δ-expansion method for nonlinear stochastic differential equations describing wave propagation in a random medium

Robert A. Van Gorder*

Department of Mathematics, University of Central Florida, P.O. Box 161364, Orlando, Florida 32816-1364, USA

(Received 18 June 2010; revised manuscript received 20 September 2010; published 19 November 2010)

We apply the δ-expansion method to nonlinear stochastic differential equations describing wave propagation in a random medium. In particular, we focus our attention on a model describing a nonlinear wave propagating in a turbulent atmosphere which has random variations in the refractive index (we take these variations to be stochastic). The method allows us to construct much more reasonable perturbation solutions with relatively few terms (compared to standard "small-parameter" perturbation methods) due to more accurate linearization used in constructing the initial approximation. We demonstrate that the method allows one to compute effective wave numbers more precisely than other methods applied to the problem in the literature. The method also picks up on the stochastic damping of the solutions quickly, holding all of the relevant data in the initial term. These properties allow for both a qualitative and a quantitative construction of physically meaningful solutions. In particular, we show that the method allows one to retain higher-order harmonics which are hard to capture with standard perturbation methods based on small parameters.

DOI: 10.1103/PhysRevE.82.056712 PACS number(s): 02.60.−x, 02.30.Mv

I. INTRODUCTION

When studying differential equations which are both (i) nonlinear and (ii) stochastic, obtaining closed form exact solutions is typically difficult if not impossible. Furthermore, specific numerical methods may work only in some cases and even then perhaps only for certain parameter regimes. This leads one to consider methods of obtaining analytical solutions via methods such as perturbation methods. However, even for deterministic equations, many perturbation solutions yield solutions valid only over restricted domain or for small values of some model parameter. Even then, one may be faced with solving an infinite number of linear *stochastic* differential equations so as to construct the perturbation solution.

In the late 1980s, Bender and colleagues introduced a type of perturbation technique, the δ-expansion method (see, for instance, [1–17]), in which one expands in powers of a nonlinearity present in a nonlinear differential equation. At first applied to problems in quantum field theory, the method found plenty of application to nonlinear differential equations in many areas of science (see, for example, [7] and the references therein). Such a method allows us to preserve more of the original nonlinear equation, which in turn allows for our perturbation solutions to converge more rapidly to the true solution (which, in the absence of exact solutions, we take to be numerical solutions).

In the present paper, we shall apply this method in order to study the propagation of nonlinear waves in a random medium; in particular, we focus our attention on a model describing a nonlinear wave propagating in a turbulent atmosphere which has random variations in the refractive index. Such variations are taken to be stochastic. We obtain results for the one-dimensional case for both the mean electric field and the general stochastic problem. The corresponding three-dimensional problem is also discussed, and the general method of linearization via the δ-expansion method is considered. We show that there are certain pros and cons involved in employing the method, and one must weigh these when choosing an appropriate perturbation method to use. In order to effectively understand the benefits of the method, we compare our results with the available "small-parameter" results present in the literature. In one case, we offer a fix which involves deformation of the original equation into a simpler, yet still nonlinear, equation. This modification allows us to capture behaviors due to the higher-order harmonics, which may be overlooked in the small-parameter perturbation solutions. Here, the zeroth-order term in the perturbation expansion is governed by a (simpler) nonlinear equation, which makes computation of the higher order iterates in the perturbation expansion a bit more complicated.

In the model describing a nonlinear wave propagating in a turbulent atmosphere which has random variations in the refractive index, both nonlinear and stochastic contributions are taken to be (relatively) small due the addition of a small parameter ϵ. One benefit to considering a perturbation method which does not rely on small model parameters, such as the δ-expansion method, is that we are free to consider (relatively) large values for model parameters. However, in doing so, one must be mindful of the physical behavior of obtained solutions under such parameter regimes. As such, we show that for sufficiently large values of some of the model parameters, solutions become nonphysical (they blow up as opposed to decaying). This does not mean that such model parameters are nonphysical, but, rather, that the model we employ is not valid for such parameter values.

Via a comparison of the δ-expansion method solutions to perturbation solutions present in the literature, we show that the δ-expansion method allows one to compute effective wave numbers more precisely than the other methods applied to the problem in the literature. Additionally, the method also picks up on the stochastic damping of the solutions quickly, holding all of the relevant data in the initial term. These

*rav@knights.ucf.edu

properties allow for both a qualitative and a quantitative construction of physically meaningful solutions. In particular, we show that the method allows one to retain higher-order harmonics which are hard to capture with standard perturbation methods based on small parameters.

II. PROPAGATION OF NONLINEAR WAVES IN A RANDOM MEDIUM

Propagation of waves in a turbulent medium includes a number of applications, such as propagation of starlight through the atmosphere, propagation of radio waves through the ionosphere, and sound wave propagation in the ocean [18,19]. Although variations in the refractive index from its mean value in a turbulent medium are very small, the wave typically propagates through a large number of refractive index inhomogeneities; thus, the cumulative effect can be very significant. Wave propagation in a random medium is usually described by stochastic differential equations and the characteristics of the medium are typically represented by stochastic coefficients (see [20–22]). A random medium is a family of media, each labeled by one value of parameter α which ranges over a space $\mathcal{A}$ in which a probability density $p(\alpha)$ determines the probability of a given value of α and therefore represents the source of the waves (which, in some cases, may be random).

Shivamoggi *et al.* [29] considered a mathematical approach to the problem of wave propagation in a random medium using a perturbation method to treat nonlinear stochastic differential equations modeling the electric field of a monochromatic nonlinear wave [see Eq. (1) below]. They noted that, for the linear problem, their procedure also yields a treatment that is mathematically rigorous while giving physically correct results.

When a propagating wave is strong enough, such as in the case of a high-power laser, it can significantly change the properties of the medium. For example, the heating of the medium by the wave causes changes in the refractive index of the medium and hence alters the propagation characteristics (see [23–25]). The wave propagation under such conditions becomes nonlinear. As in [29], we consider the electric field of such a monochromatic nonlinear wave which satisfies the restrictive one-dimensional equation,

$$\frac{d^2E}{dz^2} + k_0^2[1+\epsilon\mu(z)]E + \epsilon a E^2 + \epsilon^2 b E^3 = 0, \tag{1}$$

where k_0 is the wave number of the wave and the refractive index of the medium n is related to the parameters μ, a, and b (which need not be positive) by

$$n^2 = 1 + \epsilon\mu(z) + \epsilon\frac{a}{k_0^2}E + \epsilon^2\frac{b}{k_0^2}E^2. \tag{2}$$

The depolarization effects in the medium are ignored. Equation (1) describes a nonlinear wave propagating in a turbulent atmosphere which has small random variations in the refractive index. Here, the random variations are due to $\mu(z)$, which is taken to be a stochastic function.

