Dispersion in a thermal plasma including arbitrary degeneracy and quantum recoil

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The longitudinal response function for a thermal electron gas is calculated including two quantum effects exactly, degeneracy, and the quantum recoil. The Fermi-Dirac distribution is expanded in powers of a parameter that is small in the nondegenerate limit and the response function is evaluated in terms of the conventional plasma dispersion function to arbitrary order in this parameter. The infinite sum is performed in terms of polylogarithms in the long-wavelength and quasistatic limits, giving results that apply for arbitrary degeneracy. The results are applied to the dispersion relations for Langmuir waves and to screening, reproducing known results in the nondegenerate and completely degenerate limits, and generalizing them to arbitrary degeneracy.

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I. INTRODUCTION

The longitudinal response function for a thermal electron gas, described by the susceptibility $\chi_e^L(\omega,\mathbf{k})$, includes all relevant dispersive effects. In the absence of quantum effects, $\chi_e^L(\omega,\mathbf{k})$ for a thermal plasma can be expressed in terms of the familiar plasma dispersion function [1], denoted here by $\phi(y)$, with $y=\omega/\sqrt{2}kV_e$, where $V_e=(T_e/m_e)^{1/2}$ is the thermal speed of electrons. Two quantum effects are degeneracy and the quantum recoil. Inclusion of degeneracy corresponds to replacing the classical Maxwell-Boltzmann (MB) distribution by a Fermi-Dirac (FD) distribution. The completely degenerate limit corresponds to distribution function that is a constant below the Fermi speed, v_F, and zero for speeds $v > v_F$, or alternatively for energies $\varepsilon=\frac{1}{2}m_e v^2$ below and above the Fermi energy or temperature, $T_F=\frac{1}{2}m_e v_F^2$, respectively. The quantum recoil corresponds to a correction $\hbar k^2/2m_e$, to the resonant frequency $\omega-\mathbf{k}\cdot\mathbf{v}$. Both these effects are included in Lindhard's [2] form for the response function, which applies only in the completely degenerate limit. In this limit the plasma dispersion function is replaced by two logarithmic functions. Relatively few results are available in the intermediate range of partial degeneracy. Some results were obtained in the relativistic quantum case using expansions that apply in the nearly nondegenerate limits and in the nearly degenerate limits, respectively [3,4]. In the absence of the recoil, some results are known in the nonrelativistic counterpart of these two cases [5]. With the recent interest in quantum plasmas the more general case of arbitrary degeneracy with the quantum recoil has become of interest. Specifically, the dispersion relation for Langmuir waves has been generalized to arbitrary degeneracy, with the quantum recoil included as a perturbation [6].

Plasma dispersion is of particular interest in two limits, which we refer to as the long-wavelength and quasistatic limits. These correspond to approximating $\phi(y)$ for $y^2 \gg 1$ and $y^2 \ll 1$, respectively. The long-wavelength limit is relevant to the dispersion relation for Langmuir waves, giving dispersive corrections (to $\omega^2=\omega_p^2$) $3k^2V_e^2$ and $3k^2v_F^2/5$ in the nondegenerate and completely degenerate limits, respectively. The generalization to arbitrary degeneracy has been derived in the long-wavelength limit [6]. The quasistatic limit is relevant to screening, and to the dispersion relation for ion acoustic waves (IAWs). The screening is characterized by the Debye length, $\lambda_{De}=V_e/\omega_p$, and the Thomas-Fermi length, $\lambda_{TF}=v_F/\sqrt{3}\omega_p$, in the nondegenerate and completely degenerate limits, respectively. We are unaware of any result for the screening length for intermediate values of degeneracy.

In this paper, we consider $\chi_e^L(\omega,\mathbf{k})$ for a FD distribution of electrons, retaining the quantum recoil exactly. To derive useful results some expansion needs to be made. One type of expansion is about the nondegenerate limit or about the complete degenerate limit. Regarding $T_F \propto n_e^{2/3}$ as a parametrized form of the electrons density, n_e, these expansions are in $T_F/T_e \ll 1$ and $T_e/T_F \ll 1$, respectively, A second type of expansion is in the argument, y, of the plasma dispersion function, with $y^2 \gg 1$ corresponding to the long-wavelength limit and $y^2 \ll 1$ to the quasistatic limit. The third expansion is in the quantum recoil. Our emphasis in this paper is on the expansion about the nondegenerate limit. Formally, this involves an expansion in the parameter $\xi=e^{\mu_e/T_e} \ll 1$, where μ_e is the chemical potential of the electrons. This expansion allows the electron number density for a FD distribution to be evaluated as an infinite sum, which gives a polylogarithm function $[-Li_n(-\xi)$ with $n=3/2]$. The polylogarithm is well defined for arbitrary degeneracy, with its asymptotic form $\xi \to \infty$ corresponding to the completely degenerate limit, $\mu_e \to T_F$, $T_e \to 0$. This allows one to evaluate any integral over a FD distribution by expanding in ξ, integrating term-by-term and identifying the resulting series as the power series expansion of the polylogarithm, which applies for arbitrary ξ. We use this approach to evaluate the infinite sums in term of polylogarithms in both the long-wavelength and the quasistatic limits, giving results that apply for arbitrary degeneracy. Our results retain the quantum recoil exactly.

