Analytic description of cylindrical electromagnetic wave propagation in an inhomogeneous nonlinear and nondispersive medium

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(Received 9 July 2010; published 18 November 2010)

In a recent publication [E. Y. Petrov and A. V. Kudrin, Phys. Rev. Lett. **104**, 190404 (2010)], a method for constructing exact axisymmetric solutions of the Maxwell equations in a nonlinear nondispersive medium has been put forward. In this Brief Report, we will use the proposed method to deal with problems of wave propagation in an inhomogeneous nonlinear and nondispersive medium. The inhomogeneous factor is chosen in the form as r^{β}, where β is a certain constant. By solving the Maxwell equations an exact axisymmetric solution is obtained, starting from the corresponding solution of linear field equations, to describe the propagation of cylindrical electromagnetic waves in the medium. In the limit $\beta \to 0$, our solutions go into a nonlinear but homogeneous case, which is the same as prevenient results. We analyze the initial value problem and boundary value problem, to compare the differences between homogeneous and inhomogeneous conditions. It is found that the amplitude and frequency of the electromagnetic wave can be controlled with different β. Our results can be used for analysis of inhomogeneous ferroelectric resonators.

DOI: 10.1103/PhysRevE.82.057602 PACS number(s): 03.50.De, 02.30.Jr

Electromagnetic wave propagation in various kinds of media is a wide-ranging problem in physics [1–19]. For linear and homogeneous media, the corresponding linear Maxwell equations have been studied and applied extensively. However, the linear and homogeneous media give only an approximation of the real media. The real media are usually nonlinear and inhomogeneous. This makes wave propagation in nonlinear media to be a fundamental problem in physics [1–3]. Significant progress has been made by using analytical and numerical methods [4–6]. Wave propagation in inhomogeneous media also attracted wide attention in theory and applications [7–11]. A lot of achievements have been made theoretically and experimentally. For instance, suppression of spiral and turbulence in inhomogeneous media was investigated numerically by Chen *et al.* [9]. Levy *et al.* [7] used an inhomogeneous dielectric metamaterial with space-variant polarizability to achieve light focusing.

However, electromagnetic wave propagation in a medium with inhomogeneous and nonlinear simultaneously, as an extremely complicated problem, remains poorly studied, especially exact solutions. Recently, a method for constructing exact axisymmetric solutions of the Maxwell equations in a nonlinear nondispersive medium has been put forward [1]. It is very important that this work gives an exact solution for $\varepsilon(E)=\epsilon_0\varepsilon_1 \exp(\alpha E)$ to describe the behavior of nonlinear waves in a bounded volume which is also an extremely complicated problem. Furthermore, we find that this method could be extended to solve problems of electromagnetic wave propagation in an inhomogeneous and nonlinear medium. In what follows, we will employ this method to construct exact axisymmetric solutions of the Maxwell equations in an inhomogeneous nonlinear and nondispersive medium.

Similar to Ref. [1], we assume that the medium possesses an axis of symmetry and taken as the z axis of a cylindrical coordinate system (r,ϕ,z). If the fields are independent of ϕ and z, then the Maxwell equations can be written as

$$\frac{\partial H}{\partial r}+\frac{H}{r}=\varepsilon(E,r)\frac{\partial E}{\partial t}, \quad \frac{\partial E}{\partial r}=\mu_0\frac{\partial H}{\partial t}, \tag{1}$$

where $H\equiv H_{\phi}(r,t)$, $E\equiv E_z(r,t)$, and $\varepsilon(E,r)=dD/dE$. In our work, the function $\varepsilon(E,r)$ will be chosen in the form