The three-dimensional counterpart to Eq. (1) is given by [29] as

$$\nabla^2 E + k_0^2[1+\mu(x,y,z)]E + \beta|E|^2 E = 0, \tag{3}$$

and this equation was analyzed in several simplified forms by Shivamoggi *et al.* [29].

We first obtain perturbation solutions to Eq. (1) by use of the δ-expansion method. We then construct the relevant solution for the mean value of E, $\langle E\rangle$. We construct both δ-expansion solutions and standard perturbation solutions in this special case. Note that we keep the parameter ϵ present in the formulation. We shall compute perturbation solutions about ϵ for small values of ϵ and show that these solutions break down for large values of ϵ. Meanwhile, we show that upon applying the δ-expansion method, one obtains perturbation solutions valid for arbitrary values of $\epsilon>0$. While we compare that the two results in the mean value case as the resulting equations are deterministic, we expect that the δ-expansion approximations will perform better than standard perturbation in all cases for sufficiently small noise terms μ. In addition to considering the one-dimensional problems, we provide a method to solve the fully three-dimensional problem. The equations are, in general, too hard to solve analytically, so the δ-expansion method can be used to linearize the nonlinear equations, and the resulting system of linear equations may be solved successively up to a desired order via a numerical method.

Perturbation solution about ϵ for the one-dimensional problem: The mean value case

In [29], the mean electric field $\langle E\rangle$ was shown to satisfy the nonlinear ordinary differential equation,

$$\frac{d^2}{dz^2}\langle E\rangle + \epsilon^2\frac{k_0 c_2}{2}\frac{d}{dz}\langle E\rangle + k_0^2\left(1-\epsilon^2\frac{c_1}{2}\right)\langle E\rangle + \epsilon a\langle E\rangle^2 + \epsilon^2 b\langle E\rangle^3 = 0, \tag{4}$$

where c_1 and c_2 are constants determined by the probability distribution associated to μ. In general,

$$c_1 = \int_0^\infty \langle\mu(\xi/k_0)\mu[(\xi-\eta)/k_0]\rangle\sin(2\eta)d\eta \tag{5}$$

and

$$c_2 = \int_0^\infty \langle\mu(\xi/k_0)\mu[(\xi-\eta)/k_0]\rangle[1-\cos(2\eta)]d\eta. \tag{6}$$

It is worth mentioning that in their derivation, Shivamoggi *et al.* [29] treated ϵ as a formal parameter at this juncture and only later relied on it being "small;" this is important, as it allows for the extension of Eq. (4) for larger values of ϵ. However, we later show that for certain model parameters, this approximation breaks down. As mentioned in [29], when the wave propagation in a random medium is stationary, Gaussian, and a Markov process, then it is a Uhlenbeck-Ornstein process so that

$$\langle\mu(\xi/k_0)\mu[(\xi-\eta)/k_0]\rangle = e^{-\eta/k_0}, \tag{7}$$

and, hence,

$$c_1 = \frac{2k_0^2}{1+4k_0^2}, \quad c_2 = \frac{4k_0^2}{1+4k_0^2}. \tag{8}$$

Meanwhile, if we assume the commonly used Gaussian distribution,

$$\langle \mu(\xi/k_0)\mu[(\xi-\eta)/k_0]\rangle = e^{-\eta^2/2k_0^2}, \tag{9}$$

we obtain

$$c_1 = 0, \quad c_2 = \sqrt{\frac{\pi}{2}}k_0(1-\sqrt{2}k_0 e^{-2k_0^2}). \tag{10}$$

In order to obtain a perturbation solution about ϵ, one would consider a solution of the form

$$\langle E\rangle = E_0(z) + E_1(z)\epsilon + E_2(z)\epsilon^2 + \cdots. \tag{11}$$

Unfortunately, such a solution yields a complicated expression which is slow to converge. The higher-order corrections will be yet more complicated. Note that, due to secular terms, there are factors of $z\sin(k_0 z)$ and $z\cos(k_0 z)$ present in the second-order approximation, $F_2(z)$. So, for large z, solutions will blow up. It is clear to see that the leading trigonometric terms in the nth approximant $F_n(z)$ will be $\sin[(n+1)k_0 z]$ and $\cos[(n+1)k_0 z]$, so we generate an expansion in those modes in addition to secular contributions. Hence, our perturbation solution suffers from the fact that k_0 is a crude approximation of the true wave number k_{eff} due to the nonlinearity, valid only for very small ϵ.

For the above reasons, in [29], a solution of the form

$$\langle E\rangle = \tfrac{1}{2}(e^{ik_{\text{eff}}z}\mathcal{E} + e^{-ik_{\text{eff}}z}\mathcal{E}^*), \tag{12}$$

is assumed, where * denotes complex conjugation; we find that Eq. (4) gives

$$k_{\text{eff}}^2 = k_0^2\left(1-\frac{c_1}{2}\epsilon\right) + \frac{3b}{4}\epsilon^2|\mathcal{E}|^2 + i\frac{k_0^2 c_2}{2}\epsilon^2 + O(\epsilon^3). \tag{13}$$

In ansatz (12), Shivamoggi *et al.* [29] constructed the solution

$$\langle E\rangle = \frac{\mathcal{E}}{2}\exp\left[ik_0\left\{1+\frac{1}{4}\left(-c_1+\frac{3b}{2k_0^2}\right)\epsilon^2|\mathcal{E}|^2\right\}z\right] \times\exp\left(-\frac{k_0 c_2}{4}\epsilon^2 z\right) + \text{c.c.}, \tag{14}$$

where c.c. denotes the complex conjugate of the preceding term. Note the absence of the parameter a—its influence is hidden inside of $\mathcal{E}$. Still, for small a and small ϵ, such an expansion is valid and, indeed, is shown to be useful, in a qualitative sense, in [29]. However, in cases where ϵ is no longer small, such a solution tends to break down. Furthermore, due to the implicit appearance of the amplitude $\mathcal{E}$ in the expression, the usefulness of expression (14) for actually constructing solutions numerically is limited. Later, when we compare this result to both solutions obtained via δ-expansion and numerical simulations, we shall set $\mathcal{E}=1$, as we observe from the numerics that the amplitude for the mean value of the field is bounded in magnitude like

$$|\langle E\rangle| \le \exp\left(-\frac{k_0 c_2}{4}\epsilon^2 z\right), \tag{15}$$

which gives the strength of the damping due to the stochastic term. Note that the parameter c_2 influences such damping, while the parameter c_1 enters only into the oscillatory part of the solution. Thus, the parameter c_1 shall influence the effective wave number and thus the higher-order harmonics.

