Time-dependent closure relations for relativistic collisionless fluid equations

K. Bendib-Kalache,[1,*] A. Bendib,[1] and K. Mohammed El Hadj[1,2]
[1]*Laboratoire Electronique Quantique, Faculté de Physique, USTHB, BP32 El Alia, 16111 Bab Ezzouar, Algiers, Algeria*
[2]*Université Hassiba Benbouali de Chlef, Hay Salem Route Nationale N°19, 2000 Chlef Algeria*
(Received 4 April 2010; published 1 November 2010)

Linear fluid equations for relativistic and collisionless plasmas are derived. Closure relations for the fluid equations are analytically computed from the relativistic Vlasov equation in the Fourier space (ω,k), where ω and k are the conjugate variables of time t and space x variables, respectively. The mathematical method used is based on the projection operator techniques and the continued fraction mathematical tools. The generalized heat flux and stress tensor are calculated for arbitrary parameter ω/kc where c is the speed of light, and for arbitrary relativistic parameter $z=mc^2/T$, where m is the particle rest mass and T, the plasma temperature in energy units.

DOI: 10.1103/PhysRevE.82.056401 PACS number(s): 52.25.Fi, 52.27.Ny, 52.65.Ff, 52.25.Dg

I. INTRODUCTION

It is well-known, that the dynamic description of plasmas starting from the hydrodynamic theory has considerable advantages in comparison with the kinetic theory. Indeed, the fluid description has the advantage of involving only the space variables rather than the full phase space ones. It is equally well known that the hydrodynamic approach is faced with the closure problem. Starting from the kinetic equation, one can easily derive a set of exact moment equations. It then results naturally an infinite hierarchy of coupled hydrodynamic equations of order $N=0,1\ldots$ etc., which needs to be closed. Several methods have been reported in the literature devoted to the problem of truncating this fluid equation hierarchy. We can mention the N-moment models based on the Grad's theoretical approach [1] and where the well-known, is the *13*-moments method.

An alternative approach to the N-moment models which this work is concerned by consists in closing the hierarchy fluid equations from the kinetic theory by expressing relationships between the higher moments and lowers-order hydrodynamic ones. In practice, the first three hydrodynamic equations are kept leading to the equation of the evolution of the density n, the fluid velocity $\vec{V}$ and the temperature T. The associated closure relations correspond to the usual transport relations that they have to be calculated from the kinetic theory. The most popular closure relations are valid in the collisional range defined by a very short mean-free path as compared to the typical inhomogeneity scale length. Several works reported in the literature have been devoted to extend the validity of the closure relations to long mean-free path.

Our aim is to derive from the kinetic theory, a Landau-fluid model for arbitrary relativistic regime. More precisely, we study collisionless relativistic plasma by solving exactly the Vlasov equation for weak thermodynamic forces. Several works related to the derivation of the fluid models have been reported in the literature. Hammett and Perkins [2] and Bendib *et al.* [3] have established a set of nonrelativistic and collisionless hydrodynamic equations in one-dimensional (1D) and three-dimensional (3D) approximations, respectively. Dzhavakhishvili and Tsintadze [4] have computed the counterpart of Braginskii equations [5] in the ultrarelativistic regime. Bychenkov *et al.* [6] have calculated from the Fokker-Planck equation nonrelativistic transport coefficients for arbitrary collisionlity. We also pointed out that further studies devoted to the fluid models were performed by Schadwick *et al.* [7,8] in warm collisionless relativistic plasmas and by Hazeltine *et al.* [9,10] in relativistic magnetized plasmas.

The collisionless relativistic regime is relevant in astrophysical and laboratory plasmas. The plasma can be considered as collisionless if the particle mean-free path is greater than the typical inhomogeneity scale length and, the relativistic effects are significant if the thermal velocity or the fluid velocity V, approach the speed of light c. In this work we deal with plasmas defined by arbitrary thermal velocity and small fluid velocity $(V \ll c)$.