II. LONGITUDINAL RESPONSE FUNCTION

In an electron-ion plasma, the longitudinal response function has contributions from the electron and ion susceptibilities,

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$$K^L(\omega,\mathbf{k}) = 1 + \chi_e^L(\omega,\mathbf{k}) + \chi_i^L(\omega,\mathbf{k}). \quad (1)$$

We concentrate on the electron susceptibility, including the ions only when discussing IAWs, then assuming them to be cold and classical, corresponding to $\chi_i^L(\omega,\mathbf{k})=-\omega_{pi}^2/\omega^2$.

A relativistic quantum treatment of the electrons, which includes all quantum effects, leads to [4]

$$\chi_e^L(\omega,\mathbf{k}) = \frac{e^2}{\varepsilon_0 m_e}\int \frac{d^3\mathbf{p}}{\gamma}\frac{(1-(\mathbf{k}\cdot\mathbf{v})^2/\mathbf{k}^2c^2)f_e(\mathbf{p})}{(\omega-\mathbf{k}\cdot\mathbf{v})^2-[\hbar(\omega^2-\mathbf{k}^2c^2)/2m_e\gamma c^2]^2}. \quad (2)$$

with $\mathbf{p}=\gamma m_e\mathbf{v}$, $\gamma=(1-v^2/c^2)^{-1/2}$. Here $f_e(\mathbf{p})$ is, apart from notation, the quantum mechanical occupation number averaged over spin states and summed over electrons and positrons,

$$2\frac{\bar{n}(\mathbf{p})}{(2\pi\hbar)^3} = f_e(\mathbf{p}). \quad (3)$$

For a completely degenerate distribution one has $\bar{n}(\mathbf{p})=1$ for $v>v_F$ and $\bar{n}(\mathbf{p})=0$ for $v>v_F$; for a nondegenerate distribution, $f_e(\mathbf{p})$ is a MB distribution. In this paper we assume a FD distribution, given by Eq. (10) below. The term $[\hbar(\omega^2/c^2-\mathbf{k}^2)/2m_e\gamma]^2$ in the denominator in Eq. (2) is associated with the quantum recoil. In the absence of this recoil term, Eq. (2) is identical to the expression derived using relativistic, classical kinetic theory.

We assume the strictly nonrelativistic limit $c\to\infty$, in which case [Eq. (2)[reduces to

$$\chi_e^L(\omega,\mathbf{k}) = \frac{e^2}{\varepsilon_0 m_e}\int d^3\mathbf{p}\frac{f_e(\mathbf{p})}{(\omega-\mathbf{k}\cdot\mathbf{v})^2-\Delta_e^2}, \quad (4)$$

where the quantum recoil is included through

$$\Delta_e = \frac{\hbar\mathbf{k}^2}{2m_e}. \quad (5)$$

The integral over $d^3\mathbf{p}$ in Eq. (4) involves an integral over $|\mathbf{p}|=m_e v$ and an integral over solid angle, which may be written in terms of polar angles about the direction of $\mathbf{k}$. The integral over momentum is written in polar coordinates,

$$d^3\mathbf{p} \to 2\pi m_e^3\int_0^\infty dv v^2\int_{-1}^{+1} d\cos\theta,$$

and this integral in Eq. (4) gives

$$\int \frac{d^3\mathbf{p}}{(\omega-\mathbf{k}\cdot\mathbf{v})^2-\Delta_e^2} = 2\pi m_e^3\int_0^\infty dv v^2\int_{-1}^{+1}\frac{d\cos\theta}{(\omega-kv\cos\theta)^2-\Delta_e^2}. \quad (6)$$

The integral over $\cos\theta$ gives a logarithmic function. After writing the integral over v in terms of the energy $\varepsilon=\frac{1}{2}m_e v^2$ and expressing the distribution function in terms of the occupation number, we find

$$\chi_e^L(\omega,\mathbf{k}) = -\frac{4\pi e^2 m_e}{\varepsilon_e(2\pi\hbar)^3}\frac{1}{2k\Delta_e}\int_0^\infty d\varepsilon\bar{n}(\varepsilon)\left[\ln\frac{\omega-a\varepsilon^{1/2}+\Delta_e}{\omega+a\varepsilon^{1/2}+\Delta_e} - \ln\frac{\omega-a\varepsilon^{1/2}-\Delta_e}{\omega+a\varepsilon^{1/2}-\Delta_e}\right], \quad (7)$$

with $a=(2/m_e)^{1/2}k$. The susceptibility in the form [Eq. (7)] is the starting point for our discussion below. An alternative way of writing Eq. (7) involves the electrons temperature $T_e=m_eV_e^2$ as a parameter,

$$\chi_e^L(\omega,\mathbf{k}) = -\frac{4\pi e^2 m_e}{\varepsilon_e(2\pi\hbar)^3}\frac{T_e}{k\Delta_e}\int_{-\infty}^\infty dt t\bar{n}(\varepsilon)\ln\left(\frac{t-y_{e+}}{t-y_{e-}}\right), \quad (8)$$

with $t=v/\sqrt{2}V_e=\sqrt{\varepsilon/T_e}$, $y_{e\pm}=(\omega\pm\Delta_e)/\sqrt{2}kV_e$, and where the second logarithm is included by extending the range of integration to negative values.