III. PERTURBATION SOLUTION FOR THE ONE-DIMENSIONAL PROBLEM VIA δ EXPANSION

In this section, we shall apply the δ-expansion method in order to obtain perturbation solutions to the models describing a nonlinear wave propagating in a turbulent atmosphere which has stochastic variations in the refractive index. In particular, we first consider the one-dimensional model in Eq. (1), and we obtain the general perturbation solution which is shown to split into a deterministic part (which holds the contribution due to the nonlinearity) and a stochastic part. In order to better compare our results to those in the literature (in particular, Shivamoggi *et al.* [29], who employed the most advanced treatment to date), we restrict to the mean value case and obtain the δ-expansion solutions which are in analogy to the small-ϵ solutions given by Eq. (14); these solutions are later compared for set values of the model parameters. Finally, we discuss the general solution method for the fully three-dimensional model. For brevity, we omit many of the computational details and derivations. For details of the method, we refer the reader to the Appendix; many of the results provided here fall out of the general derivations given in the Appendix.

To apply the δ-expansion method to one-dimensional model (1), let us consider the related nonlinear ordinary differential equation,

$$\frac{d^2E}{dz^2} + k_0^2[1+\delta\epsilon\mu(z)]E + \epsilon a E^{1+\delta} + \epsilon^2 b|E|^{2\delta}E = 0. \tag{16}$$

When $\delta=1$, this equation becomes Eq. (1), while when $\delta=0$, the equation becomes linear and deterministic (nonstochastic). Let us consider a solution expansion of the form

$$E(z) = E_0(z) + E_1(z)\delta + E_2(z)\delta^2 + \cdots. \tag{17}$$

Defining the linear operator L by

$$L[U] = \frac{d^2U}{dz^2} + (k_0^2 + \epsilon a + \epsilon^2 b)U, \tag{18}$$

we find that $E_0(z)$ is determined by $L[E_0]=0$, and hence, thus,

$$E_0(z) = A_1\sin(Kz) + A_2\cos(Kz), \tag{19}$$

where

$$K \equiv \sqrt{k_0^2 + \epsilon a + \epsilon^2 b} \tag{20}$$

and A_1 and A_2 are constants of integration. Again, notice that $E_0(z)$ is completely deterministic, as we have pushed all stochastic contributions due to μ into higher-order terms. Furthermore, observe that K depends on both b (as does the corresponding result of Shivamoggi *et al.* [29]) and a—hence, in our solution both the cubic and quadratic terms in Eq. (1) are taken into account in the very first approximation. Note that the *standard* small-ϵ method (which Shivamoggi *et al.* [29] improved on) would actually omit both parameters and would have taken $K=k_0$ quite incorrect for $\epsilon\neq 0$.

In general, for $k\geq 1$, the higher-order terms are governed by linear inhomogeneous ordinary differential equations of the form

$$L[E_k] = -\mathcal{B}_k[E_0(z),E_1(z),\ldots,E_{k-1}(z)] - \epsilon k_0^2\mu(z)E_{k-1}(z) = -\mathcal{B}_k^*(z) - \epsilon k_0^2\mu(z)E_{k-1}(z). \tag{21}$$

Assuming that E_0 holds all relevant information at $z=0$, we may take $E_k(0)=0$ and $(dE_k/dz)(0)=0$. Then, inverting L, we find that

$$E_k(z) = \frac{1}{K}\int_0^z \sin[K(\tau-z)]\mathcal{B}_k^*(\tau)d\tau + \frac{\epsilon k_0^2}{K}\int_0^z \sin[K(\tau-z)]\mu(\tau)E_{k-1}(\tau)d\tau. \tag{22}$$

Note that both integrals are, in general, stochastic: the former integral due to the dependence on the stochastic approximants $E_1,E_2,\ldots,E_{k-1}$ and the latter integral explicitly so. We then have the solution

$$E(z) = A_1\sin(Kz) + A_2\cos(Kz) + \frac{1}{K}\sum_{k=1}^{\infty}\delta^k\int_0^z \sin[K(\tau-z)]\mathcal{B}_k^*(\tau)d\tau + \frac{\epsilon k_0^2}{K}\sum_{k=1}^{\infty}\delta^k\int_0^z \sin[K(\tau-z)]\mu(\tau)E_{k-1}(\tau)d\tau. \tag{23}$$

Observe that we may always partition the terms in $E(z)$ as

$$E(z) = E_{\mathrm{det}}(z) + E_{\mathrm{stoch}}(z), \tag{24}$$

where $E_{\mathrm{det}}(z)$ is the deterministic contribution and $E_{\mathrm{stoch}}(z)$ is the stochastic contribution. Now, to order δ^2, we find that [keeping all expressions in terms of the nonstochastic $E_0(z)$ for sake of simplicity]

$$E_{\mathrm{det}}(z) = E_0(z) + \frac{\delta}{K}\int_0^z \sin[K(\tau-z)][a\epsilon + b\epsilon^2E_0(\tau)] \times\ln[|E_0(\tau)|]d\tau + \frac{2\delta^2}{K}\int_0^z \sin[K(\tau-z)][a\epsilon + b\epsilon^2E_0(\tau)]\ln[|E_0(\tau)|]d\tau + \frac{2\delta^2}{K^2}\int_0^z \sin[K(\tau-z)] \times\left\{b\epsilon^2[1+|E_0(\tau)|] + \frac{a\epsilon}{E_0(\tau)}\right\}\int_0^\tau \sin[K(\sigma-\tau)] \times[a\epsilon + b\epsilon^2E_0(\sigma)]\ln[|E_0(\sigma)|]d\sigma d\tau \tag{25}$$

