies of the model suggested a continuous variation of the critical exponents along the critical line, but it became clear, after the work of Ref. [14], that the concentration-dependent critical exponents found in MC simulations are the effective ones characterizing the ap-

proach to the asymptotic regime. Note, here, that a crucial problem of the new critical exponents obtained in these studies is that the ratios β/ν and γ/ν occurring in finite-size scaling (FSS) analysis are almost identical for the disordered and pure models. In fact, for the pure 3D Ising model, accurate values are [26] $\nu=0.6304(13)$, $\beta/\nu=0.517(3)$, $\gamma/\nu=1.966(3)$, and $\alpha=0.1103(1)$. Respectively, for the site- and bond-diluted model, the most accurate sets of asymptotic exponents ($\beta/\nu, \gamma/\nu, \nu$) have been given by the extensive numerical works of Ballesteros *et al.* [16] and Berche *et al.* [19] are [0.519(3), 1.963(5), 0.6837(53)] and [0.515(5), 1.97(2), 0.68(2)].

The above estimates of critical exponents provided evidence that the 3D Ising model with quenched uncorrelated disorder belongs to a *single universality class*, distinct from that of the pure model, as also indicated by the Harris criterion, independent of the considered disorder distribution. Yet, in a very recent paper Murtazaev and Babaev [27] using MC simulations and FSS methods on the site-diluted model the above view was contradicted and these authors suggested that the model has two regimes of critical behavior universality, depending on the nonmagnetic impurity concentration. Motivated by the above contradictions and the great theoretical interest of the existence of universality classes in disordered models, we have chosen to investigate the 3D Ising model with bond disorder in order to compare all these three kinds of disorder (site-, bond-dilution and bond disorder) and to verify whether these lead to the same set of new critical exponents, as would be, in principle, expected by universality arguments [19]. In this contribution we show that, indeed, the 3D Ising model with quenched, uncorrelated bond disorder belongs to the same universality class as the site- and bond-diluted models, defining in this way a complete universality class in disordered spin models.

In the following we consider the 3D bond-disorder Ising model whose Hamiltonian with uncorrelated quenched random interactions can be written as

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} s_i s_j, \quad (1)$$

where the spin variables s_i take on the values $-1, +1$, $\langle ij \rangle$ indicates summation over all nearest-neighbor pairs of sites,

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and the ferromagnetic interactions $J_{ij} > 0$ follow a bimodal distribution of the form

$$\mathcal{P}(J_{ij}) = \frac{1}{2}[\delta(J_{ij} - J_1) + \delta(J_{ij} - J_2)], \quad (2)$$

where $J_1 + J_2 = 2$, $J_1 > J_2$, and $r = J_2/J_1$ reflects the strength of the bond randomness. Additionally, we fix $2k_B/(J_1 + J_2) = 1$ to set the temperature scale. The value of the disorder strength considered throughout this work is $r = 1/3$.

Resorting to large scale MC simulations is often necessary, especially for the study of the critical behavior of disordered systems. It is also well known that for such complex systems traditional methods become inefficient and thus in the last few years several sophisticated algorithms, some of them are based on entropic iterative schemes, have been proven to be very effective [28]. The present numerical study has been carried out by applying our recent and efficient entropic scheme [29]. In this approach we follow a two-stage strategy of a restricted entropic sampling [30] based on the Wang-Landau (WL) algorithm [31]. As we do not wish to reproduce here the details of our implementation, we give only a brief discussion on the nature of the WL method. The usual WL recursion proceeds by modifying the density of states $G(E)$ according to the rule $G(E) \rightarrow fG(E)$ and initially one chooses $G(E) = 1$ and $f = f_0 = e$. Once the accumulative energy histogram is sufficiently flat, the modification factor f is redefined as $f_{j+1} = \sqrt{f_j}$, with $j = 0, 1, \dots$ and the energy histogram reset to zero until f is very close to unity (i.e., $f = e^{10^{-8}} \approx 1.000\ 000\ 01$). Once f is close enough to unity, systematic deviations become negligible. However, the WL recursion violates the detailed balance from the early stages of the process and care is necessary in setting up a proper protocol of the recursion. In spite of the fact that the WL method has produced very accurate results in several models, it is fair to say that there is not a safe way to access possible systematic deviations in the general case. This has been pointed out and critiqued in a recent review by Janke [32]. However, from our experience and especially from our recent studies on 2D and 3D disordered spin models [33], the WL implementation followed in these papers has produced excellent results, enabling the discrimination between competing theoretical predictions on that model. Using this combined approach we performed extensive simulations for several lattice sizes $L \in \{8, 16, 24, 32, 40, 64\}$, over large ensembles of random realizations of the order of 1000–3000. Each disorder realization was simulated at least 10–20 times with different initial conditions to improve accuracy.