Another result needed below is the long-wavelength approximation, We expand the resonant denominator in Eq. (4) assuming $|\mathbf{k}\cdot\mathbf{v}|\ll\omega$. For an isotropic distribution of electrons, the odd powers of $\mathbf{k}\cdot\mathbf{v}$ average to zero, and the leading terms give

$$\chi_e^L(\omega,\mathbf{k}) = -\frac{\omega_p^2}{2\Delta_e}\sum_\pm\frac{\pm1}{\omega\mp\Delta_e}\left[1+\frac{\langle(\mathbf{k}\cdot\mathbf{v})^2\rangle}{(\omega\mp\Delta_e)^2}+\cdots\right], \quad (9)$$

with $\omega_p^2=e^2n_e/\varepsilon_e m_e$, where n_e is the electron number density, and where the angular brackets denote the average over the distribution function.

III. FERMI-DIRAC DISTRIBUTION

A quantum thermal electron gas corresponds to a FD distribution. The occupation numbers of electrons in Eq. (3), in terms of energy ε, is

$$\bar{n}(\varepsilon) = \frac{1}{\exp[(\varepsilon-\mu_e)/T_e]+1}. \quad (10)$$

The parameter μ_e/T_e is large and negative in the nondegenerate limit, and is large and positive in the completely degenerate limit. Hence, we are concerned with $\xi=e^{\mu_e/T_e}$ ranging from $\xi\ll1$ in the nondegenerate limit, to $\xi\gg1$ in the completely degenerate limit. We illustrate the form of the occupation number for ranging from the nondegenerate to the completely degenerate limit in Fig. 1. The normalization of the FD distribution may be expressed in terms of the function

$$-Li_{s+1}(-\xi) = \frac{1}{\Gamma(s+1)}\int_0^\infty dt\frac{t^s}{\xi^{-1}e^t+1}. \quad (11)$$

The polylogarithm function has the power series expansion

$$Li_n(z) = \sum_{k=1}^\infty\frac{z^k}{k^n}, \quad (12)$$

which may be regarded as an alternative definition of $Li_n(z)$ The normalization in the limit $\xi\ll1$ can be determined by first expanding the occupation number in powers of $\xi=e^{\mu_e/T_e}\ll1$,

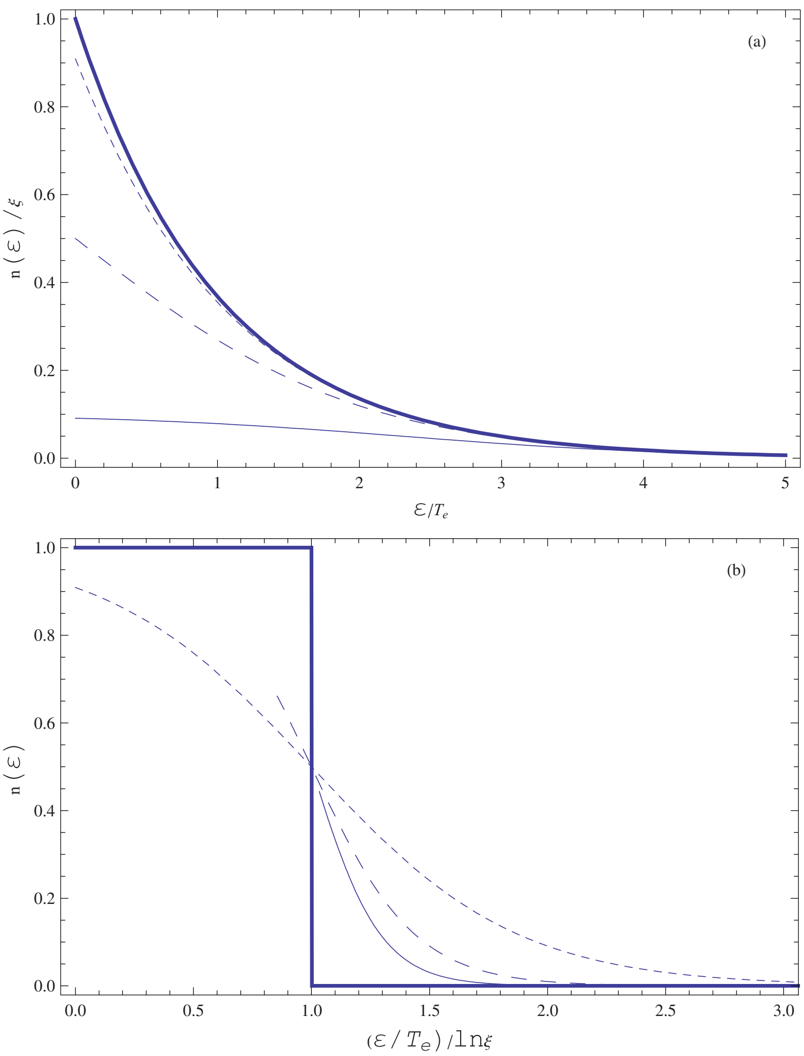

FIG. 1. (Color online) Plot of the occupation number for a FD distribution for (upper panel-a) ξ=0 (bold), 0.1 (dotted), 1 (dashed), and 10 (solid) and (lower panel-b) ξ=10 (dotted), 100 (dashed), 1000 (solid), and ∞ (bold) with a different scaling.

$$n(\varepsilon) = \sum_{s=1}^{\infty} (-)^{s-1} \xi^s e^{-s\varepsilon/T_e}, \tag{13}$$

and integrating term by term to find

$$n_e = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\bar{n}(\varepsilon) = -\frac{2(2\pi)^{3/2} m_e^3 V_e^3}{(2\pi\hbar)^3} Li_{3/2}(-\xi), \tag{14}$$

where the series is summed using Eq. (12). The result [Eq. (14)] applies for arbitrary degeneracy, and its evaluation by the power series expansion for small ξ may be regarded as a convenient way of deriving the exact integral, defined by Eq. (11).