$$\varepsilon(E,r)=\epsilon_0\varepsilon_1 r^{\beta}\exp(\alpha E), \tag{2}$$

where ε_1, α, β are certain constants. It denotes that the medium considered here is nonlinear, nondispersive, and inhomogeneous. The nonlinear factor is $\exp(\alpha E)$, and the inhomogeneous factor is r^{β}. We will show below that system of equations (1) and (2) can be integrated exactly and admits exact axisymmetric solutions in this case. Before passing to the process of constructing solutions, we will show the possibility of using the dielectric susceptibility as formula (2). References [1,3] have discussed the possibility of using $\varepsilon(E)=\epsilon_0\varepsilon_1 \exp(\alpha E)$ in detail and reached a conclusion that $\varepsilon(E)=\epsilon_0\varepsilon_1 \exp(\alpha E)$ correctly describes dielectric properties of media lacking a center of inversion in the case of weak nonlinearity. For the inhomogeneous factor, the type of the space-variant polarizability as r^{β} or $(r-a)^{\beta}$ has been used in many works [16–19]. For example, Jiang *et al.* [10] used $\varepsilon_{\phi}=kr$ and $\varepsilon_r=1/(kr)$ to bend electromagnetic waves and showed that the inhomogeneous factor is realizable. In fact, the metamaterial opens a door to realize all possible material properties by designing different cellular architectures and using different substrate materials [15]. In our work if $\beta<0$, the inhomogeneous factor has an evident singularity at $r=0$. The singularity is avoidable by excluding this point and our solutions is effective. For simplicity, we will focus on the case of $\beta \geq 0$.

We use the following ansatz in system of equations (1) and (2): $E=\alpha^{-1}[u-(\beta+2)\xi]$, $H=g_0^{-1}[v-(\beta+2)\eta]$, where $g_0^{-1}=\varepsilon_1^{1/2}(Z_0\alpha r/r_0)^{-1}$, $\xi=\ln(r/r_0)$, $\eta=(\epsilon_0\varepsilon_1\mu_0)^{-1/2}t/r_0$, and

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$Z_0=(\mu_0/\varepsilon_0)^{1/2}$. r_0 is an arbitrary constant with the dimension of length. In our work we set $r_0=1$. Then system of equations (1) and (2) can be reduced to

$$\frac{\partial u}{\partial \xi}=\frac{\partial v}{\partial \eta}, \quad \frac{\partial v}{\partial \xi}=e^u\frac{\partial u}{\partial \eta}. \tag{3}$$

Based on the application of a hodograph transformation, for Jacobian $D(u,v)/D(\xi,\eta)$ being nonzero [1], we can use u and v as independent variables and obtain

$$\frac{\partial \eta}{\partial v}=\frac{\partial \xi}{\partial u}, \quad \frac{\partial \eta}{\partial u}=e^u\frac{\partial \xi}{\partial v}. \tag{4}$$

Following Refs. [1,3], we represent the solution of the homogeneous and linear problem which records as E_0 and H_0 in the form

$$E_0=\mathcal{E}(\rho,\tau), \quad H_0=\varepsilon_1^{1/2}Z_0^{-1}\mathcal{H}(\rho,\tau), \tag{5}$$

where $\rho=r\cdot 2^{2/(\beta+2)}$, $\tau=t(\epsilon_0\varepsilon_1\mu_0)^{-1/2}\cdot 2^{2/(\beta+2)}$, and $\mathcal{E}$ and $\mathcal{H}$ satisfy the linear system:

$$\frac{\partial \mathcal{H}}{\partial \rho}+\frac{\mathcal{H}}{\rho}=\frac{\partial \mathcal{E}}{\partial \tau}, \quad \frac{\partial \mathcal{E}}{\partial \rho}=\frac{\partial \mathcal{H}}{\partial \tau}. \tag{6}$$

We write ξ and η as

$$\xi=\frac{-\alpha\mathcal{E}(w,v)+u}{\beta+2}, \quad \eta=\frac{-\alpha w\mathcal{H}(w,v)/2+v}{\beta+2}, \tag{7}$$

with $w=2\exp(u/2)$, then ξ and η satisfy Eq. (4). We can obtain

$$E=\mathcal{E}\left(\rho^{1+\beta/2}e^{\alpha E/2},2^{-2/(\beta+2)}\left[(\beta+2)\tau+\frac{Z_0\alpha\rho H}{\sqrt{\varepsilon_1}}\right]\right),$$

$$H=\frac{\sqrt{\varepsilon_1}e^{\alpha E/2}\rho^{\beta/2}}{Z_0 2^{\beta/(\beta+2)}}\times\mathcal{H}\left(\rho^{1+\beta/2}e^{\alpha E/2},2^{-2/(\beta+2)}\left[(\beta+2)\tau+\frac{Z_0\alpha\rho H}{\sqrt{\varepsilon_1}}\right]\right). \tag{8}$$