and

$$E_{\mathrm{stoch}}(z) = \frac{\epsilon k_0^2}{K}\delta\int_0^z \sin[K(\tau-z)]E_0(\tau)\mu(\tau)d\tau + \frac{2\epsilon k_0^2\delta^2}{K^2}\int_0^z \sin[K(\tau-z)]\left\{b\epsilon^2[1+|E_0(\tau)|] + \frac{a\epsilon}{E_0(\tau)}\right\}\int_0^\tau \sin[K(\sigma-\tau)]E_0(\sigma)\mu(\sigma)d\sigma d\tau + \frac{\epsilon k_0^2}{K^2}\delta^2\int_0^z \sin[K(\tau-z)]\mu(\tau)\int_0^\tau \sin[K(\sigma-\tau)] \times[a\epsilon + b\epsilon^2E_0(\sigma)]\ln|E_0(\sigma)|d\sigma d\tau + \frac{\epsilon k_0^2}{K^2}\delta^2\int_0^z \sin[K(\tau-z)]\mu(\tau) \times\int_0^\tau \sin[K(\sigma-\tau)]E_0(\sigma)\mu(\sigma)d\sigma d\tau. \tag{26}$$

Setting $\delta=1$, we recover the δ-expansion approximation to Eq. (1). Notice the appearance of natural logarithms in these expressions. These will serve to complicate the solution process when we attempt to obtain the iterates in the perturbation expansion about δ. The trade-off, we find, is that the iterates tend to converge more rapidly to the solution when we apply the δ-expansion method, compared to traditional perturbation about a small parameter.

A. Perturbation solution via δ expansion for the one-dimensional problem: The mean value case reconsidered

Let us now restrict the expression obtained in Eq. (23) to the mean value case [that is, apply the δ-expansion method to mean value (4)], so that we may more easily compare our solutions to those in the literature. Note that for us ϵ serves as a general model parameter and not a perturbation parameter, so we are free to take $\epsilon>0$ large. Solving the first of these linear equations for $F_0(z)$, we obtain

$$F_0(z) = A_1\sin(k_{\mathrm{eff}}z)\exp\left(-\frac{k_0c_2}{4}\epsilon^2z\right) + A_2\cos(k_{\mathrm{eff}}z) \times\exp\left(-\frac{k_0c_2}{4}\epsilon^2z\right), \tag{27}$$

where the effective wave number, k_{eff}, is given by

$$k_{\mathrm{eff}} = \sqrt{k_0^2 + \epsilon a - \left(\frac{k_0^2c_1}{2} - b\right)\epsilon^2 - \left(\frac{k_0c_2}{4}\right)^2\epsilon^4}. \tag{28}$$

There are a few points to note here. First of all, note that this zeroth-order term is of a similar (although more complicated) form to solution (14) obtained when we perturb the effective wave number, k_{eff}, and the additional perturbations are effectively in the amplitude. Furthermore, the exponential factor is *identical* in both the perturbation solution of Shivamoggi *et al.* [29] and the zeroth-order δ-expansion solution presented here. Thus, the very first approximate already recovers the important qualitative features of the perturbation solution of Shivamoggi *et al.* [29], namely, the damping due to the stochastic parameter. The addition of higher-order terms will then allow one to better account for the quadratic and cubic nonlinearities present in the problem.

B. Some comments on the perturbation method for the fully three-dimensional problem

In the literature, solutions for mutual coherence functions (MCFs) exist for a number of special cases [24,26,27]. However, when a nonlinear wave has traveled a long distance compared with the correlation range in the medium, it suffers a large number of independent random scatterings, each of which produces a slight random contribution and, hence, can be considered to be a Markov process, which suggests the use of a Fokker-Planck equation [19,28] to describe the wave propagation in a random medium. In [29], a general formulation for the MCF for Eq. (3) through a Fokker-Planck equation approach applied to the parabolic equation approximation was considered. The authors then proceeded to investigate the effects of the various nonlinear aspects of the wave propagation. The authors then considered a ponderomotive-force-driven model. In such a model, the ponderomotive force exerted by the electric field of the wave digs density cavities into the medium, which leads to local changes in the refractive index of the medium (which then alters the wave propagation characteristics).

Applying the δ-expansion method, we are able to linearize the nonlinear partial differential equation governing the electric field associated to the nonlinear wave. As we shall see, even the process of solving the linear problem is computationally involved. Thus, this serves as an instance where it is useful to use a perturbation method along with numerical methods in order to fully obtain the approximate solutions. The perturbation method allows one to linearize the nonlinear partial differential equation, while the numerical method would then allow one to successively solve the resulting linear boundary value problems, which take the form of the inhomogeneous Helmholtz problems.

To this end, consider the related nonlinear partial differential equation,

$$\nabla^2 E + k_0^2[1+\delta\mu(x,y,z)]E + \beta|E|^{2\delta}E = 0. \tag{29}$$

Clearly, when $\delta=1$ we recover the original Eq. (3), while when $\delta=0$ we have a linear equation. Thus, we assume a perturbation solution about $\delta=0$ or the form

$$E(x,y,z) = E_0(x,y,z) + E_1(x,y,z)\delta + E_2(x,y,z)\delta^2 + \cdots. \tag{30}$$

Provided solution (30) converges at $\delta=1$, then Eq. (30) is a solution to the original nonlinear Eq. (3). Defining the linear partial differential operator,

$$L_H[u] = \nabla^2 U + (k_0^2+\beta)U, \tag{31}$$

which is just the Helmholtz operator over $\mathbb{R}^3$, we find that

$$L_H[E_0] = 0, \tag{32}$$

$$L_H[E_1] = -2\beta\ln(|E_0|)E_0 - k_0^2\mu(z)E_0, \tag{33}$$

$$L_H[E_2] = -2\beta\ln(|E_0|)E_1 - 2E_1\beta - 2\beta[\ln(|E_0|)]^2E_0 - k_0^2\mu(z)E_1, \tag{34}$$

etc. We take the zero order term $E_0(x,y,z)$ to hold the relevant boundary or far field conditions and the remaining approximants $E_k(x,y,z)$ $(k\geq 1)$ to take into account the inhomogeneous contributions due to the nonlinearity and stochasticity of the original nonlinear problem. Note that $E_0(x,y,z)$ may be obtained via the Fourier analysis for reasonable boundary or far field conditions. In particular, one solves the homogeneous Helmholtz equation for the given boundary or far field data. However, computing the higher-order approximates, in which the inhomogeneous contributions involve $\ln|E_0|$, will be a nightmare. To make the process more tractable, we can consider a finite term approximation to $E_0(x,y,z)$ (in effect, we would truncate the Fourier series to include only so many terms that are needed to provide the desired accuracy). We must be careful here, as any errors obtained in approximating $E_0(x,y,z)$ will propagate through to $E_1(x,y,z)$ and so on. One would then proceed to solve for the higher-order approximants in expansion (30). Obviously, this would need to be done numerically, as the governing equations, while linear, are simply too complicated. For each approximant $E_k(x,y,z)$, one solves in the inhomogeneous Helmholtz equation for which the inhomogeneity is rather complicated [it will involve the natural logarithm of the approximate Fourier series solution to $E_0(x,y,z)$].