In the completely degenerate limit, $\xi \gg 1$, the asymptotic limit of the polylogarithm gives

$$-\lim_{\xi\to\infty} Li_s(-\xi)\Gamma(s+1) = (\ln \xi)^s, \tag{15}$$

The completely degenerate limit corresponds to $\mu_e \to T_F$, $\ln \xi \to T_F/T_e \gg 1$.

There are three different choices of parameters to describe the electron gas. The choice of μ_e, T_e is important for formal purposes, but the chemical potential, μ_e, is not convenient for physical interpretation. The choice of n_e, T_e is more useful for practical purpose. These choices are related by $n_e/T_e^{3/2} \propto -Li_{3/2}(-e^{\mu_e/T_e})$. A third choice is to introduce the Fermi temperature, T_F, as a parameterized version of the number density. In the completely degenerate limit one has

$$n_e = \frac{8\pi}{3}\left(\frac{m_e v_F}{2\pi\hbar}\right)^3 = \frac{8\pi}{3}\frac{(2m_e T_F)^{3/2}}{(2\pi\hbar)^3}, \tag{16}$$

and for arbitrary degeneracy one can regard Eq. (16) as a definition of T_F.

The relation between T_F/T_e and ξ is

$$-Li_{3/2}(-\xi) = \frac{4}{3\sqrt{\pi}}\left(\frac{T_F}{T_e}\right)^{3/2}. \tag{17}$$

The relation (17) is plotted in Fig. 2. The nondegenerate limit is $T_F \ll T_e$ and the completely degenerate limit is $T_F \gg T_e$. The chemical potential passes through zero in between these limits, corresponding to ξ=1. The value of T_F/T_e corresponding to μ_e=0 is close to unity. [The exact value follows from Eq. (17), with $-Li_s(-1) = (1-2^{1-s})Li_s(1)$, $Li_s(1) = \zeta(s)$, with the relevant Riemann zeta function having the value $\zeta(3/2) = 2.612\ldots$, giving $T_F/T_e = 1.01\ldots$.]

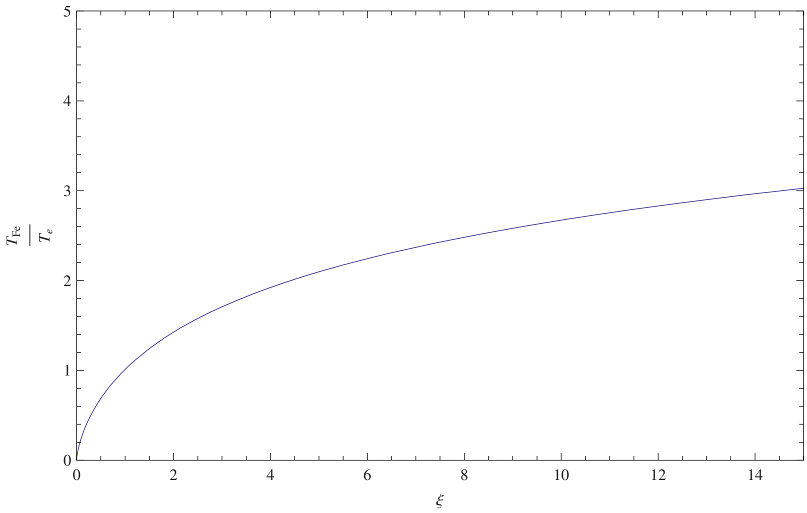

FIG. 2. (Color online) A plot of the relation between the parameters T_F/T_e, with $T_F \propto n_e^{2/3}$, and $\xi = e^{\mu_e/T_e}$. The nondegenerate limit is $T_F/T_e \ll 1$, $\xi \ll 1$, the degenerate limit is $T_F/T_e \gg 1$, $\xi \gg 1$.

IV. DIELECTRIC RESPONSE FUNCTION

The power series expansion in $\xi = e^{\mu_e/T_e}$ of the electron susceptibility is obtained by inserting Eq. (13) in Eq. (7). The leading term, $s=1$, corresponds to a MB distribution, and this case was evaluated in Ref. [8]. The higher order terms are of the same form, with the integral corresponding to a MB distribution with $T_e \to T_e/s$ in the sth term. Thus the integral that appears in the sth term is of the same form as for a MB distribution. Hence one finds

$$\chi_e^L(\omega,\mathbf{k}) = -\pi^{1/2}\frac{4\pi e^2 m_e^2 V_e^2}{\varepsilon_0 (2\pi\hbar)^3 k}\frac{1}{2\Delta_e}\sum_{s=1}^{\infty}(-)^{s-1}\frac{\xi^s}{s} \times\left[\frac{\bar{\phi}(y_{e-}^{(s)})}{y_{e-}^{(s)}} - \frac{\bar{\phi}(y_{e+}^{(s)})}{y_{e+}^{(s)}}\right], \tag{18}$$

where

$$\bar{\phi}(y) = -\frac{y}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{dte^{t^2}}{t-y} = 2ye^{-y^2}\int_0^y dte^{t^2} - iy\sqrt{\pi}e^{-y^2}, \tag{19}$$

is the conventional plasma dispersion function, with

$$y_{e\pm}^{(s)} = \sqrt{s}y_{e\pm}, \quad y_{e\pm} = \frac{\omega \pm \Delta_e}{\sqrt{2}kV_e}. \tag{20}$$

The result [Eq. (18)] gives an exact expression for $\chi_e^L(\omega,\mathbf{k})$ for all values for which the sum converges. We evaluate the sum in three important cases: the long-wavelength limit, the quasistatic limit and for Landau damping.