It is well known that, extensive disorder averaging is necessary when studying random systems, where usually broad distributions are expected leading to a strong violation of self-averaging [17,34]. A measure from the scaling theory of disordered systems, whose limiting behavior is directly related to the issue of self-averaging [17] may be defined with the help of the relative variance of the sample-to-sample fluctuations of any relevant singular extensive thermodynamic property Z as follows: $R_Z = ([Z^2]_{\text{av}} - [Z]_{\text{av}}^2) / [Z]_{\text{av}}^2$. Closely related to the above issue of self-averaging in disordered systems is the manner of averaging over the disorder.

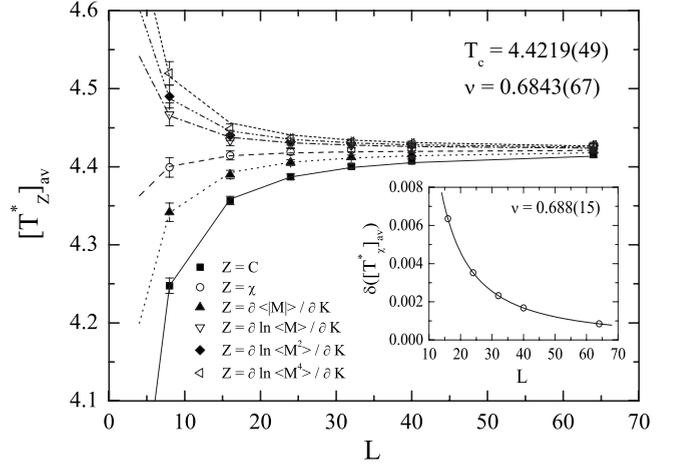


FIG. 1. Shift behavior of several pseudocritical temperatures as defined in the text. The error bars reflect the sample-to-sample fluctuations. The inset shows the FSS of the sample-to-sample fluctuations of the pseudocritical temperature of the magnetic susceptibility.

This nontrivial manner may be performed in two distinct ways when identifying the finite-size anomalies. The first way corresponds to the average over disorder realizations ($[\dots]_{\text{av}}$) and then taking the maxima ($[\dots]_{\text{av}}^*$), or taking the maxima in each individual realization first, and then taking the average ($[\dots^*]_{\text{av}}$). Closing this brief outline, let us comment on the statistical errors of our numerical data. The statistical errors of our WL scheme on the observed average behavior were found to be of small magnitude (of the order of the symbol sizes) and thus are neglected in the figures. On the other hand for the case $[\dots^*]_{\text{av}}$ the error bars shown reflect the sample-to-sample fluctuations.

We briefly present here our numerical results for the 3D random-bond Ising model. Figure 1 illustrates in the main panel the shift behavior of 6 pseudocritical temperatures estimated via the second way of averaging discussed above, i.e., by taking the average over the individual pseudocritical temperatures. The error bars reflect the sample-to-sample fluctuations.

The pseudocritical temperatures considered correspond to the peaks of the following six quantities: specific heat C , magnetic susceptibility χ , derivative of the absolute order parameter with respect to inverse temperature $K = 1/T$: $\partial \langle |M| \rangle / \partial K = \langle |M|H \rangle - \langle |M| \rangle \langle H \rangle$ [35] and logarithmic derivatives of the first ($n=1$), second ($n=2$), and fourth ($n=4$) powers of the order parameter with respect to inverse temperature $\partial \ln \langle M^n \rangle / \partial K = \langle M^n H \rangle / \langle M^n \rangle - \langle H \rangle$ [35]. Fitting our data for the whole lattice range to the expected power-law behavior $[T^*]_{Z,av} = T_c + bL^{-1/\nu}$, where Z stands for the different thermodynamic quantities mentioned above, we find the critical temperature to be $T_c = 4.4219(49)$ well below the critical temperature of the pure model and the estimate $\nu = 0.6843(67)$ for the critical exponent of the correlation length. This value is in excellent agreement with the values 0.6837(53) and 0.68(2) given by Ballesteros *et al.* [16] and Berche *et al.* [19].