IV. DISCUSSION

Here we shall discuss certain physical properties of the solutions. We show that the δ-expansion method allows us to recover the salient features of the solution to some of the nonlinear stochastic models governing wave propagation in a random medium and is in strong agreement with numerical simulations. We first obtain perturbation solutions to the one-dimensional problem [Eq. (1)] by the use of the δ-expansion method and then construct the relevant solution for the mean value of the field E, $\langle E\rangle$. In particular, we consider both δ-expansion solutions and standard perturbation solutions in the mean-field case. We find that, upon applying the δ-expansion method, one obtains perturbation solutions valid for arbitrary values of $\epsilon>0$. In fact, even the order zero approximations are quite good for small values of z. While we compare that the two results in the mean value case as the resulting equations are deterministic, we expect that the δ-expansion approximations will perform better than standard perturbation in all cases for sufficiently small variations in the noise terms μ.

A. Damping due to stochastic contributions

The exponential factors present in the perturbation solutions, both those obtained here and those obtained via standard small-parameter methods, damp the solutions, and the strength of this damping is directly tied to the properties of the stochastic function $\mu(z)$. In the absence of the stochastic factor, $\mu(z)\equiv 0$, and thus the constants $c_1=c_2=0$. Hence, the

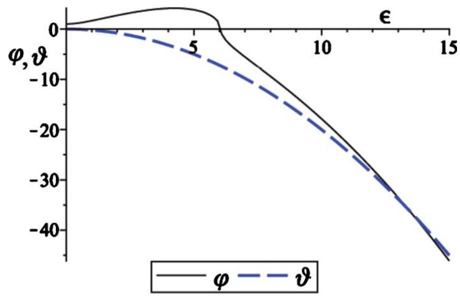

FIG. 1. (Color online) We consider the case when $k_0=1$, $c_1=2/5$, $c_2=4/5$, $a=0$, and $b=2$ and plot the signed magnitude $\varphi(\epsilon)=|k_{\text{eff}}(\epsilon)|\text{sgn}\{(k_{\text{eff}}(\epsilon))^2\}$ [which is positive if $k_{\text{eff}}(\epsilon)$ is real and negative if $k_{\text{eff}}(\epsilon)$ is complex]. We also plot the function $\vartheta(\epsilon)=-k_0c_2\epsilon^2/4$, which is the decay rate of the exponential factor. We see that $\epsilon_c\approx 6.0454$ and $\epsilon_E\approx 13.437$. For $0\leq\epsilon<\epsilon_c$ we have that k_{eff} is real valued, so the wave form solutions in terms of sines and cosines are maintained. For $\epsilon_c<\epsilon<\epsilon_E$, the solution will be in terms of a hyperbolic functions which are dominated by the decaying exponential [of decay strength $\vartheta(\epsilon)$]. However, for $\epsilon>\epsilon_E$, note that $k_{\text{eff}}(\epsilon)$ strictly dominates $\vartheta(\epsilon)$; hence, there is a possible blowup in the solutions.

zeroth-order approximant is not damped. Meanwhile, for general stochastic functions $\mu(z)$, c_2 cannot be taken to be zero, as we see in the examples in Sec. IV. Hence, in general, for $c_2\neq 0$, the wave undergoes stochastic damping. This includes the cases for which the wave propagation is an Uhlenbeck-Ornstein process or a Gaussian process, as discussed previously.

B. Region of validity for Eq. (1)

Note that we may view ϵ as a parameter measuring the strength of the combined stochastic and nonlinear contributions. For relatively large values of the parameter b, a measure of the strength of the cubic nonlinearity, compared to the parameters c_1 and c_2 due to the stochastic contribution, we maintain the wave form present in Eq. (27). If, however, the parameters c_1 and c_2 are sufficiently large, then there will exist $\epsilon_c>0$, a critical value of ϵ for which if $0\leq\epsilon\leq\epsilon_c$ then k_{eff} is real (corresponding to the standard physical solutions), if $\epsilon=\epsilon_c$ then k_{eff} is zero (corresponding to a completely damped mean field which decays to zero), and if $\epsilon_c\leq\epsilon\leq$ then k_{eff} is purely complex (nonphysical in the case where the mean field may have unbounded growth, i.e., if $|k_{\text{eff}}|$ exceeds $k_0c_2\epsilon^2/4$). Let us also denote by ϵ_E the ϵ value for which $|k_{\text{eff}}|=k_0c_2\epsilon^2/4$. To illustrate this, we plot $\text{sgn}(k_{\text{eff}})|k_{\text{eff}}|$ (the magnitude of k_{eff} with sign positive if k_{eff} is real and negative if k_{eff} is complex) in Fig. 1 for parametric values $k_0=1$, $c_1=2/5$, $c_2=4/5$, $a=0$, and $b=2$. We find that $\epsilon_c=\sqrt{18+2\sqrt{86}}\approx 6.0454$ and $\epsilon_E=\sqrt{90+10\sqrt{82}}\approx 13.437$. So, for this special case, when $0\leq\epsilon<\sqrt{18+2\sqrt{86}}$ we maintain the trigonometric wave form with stochastic decay. Meanwhile, for $\epsilon=\sqrt{18+2\sqrt{86}}$ there is strict exponential decay of the mean field. For $\sqrt{18+2\sqrt{86}}<\epsilon<\sqrt{90+10\sqrt{82}}$ the base form of the solutions will be hyperbolic functions, though they still decay due to the exponential factor. For $\epsilon=\sqrt{90+10\sqrt{82}}$, we find that the positive power factor in the hyperbolic factor exactly matches the exponential factor, so that the mean field decays to a fixed nonzero constant as $z\to+\infty$. Finally, when $\epsilon>\sqrt{90+10\sqrt{82}}$, the positive power in the hyperbolic function will dominate, meaning that the order zero solution can become unbounded.