V. LIMITING CASES

Two limiting cases for the real part of Eq. (18) are of particular interest: the long-wavelength limit, $y_e \gg 1$, and the low-frequency limit, $y_e \ll 1$. Mathematically, they correspond to the approximations $\phi(y) = 1 + 1/2y^2 + \cdots$ for $y^2 \gg 1$, and $\phi(y) = 2y^2 - 4y^4/3 + \cdots$ for $y^2 \ll 1$.

In the long-wavelength, the sth term in the sum in Eq. (18) gives

$$\frac{\bar{\phi}(y_{e-}^{(s)})}{y_{e-}^{(s)}} - \frac{\bar{\phi}(y_{e+}^{(s)})}{y_{e+}^{(s)}} = \sqrt{\frac{2}{s}}\frac{2\Delta_e kV_e}{\omega^2 - \Delta_e^2} \times\left[1 + \frac{2k^2V_e^2(3\omega^2 + \Delta_e^2)}{s(\omega^2 - \Delta_e^2)^2} + \cdots\right]. \tag{21}$$

Then Eq. (18) reduces to

$$\chi_e^L(\omega,\mathbf{k}) = -\frac{\omega_p^2}{\omega^2 - \Delta_e^2} - \omega_p^2 k^2 V_e^2 \frac{(3\omega^2 + \Delta_e^2)}{(\omega^2 - \Delta_e^2)^3}G,$$

$$G = \frac{Li_{5/2}(-\xi)}{Li_{3/2}(-\xi)}, \tag{22}$$

where the sum is performed using Eq. (12) with $n = 5/2$.

The foregoing derivation of Eq. (22) involves a double expansion, in $e^{-\varepsilon/T}$. leading to the sum over s, and then an expansion in $1/sy^2$ for the sth term in this sum. The fact that the series can be summed as a polylogarithm implies that the result [Eq. (22)] can be derived without the intermediate step involving the sum over s. An alternative, more direct, derivation of Eq. (22) follows by evaluating $\langle v^2 \rangle$ in Eq. (9) using Eq. (11). This alternative derivation confirms that Eq. (21) applies for arbitrary degeneracy. As pointed out in Ref. [6], the factor G reduces to unity in the nondegenerate limit, with $G \to 3v_F^2/5V_e^2$ in the completely degenerate limit, where Eq. (15) is used.

In Fig. 3, we plot G, defined by Eq. (22), as a function of T_F/T_e. In looking for an interpolation formula, we requires that the limits $G \to 1$ for $T_F/T_e \to 0$ and $G \to 2T_F/5T_e$ for $T_F/T_e \to \infty$ be reproduced. One class of interpolation is of the form

$$G_n = \left[1 + \left(\frac{2T_F}{5T_e}\right)^n\right]^{1/n}. \tag{23}$$

The linear interpolation, $n=1$, is a straight line, and is a poor approximation to the curve. We illustrate the interpolations for $n = 2,3,4$ in Fig. 3. For semiquantitative purposes the interpolation with $n=2$ corresponds to replacing Eq. (22) by

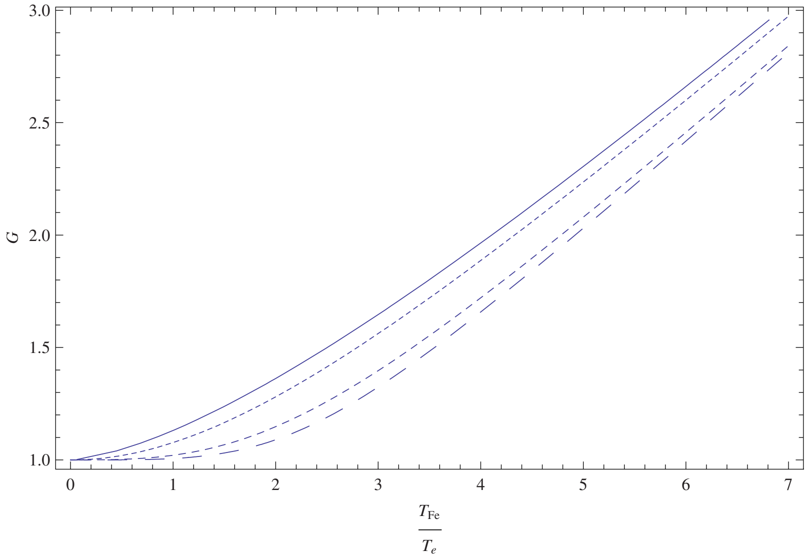

FIG. 3. (Color online) The function G, defined by Eq. (22), is plotted as a function of T_F/T_e. Approximate interpolations of the form [Eq. (23)] are plotted with $n=2$ (dotted), $n=3$ (dot-dashed) and $n=4$ (dashed), showing that the case $n=2$ gives an approximate fit.

$$\chi_e^L(\omega,\mathbf{k}) \approx -\frac{\omega_p^2}{\omega^2-\Delta_e^2} - \omega_p^2 k^2 V_e^2 \frac{(3\omega^2+\Delta_e^2)}{(\omega^2-\Delta_e^2)^3}\left[1+\left(\frac{2T_F}{5T_e}\right)^2\right]^{1/2}, \tag{24}$$