Using the above sample-to-sample fluctuations of the pseudocritical temperatures and the theory of FSS in disor-

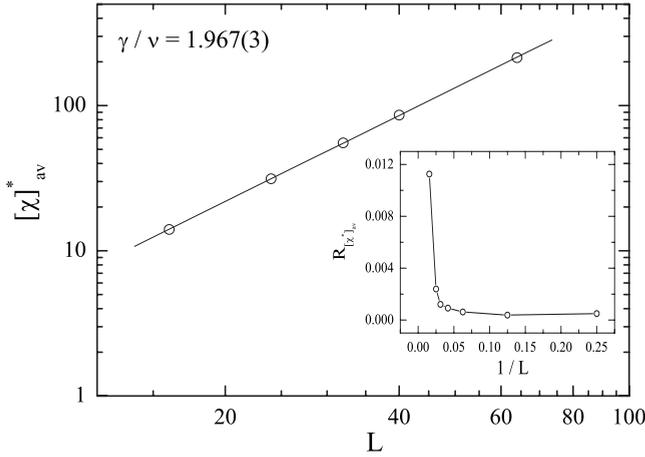


FIG. 2. FSS of the maxima of the disorder-averaged magnetic susceptibility in a log-log scale. The inset shows the limiting behavior of the ratio $R_{[\chi^*]_{av}}$.

dered systems as introduced by Aharony and Harris [34] and Wiseman and Domany [17], one may further examine the nature of the fixed point controlling the critical behavior of the disordered system. According to the theoretical predictions [17,34], the pseudocritical temperatures T_Z^* of the disordered system are distributed with a width $\delta[T_Z^*]_{av}$, that scales, in the case of a new random fixed point, with the system size as $\delta[T_Z^*]_{av} \sim L^{-1/\nu}$. In the inset of Fig. 1 we plot these sample-to-sample fluctuations of the pseudocritical temperature of the magnetic susceptibility. The solid line shows a very good power-law fitting giving the value 0.688(15) for the exponent ν , which is also in very good agreement with the value 0.6843(67) obtained via the classical shift behavior and the most accurate estimates in the literature [16,19].

In Figs. 2 and 3 we provide estimates for the magnetic exponent ratios of the model. In particular, in Fig. 2 we present the FSS of the maxima of the disorder-averaged magnetic susceptibility $[\chi^*]_{av}$ in a double logarithmic scale.

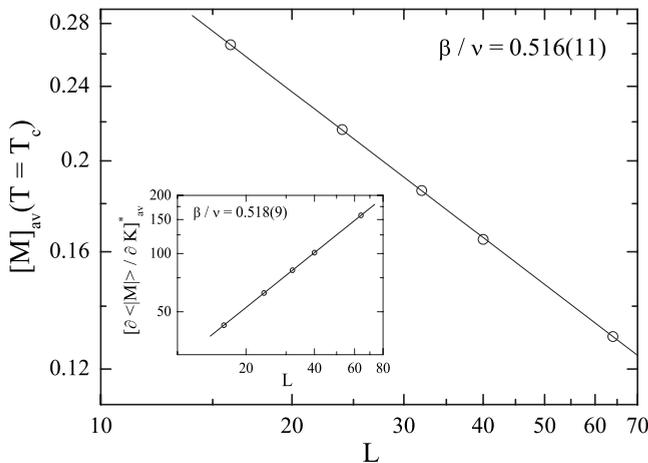


FIG. 3. FSS of the disorder-averaged order parameter at the estimated critical temperature in a log-log scale. The inset illustrates the FSS of the maxima of the disorder-averaged inverse-temperature derivative of the absolute order parameter.