These expressions give an exact solution of Maxwell equations in an inhomogeneous nonlinear and nondispersive medium. For the case $\beta\to 0$, our solution goes into solution of a nonlinear but homogeneous problem, which is the case in Ref. [1]. If $\beta\to 0$ and $\alpha\to 0$, our solution goes into solution of a linear and homogeneous problem, which is described by Eq. (5). If $\alpha\to 0$, our solution goes into solution of a linear but inhomogeneous problem.

For simplicity we begin discussion by considering $\alpha\to 0$. We obtain

$$E=\mathcal{E}[\rho^{1+\beta/2},2^{-2/(\beta+2)}(\beta+2)\tau],$$

$$H=\frac{\sqrt{\varepsilon_1}\rho^{\beta/2}}{Z_0 2^{\beta/(\beta+2)}}\mathcal{H}[\rho^{1+\beta/2},2^{-2/(\beta+2)}(\beta+2)\tau], \tag{9}$$

mark $2^{-2/(\beta+2)}(\beta+2)$ as γ, $\tilde{\rho}=\rho^{1+\beta/2}$, and $\tilde{\tau}=\gamma\tau$, the solution become

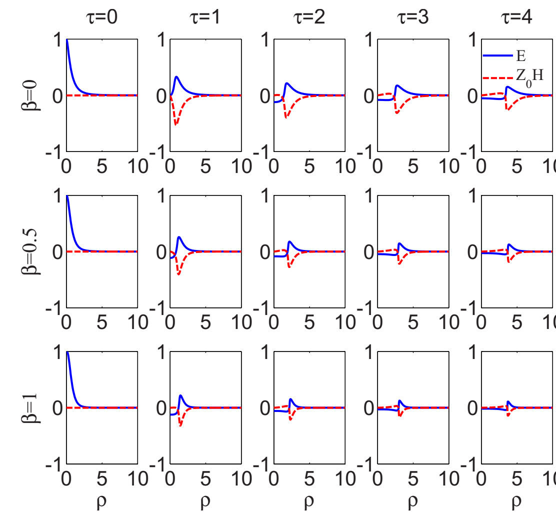

FIG. 1. (Color online) Wave propagation in an inhomogeneous and nonlinear media under the initial field distributions $\mathcal{E}(\rho,0)=(1+\rho^2)^{-3/2}$, $\mathcal{H}\equiv 0$. The electric and magnetic fields as functions of the coordinate are shown by the solid and dashed curves, respectively, at various times and β. We use parameters of $\alpha=1$ and $\varepsilon_1=2$.

$$E=\mathcal{E}(\tilde{\rho},\tilde{\tau}), \quad H=\frac{\sqrt{\varepsilon_1}}{Z_0}\left(\frac{\tilde{\rho}}{2}\right)^{\beta/(\beta+2)}\mathcal{H}(\tilde{\rho},\tilde{\tau}). \tag{10}$$

It means that problems of wave propagation in an inhomogeneous but linear medium can be easily solved by use the space-time coordinate system redefined. Consider the homogeneous and linear solutions $\mathcal{E}(\rho,\tau)$ and $\mathcal{H}(\rho,\tau)$ are periodic functions and can be written as $\mathcal{E}(\rho,\tau)=f_1(\rho)\exp(i\varpi_0\tau)$, $\mathcal{H}(\rho,\tau)=f_2(\rho)\exp(i\varpi_0\tau+\phi)$, then the inhomogeneous solutions are $E=f_1(\tilde{\rho})\exp(i\varpi_0\gamma\tau)$, $H=\sqrt{\varepsilon_1}(\tilde{\rho}/2)^{\beta/(\beta+2)}f_2(\tilde{\rho})\exp(i\varpi_0\gamma\tau+\phi)/Z_0$, so $\varpi_\beta=|\gamma|\varpi_0$, where ϖ_β is the frequency of the electromagnetic field in an inhomogeneous medium. For $\beta>0$, the frequency of the electromagnetic field in an inhomogeneous medium is higher than in a homogeneous medium.