The quasistatic limit corresponds to $y_{e\pm}\to\pm\Delta_e/\sqrt{2}kV_e$. In this limit one has $\bar{\phi}(y)=2y^2$, and

$$\frac{\bar{\phi}(y_{e-}^{(s)})}{y_{e-}^{(s)}} - \frac{\bar{\phi}(y_{e+}^{(s)})}{y_{e+}^{(s)}} \simeq 2(y_{e-}^{(s)} - y_{e+}^{(s)}) = -(s/2)^{1/2}\frac{4\Delta_e}{kV_e}. \tag{25}$$

In this case, Eq. (18) to lowest order in and expansion in $1/k^2$ reduces to

$$\chi_e^L(\omega,\mathbf{k}) = \frac{4(2\pi)^{3/2}e^2 m_e^2 V_e}{\varepsilon_0(2\pi\hbar)^3 k^2}\sum_{s=1}^{\infty}(-)^{s-1}\frac{\xi^s}{s^{1/2}} = \frac{1}{k^2\lambda_{sc}^2},$$

$$\lambda_{sc}^2 = \lambda_{De}^2\tilde{G}. \tag{26}$$

where λ_{sc} is the screening length. The function $\tilde{G}$ can be evaluated in terms of polylogarithms by summing the series,

$$\tilde{G} = \frac{Li_{3/2}(-\xi)}{Li_{1/2}(-\xi)}. \tag{27}$$

In the nondegenerate one has $\tilde{G}=1$ and $\lambda_{sc}\to\lambda_{De}$. In the completely degenerate limit, one has $\tilde{G}\to v_F^2/3V_e^2=2T_F/3T_e$, and $\lambda_{sc}\to\lambda_{FT}$, where $\lambda_{FT}=v_F/\sqrt{3}\omega_p$ is the Thomas-Fermi length. The form [Eq. (26)] gives the screening length for any intermediate degeneracy. Some authors [7] are explaining the screening and wake potentials of a test charge in a completely degenerate plasmas by using the linear dielectric response formalism.

In Fig. 4, we plot $\tilde{G}$, defined by Eq. (27), as a function of T_F/T_e. We look for an interpolation formula, analogous to Eq. (23) between the limits $\tilde{G}\to 1$ for $T_F/T_e\to 0$ and $\tilde{G}\to 2T_F/3T_e$ for $T_F/T_e\to\infty$. As shown in Fig. 4, the interpolation with $n=2$ is quite accurate. Applied to the screening length, this gives

$$\lambda_{sc}^2 = \lambda_{De}^2\left[1+\left(\frac{2T_F}{3T_e}\right)^2\right]^{1/2}. \tag{28}$$

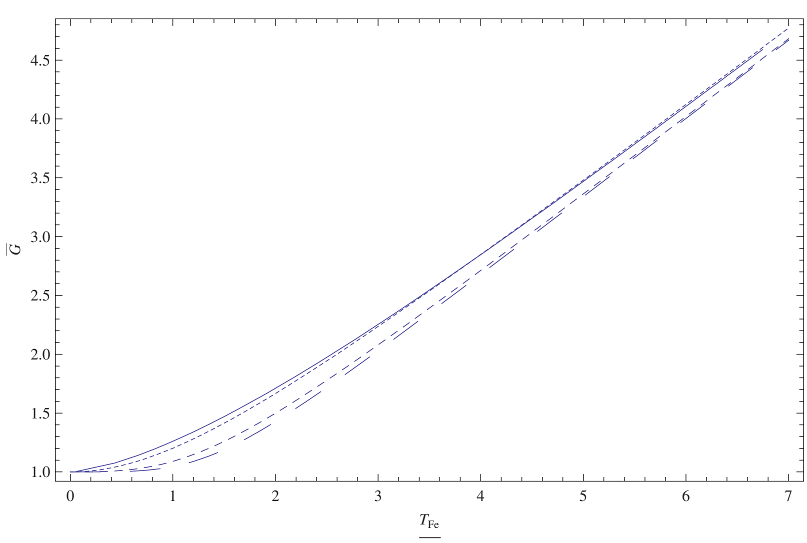

FIG. 4. (Color online) As for Fig. 3 but for the function $\tilde{G}$, defined by Eq. (27); the interpolation indicated by the dotted curve is used in Eq. (29).

The leading term [Eq. (26)] in the quasistatic is independent of the quantum recoil. The next order term in the expansion in $1/k^2$ includes the quantum recoil,

$$\chi_e^L(\omega,\mathbf{k}) = \frac{1}{k^2\lambda_{sc}^2}\left[1 - \frac{3\omega^2+\Delta_e^2}{3k^2V_e^2\tilde{G}_1}\right], \quad \tilde{G}_1 = \frac{Li_{1/2}(-\xi)}{Li_{-1/2}(-\xi)}. \tag{29}$$

However, unlike the expansions in the long-wavelength limit, the expansion in $y^2 \ll 1$ is incompatible with the expansion in s in the quasistatic limit: for sufficiently large s, one has $y_s > 1$ even for $y \ll 1$. The mathematical difficulties in treating this limit are evident from the forms Eqs. (7) or (8), where the logarithmic singularities in the integrand preclude evaluating the integral by a power series expansion of the logarithm. The result in the completely degenerate limit may be derived by performing the integral exactly, leading to logarithmic functions with argument $(\omega \pm \Delta_e)/kv_F$, involving the Fermi speed, v_F, before taking the quasistatic limit.