The relativistic collisionless fluid equations provide a convenient reduced description for many physical problems than the full kinetic equations which are more complicated to solve. The latter deal with pole integrals in the complex plane, which are not easy to handle. In particular, they can be used to compute the dispersion relations of the plasma modes (the Langmuir waves, acoustic waves, etc.) in astrophysical plasmas and laser-created plasmas in inertial confinement fusion.

In astrophysical plasmas, we can mention for instance that the application of main interest is Langmuir wave propagation and electron Landau damping in the solar wind at 1AU where spacecraft measurements of the distribution functions are readily available and wave-particle interactions may play a role in the formation of the super-halo (see Ref. [11] and references therein). On the other hand in laser-created plasmas, it is well-known that the stimulated Raman and Brillouin scattering (SRS and SBS, respectively) play a crucial role in inertial confinement fusion. The growth rate and the nonlinear saturation of these instabilities depend strongly on the damping of the Langmuir and acoustic waves [12,13]. In future fusion experiments the plasma temperature could reach high values, up to 15 keV. In these physical conditions the dispersion relations of the SRS and SBS have to be revisited taking into account the collisionless relativistic ef-

*karimabendib@hotmail.com

fects [14]. In addition, with the recent advances in laser technology where very short pulses with duration less than 1 ps have made possible to deliver laser pulses with intensities exceeding, $I \geq 10^{19}$ W/cm^2, the mechanisms of interaction laser-plasma access a novel physical regime. In particular, it has been shown that in fast ignition scheme for inertial confinement fusion, as well the electrons as the ions of the plasma are heated randomly to extremely high temperature and correlatively their mean-free paths increase significantly [15,16]. As a consequence, there has been a considerable interest to study these plasmas in particular the SRS and the relativistic modulational instabilities, in the framework of the collisionless relativistic theory [17].

This work is devoted to compute the closure relations for relativistic and collisionless hydrodynamic equations. It is an extension to the nonstationary approximation of our previous work presented in Ref. [18], and it is organized as follows. In Sec. II, we present the equations of the model and the analytical solution of the problem. In Sec. III, we establish a set of closed collisionless relativistic hydrodynamic equations by computing the generalized transport coefficients. An application of the fluid model to calculate the plasma response function is also performed. We give a summary of the results in Sec. IV.

II. EQUATIONS OF THE MODEL

As usual, to obtain the hydrodynamic equations, one must proceed from the plasma kinetic equation for the particle distribution function $f_p(t,\vec{r},\vec{p})$ defined in the phase space $(\vec{r},\vec{p})$ at time t. In this work, the plasmas of interest are relativistic and collisionless, so the starting kinetic equation is the relativistic Vlasov equation. In the laboratory frame it reads

$$\frac{\partial f_P}{\partial t} + \frac{c^2}{\varepsilon}\vec{p}\cdot\frac{\partial f_P}{\partial \vec{r}} + q\left(\vec{E} + \frac{c^2}{\varepsilon}\vec{p}\times\vec{B}\right)\cdot\frac{\partial f_P}{\partial \vec{p}} = 0, \qquad (1)$$

where $\varepsilon = \gamma mc^2$, is the total energy of a particle with mass m and charge q, $\vec{p} = \gamma m\vec{v}$, is the relativistic momentum vector, $\vec{v}$ is the velocity vector, c is the speed of light in vacuum and $\gamma = 1/\sqrt{1-(v^2/c^2)}$ is the relativistic factor. $\vec{E}$ and $\vec{B}$ are, respectively, the electric and the magnetic field. Before deriving the fluid equations, let us introduce the definition of the hydrodynamic variables, n, $\vec{V}$ and T, i.e.,

$$n(\vec{r},t) = \int f_P(\vec{r},\vec{p},t)d\vec{p}, \quad \vec{V}(\vec{r},t) = \frac{1}{n(\vec{r},t)}\int \frac{\vec{p}}{m\gamma} f_P(\vec{r},\vec{p},t)d\vec{p},$$