Now we consider the case $\alpha\neq 0$. We use the same particular examples as Ref. [1] and focus on the differences between homogeneous and inhomogeneous conditions. For initial value problem, we consider the initial field distributions are $\mathcal{E}(\rho,0)=\zeta(1+\rho^2)^{-3/2}$, $\mathcal{H}(\rho,0)=0$, and for boundary value problem, we consider $\mathcal{E}(1,\tau)=0$ and amplitude factor of the wave is ζ, where ζ is a constant. The solution can be easily obtained.

Solution of initial value problem:

$$E=\zeta\,\mathrm{Re}\{(1-i\theta)[(1-i\theta)^2+\rho^{2+\beta}e^{\alpha E}]^{-3/2}\},$$

$$H=\frac{\zeta\sqrt{\varepsilon_1}e^{\alpha E/2}\rho^{1+\beta/2}}{Z_0 2^{\beta/(\beta+2)}}\mathrm{Re}\{i[(1-i\theta)^2+\rho^{2+\beta}e^{\alpha E}]^{-3/2}\}, \tag{11}$$

where $\theta=2^{-2/(\beta+2)}[(\beta+2)\tau+Z_0\alpha\rho H/\sqrt{\varepsilon_1}]$, and initial conditions satisfied the nonlinear and inhomogeneous problem is $E=\zeta[1+\rho^{2+\beta}e^{\alpha E}]^{-3/2}$, $H\equiv 0$.

Figure 1 shows results of numerical calculations of E and H by formulas (11) in the case $\alpha=1$, $\zeta=1$ and $\varepsilon_1=2$. The

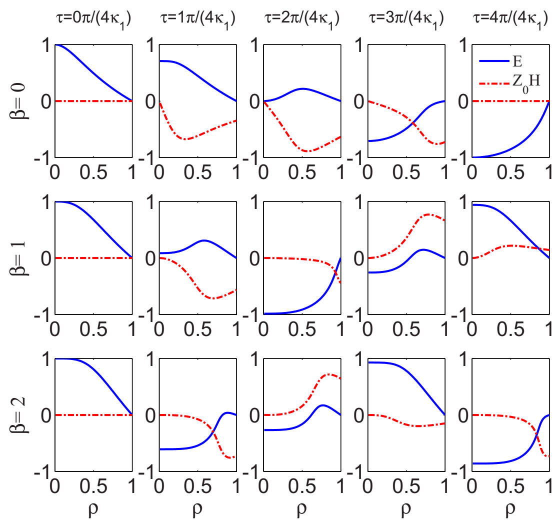

FIG. 2. (Color online) Wave propagation in a inhomogeneous and nonlinear media under the boundary value problem consider $\mathcal{E}(1,\tau)=0$, electric and magnetic fields as functions of ρ in the $n=1$ mode. The parameters are same as in Fig. 1.

electric and magnetic fields as functions of the coordinate ρ are shown by the solid and dashed curves, respectively, at various τ and β. It is obvious that amplitude of the electromagnetic wave can be controlled by using different β. The amplitude of the electromagnetic wave in an inhomogeneous media is smaller than in a homogeneous one. Reference [1] showed that even for a sufficiently strong nonlinearity, the difference between the initial conditions is small, which is also effective for the case when the medium is inhomogeneous.

Solution of boundary value problem:

$$E=\zeta J_0(\kappa_n\rho^{1+\beta/2}e^{\alpha E/2})\cos(\kappa_n\theta),$$

$$H=\frac{-\zeta\sqrt{\varepsilon_1}e^{\alpha E/2}\rho^{\beta/2}}{Z_0 2^{\beta/(\beta+2)}}J_1(\kappa_n\rho^{1+\beta/2}e^{\alpha E/2})\sin(\kappa_n\theta). \quad (12)$$