Finally, we note that the sum over s in Eq. (18) can also be performed explicitly for the imaginary part of the response function. By inspection, the sum corresponds to $n=1$ in Eq. (11), giving

$$\mathrm{Im}\,\chi_e^L(\omega,\mathbf{k}) = \left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_p^2}{kV_e}\frac{1}{2\Delta_e}\left[\frac{Li_1(-\xi e^{-y_{e-}^2}) - Li_1(-\xi e^{-y_{e+}^2})}{Li_{3/2}(-\xi)}\right]. \tag{30}$$

One has $Li_1(z) = -\ln(1-z)$, and hence Eq. (30) has the alternative form

$$\mathrm{Im}\,\chi_e^L(\omega,\mathbf{k}) = -\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_p^2}{kV_e}\frac{1}{2\Delta_e}\frac{1}{Li_{3/2}(-\xi)}\ln\left(\frac{1+\xi e^{-y_{e-}^2}}{1+\xi e^{-y_{e+}^2}}\right). \tag{31}$$

VI. LANGMUIR WAVES

In treating Langmuir waves we neglect the ions, which only act as a fixed, neutralizing background. The dispersion relation follows from $K^L(\omega,\mathbf{k})=0$ with $\chi_e^L(\omega,\mathbf{k})$ given by Eq. (22). To lowest order in $k^2V_e^2/\omega^2, \Delta_e^2/\omega^2$ this gives

$$\omega_L^2(k) = \omega_p^2 + 3k^2V_e^2G + \hbar^2k^4/4m^2, \tag{32}$$

where we neglect the relativistic term in the quantum recoil, restricting the validity of Eq. (32) to nonrelativistic phase speeds, $\omega_L^2(k) \ll k^2c^2$. In the nondegenerate limit, $G\to 1$, Eq. (32) gives the familiar dispersion relation for Langmuir waves with the quantum recoil included, $\omega_L^2(k) = \omega_p^2 + 3k^2V_e^2 + \hbar^2k^4/4m^2$. In the completely degenerate limit, $G \to 2v_F^2/5V_e^2$, and Eq. (32) reproduces the known dispersion relation in this limit [9]. Shukla and Eliasson [10] derived the dispersion relation (32) in the completely degenerate limit, with an extra term attributed to the finite width of the electron wave function in a dense Fermi plasma. Our interpolation formula between these limits gives the approximate dispersion relation

$$\omega_L^2(k) \approx \omega_p^2 + 3k^2[V_e^4 + (v_F^2/5)^2]^{1/2} + \hbar^2k^4/4m^2, \tag{33}$$

for arbitrary degeneracy.

The dispersion relation $K^L(\omega,\mathbf{k})=0$ with $\chi_e^L(\omega,\mathbf{k})$ given by Eq. (22) is a cubic equation in ω^2, and so has three solutions for ω^2. For $k^2\to 0$, besides the Langmuir mode at $\omega^2 = \omega_p^2 + \Delta_e^2$, there is a double solution at $\omega^2 = \Delta_e^2$. For $k^2 \neq 0$ the double solution becomes a complex conjugate pair of solutions, one of which is intrinsically growing. However, this is inconsistent with the requirement that a thermal plasma be stable. In fact, these solutions have a large phase speed, inconsistent with the assumption that the phase speed is nonrelativistic, made in deriving Eq. (22). We suggest that these additional solutions are spurious, and that a more careful analysis is needed to identify any intrinsically new modes associated with the quantum recoil.

The absorption coefficient for Landau damping of Langmuir waves follows from

$$\gamma_L(k) = \frac{\omega_L^3(k)}{\omega_p^2}\mathrm{Im}\,\chi_e^L(k), \tag{34}$$

with the imaginary part of the response function, $\mathrm{Im}\,\chi_e^L(k)$, given by setting $\omega = \omega_L(k)$ in Eq. (31). The resulting expression is

$$\gamma_L(k) = -\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_L^3(k)}{kV_e}\frac{1}{2\Delta_e}\frac{1}{Li_{3/2}(-\xi)}\ln\left(\frac{1+\xi e^{-y_{e-}^2}}{1+\xi e^{-y_{e+}^2}}\right). \tag{35}$$

The logarithmic factor in Eq. (35) is the difference between the logarithms of the occupation number for the FD distribution at $v = (\omega \pm \Delta_e)/k$. The physical interpretation is facilitated by considering the nondegenerate limit. In this limit one has

$$-\frac{1}{Li_{3/2}(-\xi)}\ln\left(\frac{1+\xi e^{-y_{e-}^2}}{1+\xi e^{-y_{e+}^2}}\right) \approx e^{-(\omega-\Delta_e)^2/2k^2V_e^2} - e^{-(\omega+\Delta_e)^2/2k^2V_e^2}. \tag{36}$$

The difference on the RHS of Eq. (36) may be interpreted in terms of the rate of true absorption, involving transitions $\varepsilon - \omega, \mathbf{p} - \hbar\mathbf{k} \to \varepsilon, \mathbf{p}$, exceeding the rate of stimulated emission involving transitions, $\varepsilon + \omega, \mathbf{p} + \hbar\mathbf{k} \to \varepsilon, \mathbf{p}$, leading to a net absorption for a MB distribution. The damping is less for a degenerate distribution due to the Pauli exclusion principle precluding transitions when the final state is occupied, with the occupation number being the probability that a state is occupied.