$$\int (\varepsilon - mc^2) f_P(\vec{r},p,t)d\vec{p} = n(\vec{r},t)[mc^2(G(z)-1) - T(\vec{r},t)], \qquad (2)$$

where $G(z) = K_3(z)/K_2(z)$, $K_n(z)$ being the modified Bessel function of the nth order and the argument is $z = mc^2/T$. Here, the temperature is expressed in energy units, used throughout this work. Multiplying Eq. (1) by 1, $\vec{p}$ and $(\varepsilon - mc^2)$, and integrating upon the momentum space, the resulting moment equations are, respectively, the continuity equation and the momentum and energy balance equations

$$\frac{\partial}{\partial t}(\gamma_V n) + \vec{\nabla}\cdot(\gamma_V n\vec{V}) = 0 \qquad (3)$$

$$\gamma_V n\frac{d}{dt}(\gamma_V mGV_i) = -\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_k}(S_{im}S_{kn}\Pi_{mn}) + \gamma_V nq[\vec{E} + (\vec{V}\wedge\vec{B})]_i - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\gamma_V S_{ik}V_m\Pi_{km} + \gamma_V\left(S_{ik} + \frac{1}{c^2}\gamma_V V_i V_k\right)q_k\right]$$
$$- \frac{1}{c^2}\frac{\partial}{\partial x_k}[\gamma_V(S_{im}V_k + S_{km}V_i)q_m] \qquad (4)$$

$$n\frac{d}{dt}(mc^2G - T) - T\frac{dn}{dt} = -\frac{\partial}{\partial x_k}(\gamma_V^{-1}S_{km}q_m) - S_{im}S_{kn}\Pi_{mn}\frac{\partial V_i}{\partial x_k} - \frac{1}{c^2}\left[\gamma_V S_{ik}V_m\Pi_{km} + \gamma_V\left(S_{ik} + \frac{1}{c^2}\gamma_V V_i V_k\right)q_k\right]\frac{\partial V_i}{\partial t}$$
$$- \frac{1}{c^2}\gamma_V(S_{im}V_k + S_{km}V_i)q_m\frac{\partial V_i}{\partial x_k} - \frac{1}{c^2}\frac{\partial}{\partial t}(q_i V_i) \qquad (5)$$

where $S_{ik} = \delta_{ik} + (\gamma_V - 1)\frac{V_i V_k}{V^2}$, $\gamma_V = 1/\sqrt{1-(V^2/c^2)}$ and δ_{ik} is the Kronecker symbol. In these equations, $P = nT$ is the scalar pressure, Π_{ij} is the viscous tensor and $\vec{q}$ is the heat flux, defined in the rest system of the plasma, by the following expressions:

$$P = \frac{c^2}{3}\int \frac{p'^2}{\varepsilon'} f_P d\vec{p}', \qquad (6)$$

$$\Pi_{ij} = c^2\int \frac{1}{\varepsilon'}\left(p'_i p'_j - \frac{p'^2}{3}\delta_{ij}\right) f_P d\vec{p}', \qquad (7)$$

and

$$\vec{q} = c^2\int \vec{p}' f_P d\vec{p}', \qquad (8)$$

the notation "prime" denotes the rest system of the plasma. Note here, that in comparison with Refs. [4,19], Eqs. (3)–(5) do not involve collisional terms. The computation of Π_{ij} and $\vec{q}$ from the kinetic theory in terms of lower moments constitutes the closure relation for these fluid equations.

In the following it is more convenient to rewrite Eq. (1) using the random momentum $\vec{p}'$ and the corresponding total energy $\varepsilon'=\gamma' mc^2$. This, can be done from the computation of the total time derivative of $f_P(\vec{p}',\vec{r},t)$. The transition from the laboratory frame to the rest system of the plasma, for the momentum and the energy of the particle can be made via the Lorentz transformations [20], $p_i=p_i' S_{ik}+c^{-2}\gamma_V\varepsilon' V_i$ and $\varepsilon=\gamma_V(\varepsilon'+p_i' V_i)$. On the other hand, as we confine ourselves to unmagnetized plasmas ($\vec{B}=0$) with nonrelativistic flow velocities ($V\ll c$), Eq. (1) expressed with the random variables takes the following form:

$$\frac{\partial f_P}{\partial t}+\left(\frac{c^2}{\varepsilon}\vec{p}+\vec{V}\right)\cdot\frac{\partial f_P}{\partial \vec{r}}+\left(q\vec{E}-\frac{\varepsilon}{c^2}\frac{d\vec{V}}{dt}\right)\cdot\frac{\partial f_P}{\partial \vec{p}}-p_k\frac{\partial V_i}{\partial x_k}\frac{\partial f_P}{\partial p_i}=0, \tag{9}$$

where for the sake of clarity, the notation "prime" is removed.

In the linear theory, the distribution function is assumed to be close to the equilibrium distribution function $f^{(0)}$, i.e., $f_p=f^{(0)}+f$ where $f\ll f^{(0)}$. In this paper, we consider that the equilibrium distribution function is the global Maxwell-Boltzmann-Jüttner function [21] called MBJ hereafter,

$$f^{(0)}=F_{MBJ}=\frac{n_0 z_0}{4\pi m^3c^3K_2(z_0)}\exp(-z_0\gamma) \tag{10}$$

where the subscript "0," stands for the equilibrium state, defined by the density n_0, the temperature T_0, (the corresponding relativistic parameter is $z_0=mc^2/T_0$), and zero mean velocity and electric field ($\vec{V}_0=0$, $\vec{E}_0=0$). The perturbed state is described by the distribution f, the density n, the temperature T, the fluid velocity $\vec{V}$ and the electric field $\vec{E}$. Furthermore, we consider that the inhomogeneity is along the x-direction with longitudinal fluid velocity and electric field i.e., $f=f(x,\vec{p},t)$, $n=n(x,t)$, $T=T(x,t)$, $\vec{V}=V(x,t)\vec{x}$, and $\vec{E}=E(x,t)\vec{x}$.

Using these assumptions and taking the spatial $(x\leftrightarrow k)$ and the temporal $(t\leftrightarrow\omega)$ Fourier transforms of the linearized form of Eq. (9), we obtain the following equation:

$$-i\omega\tilde{f}+ikc^2\frac{p_x}{\varepsilon}\tilde{f}-q\tilde{E}\frac{p_x}{m\varepsilon}z_0F_{MBJ}-i\omega\frac{p_x}{mc^2}z_0F_{MBJ}\tilde{V}+\frac{p_x^2}{m\varepsilon}z_0F_{MBJ}ik\tilde{V}=\lim_{\nu\to 0}\nu(\tilde{f}_{MBJ}-\tilde{f}), \tag{11}$$

where

$$\tilde{f}_{MBJ}=\left(\frac{\tilde{n}}{n_0}+(1-z_0G(z_0))\frac{\tilde{T}}{T_0}\right)\mu_0\exp(-z_0\gamma)+\frac{\tilde{T}}{T_0}z_0\gamma\mu_0\exp(-z_0\gamma) \tag{12}$$

is the perturbed MBJ, $\mu_0=\frac{n_0z_0}{4\pi m^3c^3K_2(z_0)}$ and the notation "∼" means that the corresponding quantities are written in the Fourier space. Following the method developed in Ref. [18], to solve the Vlasov equation, we have added formally in Eq. (11) a Krook collision term in the limit of a vanishing collision frequency $\nu\to 0$. This method is equivalent to the one used by Landau [22] to calculate the dispersion relation of plasma waves in collisionless plasmas. Indeed, Landau was the first who predicted that the nonrelativistic plasma waves are damped even in the absence of collisions. At this end, he solved the Vlasov-Poisson set of equations as an initial value problem. To account for the initial value of the plasma wave field, he added an infinitesimal positive imaginary part to the real frequency, i.e., $\omega\to\omega+i0^+$. The corresponding path of integration of the integral containing the collisionless propagator $(kp_x/m-\omega)^{-1}$ is detoured around the singularity $\frac{p_x}{m}=\frac{\omega}{k}$. This rule is known as the Landau prescription. The damping coefficient of the waves (the so-called Landau damping) can be found also with an alternative approach which consists to add formally in the Vlasov equation a small dissipative term, $\nu\tilde{f}(\vec{p},\vec{k},\omega)$ in the limit $\nu\to 0^+$ (see Ref. [23]). This dissipation is not similar to the classical one due to the collisions between particles (finite value of ν), since it is not associated with the entropy increase. Therefore the Krook collision operator in the right hand side of Eq. (11) does not describe physical collisions rather, it describes purely collisionless dissipation.