Figure 2 shows results of numerical calculations of normalized field E and H by formulas (12) in the lowest mode with $n=1$ and $\kappa_1\approx 2.4048$. Other parameters remain the same as Fig. 1. It is a very interesting phenomenon that the waveform with different β is entirely different at the same time while very similar at some specifically different time. For instance, the waveform of $\beta=0$, $\tau=3\pi/(4\kappa_1)$ is very similar to the one of $\beta=2$, $\tau=1\pi/(4\kappa_1)$, the waveform of $\beta=0$, $\tau=4\pi/(4\kappa_1)$ is very similar to the one of $\beta=1$, $\tau=2\pi/(4\kappa_1)$, and the waveform of $\beta=1$, $\tau=3\pi/(4\kappa_1)$ is very similar to the one of $\beta=2$, $\tau=2\pi/(4\kappa_1)$. So we surmise that the dissimilar waveform at the same time may arise from the frequency of the oscillograms. For the given mode, the frequency of the electromagnetic oscillation is coordinate independent. It is a function of β. In case of the incipient waveform is similar, the phase position of the electromagnetic oscillation can be described as $\phi\sim\varpi_\beta\tau$. The phase position can be used to describe the waveform of electromagnetic oscillation. At a given τ, $\phi_1\sim\varpi_{\beta_1}\tau$ is very different from $\phi_2\sim\varpi_{\beta_2}\tau$ except $|\phi_1-\phi_2|=2k\pi,(k=0,1,2,\ldots)$ which is hard to reach. At different τ, $|\phi_1-\phi_2|=2k\pi$ can be achieved more easily. The simplest way is $\tau_1/\tau_2=|\varpi_2/\varpi_1|$. Figure 3 shows oscillograms of the fields in the $n=1$ mode at $\rho=0.7$, as functions of time. The analysis of the frequency of wave propagation in inhomogeneous but linear media are still effective for weak nonlinear media. That is $\varpi_\beta=|\gamma|\varpi_0$. It shows that our foregoing surmise is right. Similar results will be obtained from higher modes.

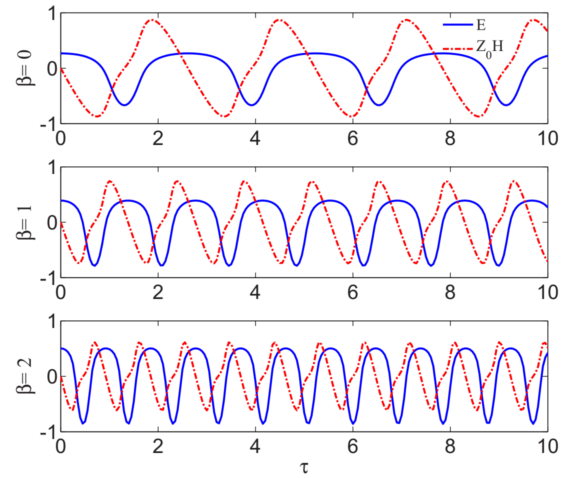

FIG. 3. (Color online) Oscillograms of the fields in the $n=1$ mode at $\rho=0.7$, as functions of time and at various β.

Figure 4 shows results of numerical calculations of normalized field E and H by formulas (12) in $n=2$ mode with $\kappa_2\approx 5.5201$. The waveform of $\beta=1$, $\tau=3\pi/(4\kappa_2)$ is very similar to the one of $\beta=2$, $\tau=2\pi/(4\kappa_2)$, and the waveform of $\beta=1$, $\tau=4\pi/(4\kappa_2)$ is very similar to the one of $\beta=2$, $\tau=3\pi/(4\kappa_1)$. It implies that the waveform of electromagnetic oscillation also can be described by a phase position and foregoing discussion remain effective. Nonlinear and inhomogeneous effects become more pronounced with increasing n and depend significantly on ρ, τ, and β, which is in accordance with the result in Ref. [1]. Figure 5 shows