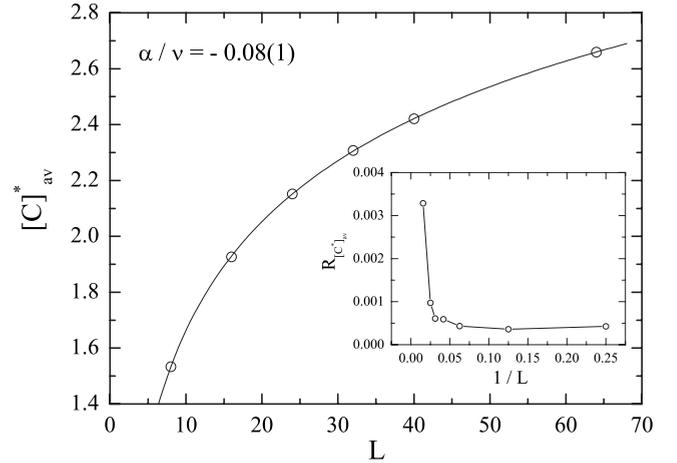


FIG. 4. FSS of the maxima of the disorder-averaged specific heat. The inset shows the limiting behavior of the ratio $R_{[C^*]_{av}}$.

The solid line presents a linear fitting using the total lattice range spectrum, giving the estimate 1.967(3) for the ratio γ/ν ($[\chi^*]_{av} \sim L^{\gamma/\nu}$). The inset shows the ratio R_Z , where $Z = [\chi^*]_{av}$ as a function of the inverse lattice size indicating a strong violation of self-averaging of the magnetic properties of the 3D Ising model with bond disorder. Respectively, in Fig. 3 we plot the disorder-averaged magnetization at the estimated critical temperature, as a function of the lattice size L , also in a log-log scale. The solid line is a linear fitting ($[M]_{av}(T=T_c) \sim L^{-\beta/\nu}$) giving the value $\beta/\nu = 0.516(11)$. Additional estimate for the ratio β/ν can be obtained from the FSS of the derivative of the absolute order parameter with respect to inverse temperature which is expected to scale as $L^{(1-\beta)/\nu}$ with the system size [35]. Thus, in the corresponding inset of Fig. 3 we plot the data for $\partial\langle|M|\rangle/\partial K$ averaged over disorder as a function of L , also in a double logarithmic scale. The solid line is a linear fitting that gives an estimate of 0.518(9) for the ratio β/ν . Thus, overall the values for the ratios of the magnetic exponents are in excellent agreement with the best estimates for the site- $[\beta/\nu = 0.519(3)$ and $\gamma/\nu = 1.963(5)]$ [16] and bond-diluted $[\beta/\nu = 0.515(5)$ and $\gamma/\nu = 1.97(2)]$ [19] cases reinforcing the scenario of a single distinctive universality in the 3D Ising model with quenched uncorrelated disorder, independent of the disorder distribution.

Using our estimates for the critical exponents and the Rushbrooke relation ($\alpha/\nu + 2\beta/\nu + \gamma/\nu = 2/\nu$), we estimate the ratio α/ν to be $-0.085(10)$. In Fig. 4 we present the FSS of the maxima of the disorder-averaged specific heat data as a function of the linear size L . The solid line shows a typical power-law fitting attempt of the form $[C^*]_{av} \sim L^{\alpha/\nu}$, that gives a value $\alpha/\nu = -0.08(1)$.

This is a further reliability test of our numerical method and extended simulations. The corresponding inset of Fig. 4 presents the ratio $R_{[C^*]_{av}}$ as a function of the inverse linear size. As in the inset of Fig. 2, also here a violation of self-averaging is observed, yet this is, in absolute numbers, much smaller than that of the magnetic susceptibility.

Summarizing, we have presented in this Brief Report concrete evidence that the critical behavior of the 3D Ising

model with quenched uncorrelated disorder is controlled by a new random fixed point, independent of the way randomness is implemented in the system. This result has been obtained through extensive numerical simulations and classical finite-size scaling techniques. Particular interest was paid to the sample-to-sample fluctuations of the pseudocritical temperatures model and their scaling behavior that was used as a successful alternative approach to criticality that verified the above scenario. Although we acknowledge that Ballesteros *et al.*

al. [16] and Berche *et al.* [19] were the first to support numerically the above view, we believe that the present contribution puts a further significant step on this intensively debated issue of universality in disordered spin models. Very interesting would be also to study more complicated models, where disorder couples to the order parameter, and one such prominent candidate is the 3D random-field Ising model [36]. For these type of models, the existence of universality classes has been severely questioned [37].

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