VII. SCREENING AND ION ACOUSTIC WAVES

The characteristic length for screening changes from the Debye length in a nondegenerate plasma to the Thomas-Fermi length in a completely degenerate plasma. The expression (26) gives an expression for λ_{sc} for arbitrary degeneracy, reproducing these two limiting cases.

The properties of ion acoustic waves (IAWs) in a nondegenerate plasma are modified by the change in the screening length. In a simple model for IAWs, in which the ions are treated as cold, the conventional dispersion relation for IAWs is modified by the replacement of λ_{De} by λ_{sc}. This gives

$$\omega^2 = \omega_s^2(k), \quad \omega_s^2(k) = \frac{k^2 V_{ds}^2}{1 + k^2\lambda_{sc}^2}. \tag{37}$$

where $V_{ds}=\omega_{pi}\lambda_{sc}$ is the counterpart of the ion sound speed in a degenerate plasma, with the screening length given by Eq. (26). For nondegenerate electrons one has $\tilde{G}\to 1$, $V_{ds} \to V_s=\omega_{pi}\lambda_{De}$, and Eq. (37) is the conventional dispersion relation for IAWs. In the opposite limit of degenerate electrons, one has $V_{sc}\to\omega_{pi}\lambda_{TF}$, where $\lambda_{TF}^2=v_F^2/3\omega_{pi}^2$ defines the Thomas-Fermi length.

The inclusion of the quantum recoil leads to a small correction to the dispersion relation for IAWs [11]. By using Eq. (31) for electron with $\omega=\omega_s(k)$ and

$$\mathrm{Im}\ \chi_i^L(\omega,\mathbf{k}) = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega\omega_{pi}^2}{k^3V_i^3} e^{-\omega^2/2k^2V_i^2}, \tag{38}$$

for MB ions, the resulting expression of absorption coefficient for Landau damping of IAW is

$$\gamma_s(k) = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_s^3(k)}{\omega_{pi}^2} \times\left[\frac{\omega_s(k)\omega_{pi}^2}{k^3V_i^3} e^{-\omega_s^2(k)/2k^2V_i^2} - \frac{\omega_p^2}{kV_e}\frac{1}{2\Delta_e}\frac{1}{Li_{3/2}(-\xi)} \times\ln\left(\frac{1+\xi\exp\{-[\omega_s(k)-\Delta_e]^2/2k^2V_e^2\}}{1+\xi\exp\{-[\omega_s(k)+\Delta_e]^2/2k^2V_e^2\}}\right)\right] \tag{39}$$

In the nondegenerate limit, using Eq. (36), the damping Eq. (39) is modified and takes the from of expression (17) of [11].

VIII. DISCUSSION AND CONCLUSIONS

In this paper, we derive an expression for the longitudinal response function for a partially degenerate electron gas, including the quantum recoil. This result is needed to discuss kinetic effects in a quantum plasma, complementing treatments based on quantum fluid theory [12,13]. We include the quantum recoil by starting from a relativistic quantum form for the response function, which includes all relativistic and quantum effects, and taking the nonrelativistic limit. The inclusion of degeneracy is straightforward when the degeneracy is weak: one expands the FD distribution function in powers of the small parameter $\xi=e^{\mu_e/T_e}$, and evaluates each term separately. Each terms in the expansion can be written in terms of the familiar plasma dispersion function, $Z(y)$, leading to the result [Eq. (18)]. It is straightforward to treat the long-wavelength limit by expanding in inverse powers of the phase speed ω/k. One approach is to expand $Z(y)$ in powers of $1/y$ and to sum the infinite series in ξ, in terms of polylogarithms, for each term in the expansion $1/y$. Another approach is to apply the expansion [Eq. (9)] to a FD distribution, evaluating the moments of the velocity in terms of polylogarithms directly. Both approaches lead to a result that applies for arbitrary degeneracy. The long-wavelength limit is needed to derive the dispersion relation for Langmuir waves, and these results allow us to derive a dispersion relation that includes both the quantum recoil and arbitrary degeneracy.

The quasistatic limit is relevant to the dispersion relation for ion acoustic waves (IAWs). This limit corresponds to $\omega \to 0$. As in the long-wavelength limit, the resulting infinite series in ξ can be summed in terms of a polylogarithm. This leads to an expression for the screening distance that reduces to the Debye length in the nondegenerate limit and to the Thomas-Fermi length in the completely degenerate limit.

We consider interpolation formulas of the form [Eq. (23)] that reproduce both the nondegenerate and completely degenerate limits exactly. For semiquantitative purposes the interpolation [Eq. (23)] with $n=2$ is appropriate for the long-wavelength limit. An analogous interpolation formula [Eq. (29)] is suitable in estimating the screening length for arbitrary degeneracy.

The expansion of the imaginary part of the response function in powers of ξ can be summed exactly, with the polylogarithm reducing to a logarithm in this case. This leads to formula (31) that includes both degeneracy and the quantum recoil exactly. This allows these effects to be included in Landau damping, and specifically in the absorption coefficient [Eq. (35)] for Landau damping of Langmuir waves. The absorption coefficient [Eq. (35)] has a simple interpretation in terms quantum mechanical transitions for a MB distribution, with the interpretation being modified for a FD distribution due to the Pauli exclusion principle.

A natural extension of the results derived in this paper is to define a plasma dispersion function for a FD distribution and to write the response tensor in terms of this function, so that both the quantum recoil and degeneracy are included exactly. We define such a function and explore its properties is a subsequent paper [14].

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