On the other hand, as usual in the transport theory one expand the distribution function in the orthogonal polynomials basis. Due to the axial symmetry of the problem along the x axis, the Legendre polynomial basis $[P_n(\mu)]$ where $\mu=p_x/p$, is the most appropriate. The expansion of $\tilde{f}$ and Eq. (11) on this basis gives

$$\tilde{f}(\omega,k,\vec{p})=\sum_{n=0}^{\infty}P_n(\mu)\tilde{f}_n(\omega,k,p) \tag{13}$$

$$-i\omega\tilde{f}_0+ikc\frac{(\gamma^2-1)^{1/2}}{\gamma}\frac{\tilde{f}_1}{\sqrt{3}}+\frac{z_0\mu_0}{3}\left(\frac{\gamma^2-1}{\gamma}\right)\exp(-z_0\gamma)ik\tilde{V}=\lim_{\nu\to 0}\nu(\tilde{f}_{MBJ}-\tilde{f}_0) \tag{14}$$

$$-i\omega\tilde{f}_1+ikc\frac{(\gamma^2-1)^{1/2}}{\gamma}\left(\frac{2}{\sqrt{15}}\tilde{f}_2+\frac{1}{\sqrt{3}}\tilde{f}_0\right)=\frac{z_0\mu_0q}{\sqrt{3}mc}\frac{(\gamma^2-1)^{1/2}}{\gamma}\exp(-z_0\gamma)\tilde{E}++\frac{z_0\mu_0}{\sqrt{3}}i\omega(\gamma^2-1)^{1/2}\exp(-z_0\gamma)\frac{\tilde{V}}{c} \tag{15}$$

$$-i\omega\tilde{f}_2+ikc\frac{(\gamma^2-1)^{1/2}}{\gamma}\left(\frac{3}{\sqrt{35}}\tilde{f}_3+\frac{2}{\sqrt{15}}\tilde{f}_1\right)=-\frac{2}{3\sqrt{5}}\left(\frac{\gamma^2-1}{\gamma}\right)z_0\mu_0\exp(-z_0\gamma)ik\tilde{V} \quad (16)$$

$$-i\omega\tilde{f}_n+ikc\frac{(\gamma^2-1)^{1/2}}{\gamma}\left(\frac{n+1}{\sqrt{2n+3}}\frac{\tilde{f}_{n+1}}{\sqrt{2n+1}}+\frac{n}{\sqrt{2n-1}}\frac{\tilde{f}_{n-1}}{\sqrt{2n+1}}\right)=0 \quad n\geq 3, \quad (17)$$

where the relation $p=mc(\gamma^2-1)^{1/2}$ and the recursive formula [24]

$$\mu P_n(\mu)=\frac{n+1}{\sqrt{2n+1}}\frac{P_{n+1}(\mu)}{\sqrt{2n+3}}+\frac{n}{\sqrt{2n+1}}\frac{P_{n-1}(\mu)}{\sqrt{2n-1}} \quad (18)$$

were used. Equations (14)–(17) correspond to a set of infinite algebraic equations. We will now proceed to their analytical solution. First, we introduce the projection operator P associated to the Krook collision operator, and its complement orthogonal Q, such as $Q=1-P$ and $PQ=0$. The operator P is defined by

$$P[\nu(\tilde{f}_{MBJ}-\tilde{f}_0)]=0 \quad (19)$$

and its calculation is widely presented in Refs. [18,25]. We just give here its expression

$$P(\tilde{f}_0)=[I_3M_0^{1,0,1/2}-I_2M_0^{1,1,1/2}]\frac{\exp(-z_0\gamma)}{I_1I_3-I_2I_4}+[-I_4M_0^{1,0,1/2}+I_1M_0^{1,1,1/2}]\frac{\gamma\exp(-z_0\gamma)}{I_1I_3-I_2I_4}. \quad (20)$$