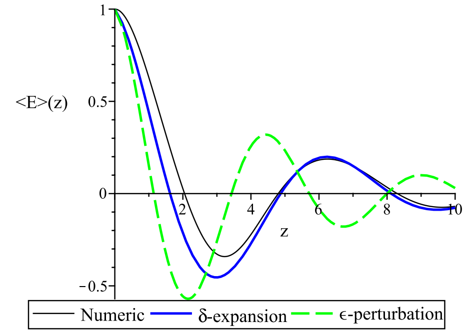

FIG. 2. (Color online) Plotted are the numerical solution (solid black line), the perturbation solution of the type obtained by Shivamoggi *et al.* [29] [Eq. (11)] (dashed green line), and the order zero δ-expansion solution [solid blue (gray) line] to the equation for the mean field [Eq. (4)] under the above conditions $\langle E\rangle(0)=1$ and $\frac{d}{dz}\langle E\rangle(0)=0$, $k_0=1$, $b=1$, $a=-1$, and $\epsilon=1$.

C. Comparison with the method of Shivamoggi *et al.*

For sake of demonstration, let us consider Gaussian case (9) [29]. Then,

$$k_{\text{eff}}=\sqrt{k_0^2+a\epsilon+b\epsilon^2-\frac{\pi k_0^4}{32}(1-\sqrt{2}k_0e^{-2k_0^2})^2\epsilon^4}. \quad (35)$$

Let us also require that $\langle E\rangle(0)=1$ and $\frac{d}{dz}\langle E\rangle(0)=0$ so that we may determine the values of the constants A_1 and A_2 in Eq. (27). In Fig. 2, we plot the numerical solution (via a Runge-Kutta method) to the equation for the mean field [Eq. (4)], along with both the regular perturbation solution and the solution obtained via the δ-expansion method, up to order zero, under the above conditions. Note that we restrict $k_0=1$, $b=1$, $a=0$, and $\epsilon=1$. Furthermore, we plot both the regular perturbation solution [of type (11)] and the solution obtained via the δ-expansion method up to order zero. Both perturbation methods yield reasonably good approximations, even at order zero, for small ϵ. However, as we can see from Fig. 2, the solution obtained by regular perturbation methods breaks down for larger values of ϵ, while the solution obtained by the δ-expansion method still captures the qualitative features of the numerical solution even at the order zero approximation.

D. Modification of the method for large $a>0$

Note that both the small-ϵ type perturbation and the δ-expansion method solutions outlined above become less accurate as we consider larger values of $a>0$, which contribute to the higher-order harmonics. Thus, for $a>0$, we

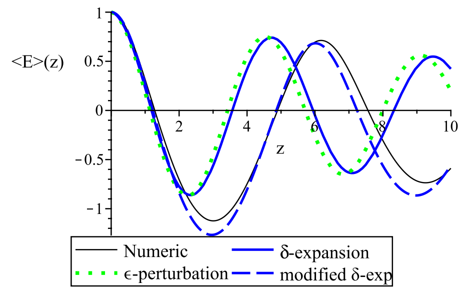

FIG. 3. (Color online) Plotted are the numerical solution (solid black line), the perturbation solution of the type obtained in Shivamoggi *et al.* [29] [Eq. (11)] (dotted green line), the order zero δ-expansion solution [solid blue (gray) line], and the *modified* δ-expansion method solution of order zero (dashed blue line) to the equation for the mean field [Eq. (4)]. The modified δ-expansion solution is obtained from solving Eq. (37), and this method results in a better fit to the numerical solution at order zero. Here we have set $\langle E\rangle(0)=1$ and $\frac{d}{dz}\langle E\rangle(0)=0$, $k_0=1$, $b=1$, $a=1$, and $\epsilon=0.5$.

need to compute further terms in the δ-expansion solution in order to capture the qualitative features of the true solution. Observe that we can obtain a more accurate order zero approximation using the δ-expansion method if we are willing to complicate the auxiliary operator L_2 a bit more. To this end, consider the related problem

$$\frac{d^2}{dz^2}\langle E\rangle+\epsilon^2\frac{k_0c_2}{2}\frac{d}{dz}\langle E\rangle+k_0^2\left(1-\epsilon^2\frac{c_1}{2}\right)\langle E\rangle + \epsilon a\,\mathrm{sgn}(\langle E\rangle)\langle E\rangle^{2\delta}+\epsilon^2 b|\langle E\rangle|^{2\delta}\langle E\rangle=0 \qquad (36)$$

and again assume an expansion in δ. Here, $\mathrm{sgn}(x)=1$ if $x>0$, $\mathrm{sgn}(x)=-1$ if $x<0$, and $\mathrm{sgn}(x)=0$ if $x=0$. Then, E_0 is governed by $L_3[E_0]=0$, where

$$L_3[U]=\frac{d^2U}{dz^2}+\epsilon^2\left(\frac{k_0c_2}{2}\right)\frac{dU}{dz} + \left(k_0^2\left\{1-\frac{c_1}{2}\epsilon^2\right\}+\epsilon a\,\mathrm{sgn}(U)+\epsilon^2 b\right)U \qquad (37)$$

is now an auxiliary *nonlinear* differential operator due to the $\mathrm{sgn}(U)$ term. Solving for E_0, we compare this approximation method with both the previous δ-expansion solution (discussed in Sec. IV C) and the regular perturbation solution [of type (11)] in Fig. 3. We see that by employing a nonlinear auxiliary operator, we are better able to approximate the behavior of the numerical solutions in the large-ϵ regime *and* large $a>0$ regime. Thus, in attempting to linearize a nonlinear differential equation, we may still end up throwing out too much information. Combining the δ-expansion method with numerical methods, we are able to obtain very good solutions using reasonable nonlinear operators such as L_3. While we were able to solve the nonlinear equation [Eq. (4)] numerically (in a direct manner via a Runge-Kutta method), note that this is not always the case. So, in particularly challenging nonlinear equations, one may attempt to employ both the δ-expansion method and numerical methods to obtain solutions which might not be possible if only direct numerical methods are available.