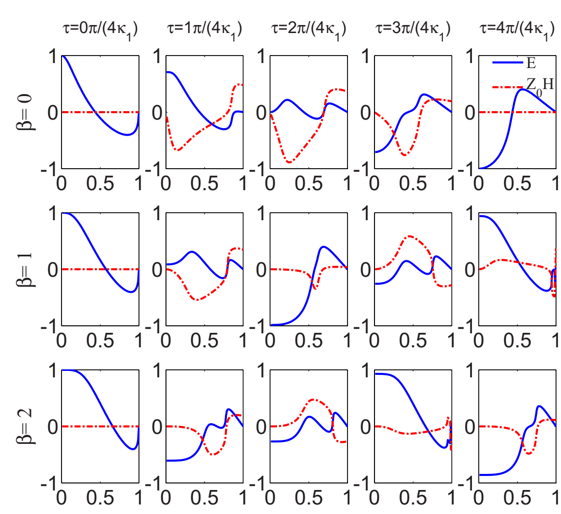

FIG. 4. (Color online) Wave propagation in a inhomogeneous and nonlinear media under the boundary value problem $\mathcal{E}(1,\tau)=0$, electric and magnetic fields as functions of ρ in the $n=2$ mode.

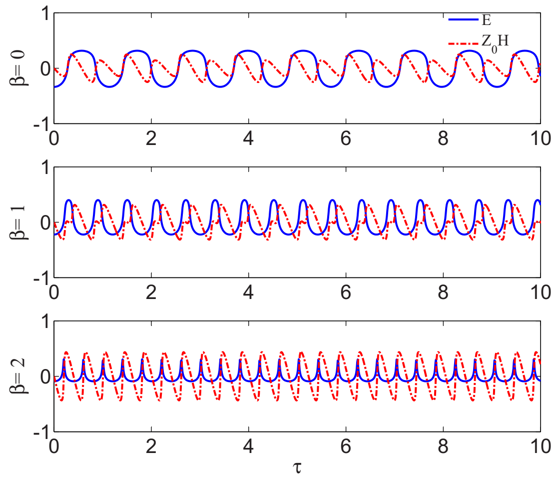

FIG. 5. (Color online) Oscillograms of the fields in the $n=2$ mode at $\rho=0.7$, as functions of time and at various β.

oscillograms of the fields in the $n=2$ mode at $\rho=0.7$, as functions of time. If $\beta=0$, the field E varies at the frequency ϖ_0 while the field H at the second harmonic $2\varpi_0$. However, for $\beta=2$, the second harmonic is disappeared, and the fields E and H are all have the same frequency ϖ_0.

Propagation of cylindrical and spherical waves in inhomogeneous media is of great interest for theory and application, however, remain poorly studied [1,14]. It can be used in fiber optics, geophysical prospecting, radio propagation, study of radar cross section, and microstrip antenna, as Ref. [14] have suggested. Our work is an interesting extension of the recent publication Ref. [1], which put forward an important technique to describe the behavior of nonlinear cylindrical electromagnetic waves in a bounded volume. Electromagnetic wave propagation in a medium with inhomogeneous and nonlinear simultaneously, as an extremely complicated problem, remains poorly studied, especially exact solutions. Integrable systems have a significant impact on both theory and phenomenology [20]. Exact solutions play an important role in understanding the physical processes and it is very important for the development of new computational asymptotic methods. Our work shows that the important technique suggested by Petrov and Kudrin can be extended to deal with problems of cylindrical electromagnetic waves propagation in an inhomogeneous nonlinear and nondispersive medium. The rigorous solution we obtained is a rigorous solution which describes electromagnetic wave propagation in a medium with inhomogeneous and nonlinear simultaneously.

In conclusion, we have used the proposed method in Ref. [1] to deal with wave propagation in an inhomogeneous nonlinear and nondispersive medium. An exact axisymmetric solution has been obtained, starting from the corresponding solution of linear field equations. We have analyzed the initial value problem and boundary value problem and found that the amplitude and frequency of the electromagnetic wave can be controlled with different β. Our result has significant advantages over the direct numerical solution of that system and provides a very convenient basis for analysis of inhomogeneous and nonlinear resonators, e.g., inhomogeneous ferroelectric resonators.

The work was supported in part by the National Science Foundation (NSF) of China (Grant Nos. 10874050, 10975054, 91021011, 11005057, and 10634060), the National Fundamental Research Program of China (Grant No. 2005CB724508), and the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 200804870051). H.X., L.-G.S., and P.H. are also supported in part by the Ministry of Education of China. The authors acknowledge Professor Ying Wu for his enlightening suggestions and anonymous referees for their insightful comments.

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