In Eq. (20), $I_1=K_2(z_0)/z_0$, $I_2=K_2(z_0)[-1+z_0G(z_0)]/z_0^2$, $I_3=K_2(z_0)[1+z_0+(3-z_0)G(z_0)]/z_0^2$, $I_4=K_2(z_0)[-1-z_0+z_0G(z_0)]/z_0^2$ and we have used the notation for the moments

$$M_n^{i,j,k}=\int_1^\infty\gamma^i(\gamma-1)^j(\gamma^2-1)^k\tilde{f}_n d\gamma. \quad (21)$$

Multiplying Eq. (14) by the operator Q, we obtain the isotropic part of the Vlasov equation which takes into account the conservative properties of the kinetic equation [see Eqs. (A4) and (A5)],

$$\begin{aligned}-i\omega\tilde{f}_0&+\frac{ikc}{\sqrt{3}}\frac{(\gamma^2-1)^{1/2}}{\gamma}\tilde{f}_1\\ &=-i\omega\tilde{f}_{MBJ}-\frac{z_0}{3}\left(\frac{\gamma^2-1}{\gamma}\right)\mu_0e^{-z_0\gamma}ik\tilde{V}\\ &+\frac{ikc}{\sqrt{3}}\left(\frac{I_3-\gamma I_4}{I_1I_3-I_2I_4}\right)M_1^{0,0,1}-\frac{ikc}{\sqrt{3}}\left(\frac{I_2-\gamma I_1}{I_1I_3-I_2I_4}\right)M_1^{0,1,1}\\ &+\left(\frac{I_3-\gamma I_4+(1-G(z_0))(I_2-\gamma I_1)}{I_1I_3-I_2I_4}\right)I_1\mu_0e^{-z_0\gamma}ik\tilde{V}.\end{aligned} \quad (22)$$

The next step consists to solve the set of infinite Eqs. (17) which correspond to a recursive relation between the components $\tilde{f}_{n-1}$, $\tilde{f}_n$, and $\tilde{f}_{n+1}$, for $n\geq 3$. To do so, we use the continued fraction to invert the operator $(-i\omega+ikc\frac{(\gamma^2-1)^{1/2}}{\gamma})$ in the $P_n(\mu)$-basis. It results after some algebra

$$\tilde{f}_3=-\frac{3}{\sqrt{35}}ikc\frac{(\gamma^2-1)^{1/2}}{\gamma}F_3\tilde{f}_2, \quad (23)$$

where here, F_3 is the infinite continued fraction of order 3, defined by the following recursive formula:

$$F_n=\left(-i\omega+k^2c^2\frac{\gamma^2-1}{\gamma^2}\frac{(n+1)^2}{4(n+1)^2-1}F_{n+1}\right)^{-1}. \quad (24)$$

We should remark that Eq. (23) is the exact solution of the set of infinite Eqs. (17). It gives a relation between the components $\tilde{f}_3$ and $\tilde{f}_2$ that incorporates the contribution of all the components $\tilde{f}_n$ with $n>3$.