V. CONCLUSIONS

Our results show the attenuation of the coherent wave solution due to the stochastic inhomogeneities in the medium. Indeed, as the strength of the stochastic term increases in magnitude (due to the parameter ϵ), the effect is amplified. Fixing $\epsilon=1$, in Fig. 2 we plot the numerical solution (via a Runge-Kutta method) to the equation for the mean field [Eq. (4)], along with both the small-ϵ perturbation solution and the solution obtained via the δ-expansion method, up to order zero. The solution obtained by the small-ϵ perturbation method of Shivamoggi *et al.* [29] breaks down for this larger value of ϵ, while the solution obtained by the δ-expansion method still captures the qualitative features of the numerical solution, even at the order zero approximation, for sufficiently small z. However, for larger z, the agreement gradually breaks down. That said, the δ-expansion solution properly captured the decay rate of the oscillating solutions, which the regular perturbation expansion in ϵ misses at order zero. The higher-order corrections for the δ-expansion solution will serve to better account for the change in the period of oscillation due to the nonlinearity. Furthermore, by employing a nonlinear auxiliary operator, we are better able to approximate the behavior of the numerical solutions, in the large-ϵ regime, even at the order zero term. Combining the δ-expansion method with numerical methods, we are able to obtain very good solutions via the modified δ-expansion approach (see Fig. 3).

Regarding the specific example considered in the mean value case, note also that our method shows that both the cubic and quadratic terms should influence the effective wave number, along the lines of Eq. (28). This is in contrast to the results of Shivamoggi *et al.* [29] [e.g., the result listed in Eq. (14)] as their perturbation results include only the cubic nonlinearity contribution (due to the structure of their perturbation solutions). As we show in the Gaussian example, the quadratic contribution should be taken in to account if one is to more properly deduce the behavior of the wave solution to the model for ϵ much larger than zero. We thus view the δ expansion results presented here as an extension to those of Shivamoggi *et al.* [29], which themselves were an improvement over standard perturbation results. The construction of higher-order harmonics from our results will provide more realistic solutions to both the mean value [Eq. (4)] and general one-dimensional [Eq. (1)] models.

As we have shown, the δ-expansion method is a promising tool for the perturbative study of differential equations which are both nonlinear and stochastic. The primary benefit to the method is that it involves a linearization that more closely approximates the original nonlinear equation. The primary drawback is that the method often involves logarithms of the order zero approximation, rendering the method difficult to apply for successively higher terms in many cases and inapplicable in other cases where the order zero approximation is not properly behaved. However, in cases where the iterates converge rapidly (so that relatively few higher-order terms are needed in the construction of an approximate solution), the method proves useful. Furthermore, when dealing with stochastic differential equations, the method allows one to shift the stochastic contribution into higher-order terms, allowing one to construct a deterministic order zero approximation. In the case of linear stochastic contributions, the first-order correction separates into a term due to the nonlinearity and a term due to the stochasticity.

In addition to the ability to construct approximate solutions for a nonlinear differential equation, perturbation methods allow us a "check" to numerical solutions over the parameter domains for which the respective solutions both exist. Even more than that, the method may be coupled with numerical methods, so as to offer an analytic-numerical method. One would first linearize the original nonlinear equation via the perturbation method. Then, one would solve the resulting linear inhomogeneous differential equations (for as many iterates as are required to obtain a desired accuracy). Such a method would be particularly effective here: in employing the δ-expansion method, we observe that the inhomogeneities resulting from the manner in which we construct our expansions is, in many cases, more complicated in structure than in the standard perturbation methods that employ some small model parameter. Particularly, in the case of differential equations with power-law nonlinearity, logarithms of the order zero iterate will appear in the inhomogeneities, rendering direct solution tedious if not impossible. That said, the δ-expansion method often gives fairly accurate solutions in relatively few iterations (compared to small-parameter perturbation methods). Pairing the δ expansion with numerical methods appropriate for the linearized equations allows one to construct an analytic-numerical solution in cases where no available numerical methods are useful for the original nonlinear problem.

Note also that, while we were able to solve the nonlinear equation [Eq. (4)] numerically (in a direct manner, via a Runge-Kutta method), there are many strongly nonlinear equations which will not yield to standard numerical methods. Thus, the δ-expansion method may be combined with existing numerical routines, making for an effective tool with which to study nonlinear phenomenon.

ACKNOWLEDGMENTS

The author appreciates the comments of the reviewers, which have led to definite improvement in the paper.

APPENDIX: δ expansion applied to stochastic operators

For sake of demonstration, consider the nonlinear differential equation

$$L[u] + N[u] + M(\alpha)[u] = g(x), \tag{A1}$$

where L is a linear operator, N is a nonlinear operator, $M(\alpha)$ is a stochastic operator with stochastic parameter(s) $\alpha \in \mathcal{A} \subset \mathbb{R} (\alpha \in \mathcal{A} \subset \mathbb{R}^n)$, $x \in \mathcal{D}$ (the problem domain), and $u : \mathcal{D} \to \mathbb{R}$. Introduce the related differential equation and operator $\tilde{N}[u;\delta]$ such that

$$L[u] + \tilde{N}[u;\delta] + \delta M(\alpha)[u] = g(x), \tag{A2}$$

where $\tilde{N}[u;1] = N[u]$ while $\tilde{N}[u;0] = L_1[u]$. Consider a perturbation solution

$$\tilde{u} = u_0 + u_1\delta + u_2\delta^2 + \cdots = u_0 + \sum_{k=1}^{\infty} u_k \delta^k. \tag{A3}$$

Observe that u_0 is governed by

$$\tilde{L}[u_0] = L[u_0] + L_1[u_0] = g(x) \tag{A4}$$

and thus u_0 is deterministic (nonstochastic). The higher-order corrections are determined by solutions to

$$\tilde{L}[u_k] = -\mathcal{F}_{k-1}(u_0, u_1, \ldots, u_{k-1}) - M(\alpha)[u_{k-1}] \tag{A5}$$

for $k \geq 1$, where the $\mathcal{F}_k$'s appear as coefficients in the expansion

$$N[\tilde{u};\delta] = \sum_{k=0}^{\infty} \mathcal{F}_k(u_0, u_1, \ldots, u_k)\delta^k. \tag{A6}$$

Recursively, we find that

$$u_k = \mathcal{O}_k(x;\alpha)[u_0], \tag{A7}$$

where, in general, the $\mathcal{O}_k$'s are very complicated nonlinear functions of u_0, its derivatives, and its integrals. Then,

$$\tilde{u} = u_0 + \sum_{k=1}^{\infty} \mathcal{O}_k(x;\alpha)[u_0]\delta^k. \tag{A8}$$

In order to deduce the mean behavior of u (so as to average out the stochastic contribution, resulting in a deterministic solution) note that

$$\langle f \rangle = \int_{\mathcal{A}} f(\alpha)p(\alpha)d\alpha, \tag{A9}$$

where p is the probability density function for the stochastic parameter $\alpha \in \mathcal{A}$. Meanwhile, for an operator $\mathcal{J}(x;\alpha)[f]$, we similarly define

$$\langle \mathcal{J}(x;\alpha)[f] \rangle = \int_{\mathcal{A}} \mathcal{J}(x;\alpha)[f]p(\alpha)d\alpha. \tag{A10}$$

Then, we arrive at the expression

$$\langle \tilde{u} \rangle = \langle u_0 \rangle + \sum_{k=1}^{\infty} \langle \mathcal{O}_k(x;\alpha)[u_0] \rangle \delta^k = u_0 + \sum_{k=1}^{\infty} \mathcal{O}^*(x)[u_0]\delta^k \tag{A11}$$

for the mean value of the perturbation solution $\tilde{u}$, where

$$\mathcal{O}^*(x) = \langle \mathcal{O}_k(x;\alpha) \rangle. \tag{A12}$$

Thus, one benefit of the perturbation approach is that it allows one to shift the stochastic contribution into the higher-order terms and thus, in our notation, into the operators $\mathcal{O}_k$. Furthermore, we required no small parameters in the original problem, thanks to the introduction of the book keeping parameter δ.

Note that we may also account for nonlinear operators $M(\alpha)[u]$ more effectively if we are willing to take on additional complications in the auxiliary linear operator. What one would do is to consider the related differential equation,

$$L[u] + \widetilde{N}[u;\delta] + \widetilde{M}(\alpha)[u;\delta] = g(x), \tag{A13}$$

where $\widetilde{M}(\alpha)[u;1]=M(\alpha)[u]$ while $\widetilde{M}(\alpha)[u;0]=L_2[u]$ is a linear deterministic operator; the construction of a particular $\widetilde{M}(\alpha)[u;0]$ will, of course, depend on the specific application at hand. The linearized equation for the first term in the δ-expansion approximation would then be governed by

$$\widetilde{L}[u_0] = L[u_0] + L_1[u_0] + L_2[u_0] = g(x). \tag{A14}$$

Observe that the order zero term u_0 is still nonstochastic. Finally, in order to account for stochastic inhomogeneities, let us assume that $g(x)$ is stochastic. We introduce the related differential equation

$$L[u] + \widetilde{N}[u;\delta] + \widetilde{M}(\alpha)[u;\delta] = \delta g(x). \tag{A15}$$

Clearly, when $\delta=1$, we recover the original stochastic nonlinear differential equation, while, when $\delta=0$, we obtain a linear nonstochastic differential equation. We have effectively pushed the stochastic inhomogeneity into the higher-order terms.

[1] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. D **37**, 1472 (1988).

[2] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. Lett. **58**, 2615 (1987).

[3] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Lett. B **205**, 493 (1988).

[4] C. M. Bender and H. F. Jones, Phys. Rev. D **38**, 2526 (1988).

[5] C. M. Bender and K. A. Milton, Phys. Rev. D **38**, 1310 (1988).

[6] C. M. Bender, *Nonperturbative Perturbation Theory, Published in Hadron Structure '87*, Proceedings of the Conference Hadron Structure '87, edited by D. Krupa (Slovak Academy of Sciences, Bratislava, 1988), Vol. 14.

[7] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., J. Math. Phys. **30**, 1447 (1989).

[8] C. M. Bender, F. Cooper, and K. A. Milton, Phys. Rev. D **39**, 3684 (1989).

[9] C. M. Bender, F. Cooper, and K. A. Milton, Phys. Rev. D **40**, 1354 (1989).

[10] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., J. Math. Phys. **31**, 2722 (1990).

[11] C. M. Bender and A. Rebhan, Phys. Rev. D **41**, 3269 (1990).

[12] C. M. Bender, S. Boettcher, and K. A. Milton, J. Math. Phys. **32**, 3031 (1991).

[13] C. M. Bender, F. Cooper, G. Kilkup, P. Roy, and L. M. Simmons, Jr., J. Stat. Phys. **64**, 395 (1991).

[14] C. M. Bender, S. Boettcher, and K. A. Milton, Phys. Rev. D **45**, 639 (1992).

[15] C. M. Bender, F. Cooper, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. D **45**, 1248 (1992).

[16] C. M. Bender, K. A. Milton, and M. Moshe, Phys. Rev. D **45**, 1261 (1992).

[17] B. Abraham-Shrauner, C. M. Bender, and R. N. Zitter, J. Math. Phys. **33**, 1335 (1992).

[18] V. Tatarskii, *Wave Propagation in a Turbulent Medium* (Dover, New York, 1967).

[19] L. Chernov, *Wave Propagation in a Random Medium* (Dover, New York, 1969).

[20] R. C. Bourret, Nuovo Cimento **26**, 1 (1962).

[21] J. B. Keller, *Stochastic Equations and Wave Propagation in Random Media*, Vol. 16 of Proc. Symp. Appl. Math. (Am. Math. Soc., McGraw-Hill, New York, 1964), pp. 145–170.

[22] N. G. Van Kampen, Phys. Rep. **24**, 171 (1976).

[23] J. W. Stohbehn, *Propagation of Laser Beams in the Atmosphere* (Springer, Berlin, 1978).

[24] B. J. Uscinski, *The Elements of Wave Propagation in Random Media* (McGraw-Hill, New York, 1979).

[25] K. S. Gochelashvily, A. N. Starodumov, I. V. Chashei, and V. I. Shishov, J. Opt. Soc. Am. A **2**, 2313 (1985).

[26] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (IEEE Press, New York, 1997).

[27] L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media* (SPIE Press, New York, 1998).

[28] U. Frisch, in *Probabilistic Methods in Applied Mathematics*, edited by A. T. Bharucha-Reid (Academic Press, New York, 1968).

[29] B. K. Shivamoggi, L. C. Andrews, and R. L. Phillips, Physica A **275**, 86 (2000).