The expansion of a distribution function into orthogonal polynomial basis is frequently used in the transport theory to solve kinetic equations. In the classical transport theory it consists to keep only the two first polynomials. This truncation is fully justified in this limit. Indeed, the ratio of the first anisotropic component $\tilde{f}_1$ to the isotropic component $\tilde{f}_0$, is of the order of the Knudsen number (the ratio of the mean-free-path to the gradient inhomogeneity scale length) which is very small in the collisional limit. By increasing the Knudsen number, the kinetic effects are more efficient, and one must keep more terms in the polynomial expansion. In this work we deal with the collisionless limit and we have kept the entire infinite series of the Legendre polynomials. The summation procedure of this polynomial expansion is performed with the use of infinite continued fractions through Eq. (23). We have checked numerically that the infinite continued fractions [Eq. (24)] are always convergent whatever the values of the phase velocity ω/k and the relativistic factor γ, and that the convergence is very fast.

Equations (15), (16), (22), and (23) represent the basic equations of our model. They constitute a set of linear equations for the function $\tilde{f}_0-\tilde{f}_3$ in terms of the electric field $\tilde{E}$ and the generalized forces, i.e., the gradients of the density $\tilde{n}$, the temperature $\tilde{T}$ and the flow velocity $\tilde{V}$. Their solution is straightforward but particularly lengthy; so for clarity we have relegated their computation in Appendix A.

If we solve the linearized Vlasov equation as an initial value-problem [22], we must rewrite it in the Laplace-Fourier space $(x\leftrightarrow k, t\leftrightarrow s)$ and the counterpart equation of Eq. (11) reads

$$s\tilde{f}(\vec{p},k,s) - \tilde{f}(\vec{p},k,t=0) + ikc^2\frac{p_x}{\varepsilon}\tilde{f}(\vec{p},k,s) - q\tilde{E}(k,s)\frac{p_x}{m\varepsilon}z_0F_{MBJ} + s\frac{p_x}{mc^2}z_0F_{MBJ}\tilde{V}(k,s) - \frac{p_x}{mc^2}z_0F_{MBJ}\tilde{V}(k,t=0) + \frac{p_x^2}{m\varepsilon}z_0F_{MBJ}ik\tilde{V}(k,s) = 0 \tag{25}$$

In Ref. [6], Eq. (25) was solved by Bychenkov *et al.* in the non relativistic regime, and for arbitrary collisionality. Following their method we assume that the initial perturbation of the distribution function corresponds to the perturbed MBJ, i.e., $\tilde{f}(\vec{p},k,t=0)=\tilde{f}_{MBJ}(\vec{p},k,t=0)$ and that the distribution function has the same density and energy that the perturbed MBJ [Eqs. (A6) and (A7)]. Expanding Eq. (25) on the Legendre polynomials basis and keeping only its isotropic part, we readily obtain

$$s\tilde{f}_0(\vec{p},k,s) - \left\{\frac{\tilde{n}(t=0)}{n_0} + [1 + z_0\gamma - z_0G(z_0)]\frac{\tilde{T}(t=0)}{T_0}\right\}F_{MBJ} + ikc^2\frac{p}{\sqrt{3}\varepsilon}\tilde{f}_1(\vec{p},k,s) + \frac{p^2}{3m\varepsilon}z_0F_{MBJ}ik\tilde{V}(k,s) = 0. \tag{26}$$

The next step is the computation of the initial hydrodynamic variables $\tilde{n}(t=0)$ and $\tilde{T}(t=0)$. At this end, Eq. (26) is multiplied by $\gamma\sqrt{\gamma^2-1}$ and $\gamma(\gamma-1)\sqrt{\gamma^2-1}$ and is integrated upon the variable γ, obtaining two algebraic equations for the variables $\tilde{n}(t=0)$ and $\tilde{T}(t=0)$ that we can easily calculate. Inserting back these expressions into Eq. (26), we recover exactly Eq. (22), derived above with the use of, the collision operator in the limit $\nu \to 0$ and its associated projection operators.

III. CLOSED RELATIVISTIC HYDRODYNAMIC EQUATIONS

This section is devoted to the computation of the generalized transport coefficients with the use of the model equation derived in Sec. II. These coefficients constitute the closure relations of the linear collisionless and relativistic fluid equations.

A. Generalized relativistic transport coefficients

We remind that in collisionless momentum and energy balance fluid equations, the transport coefficients are the stress tensor [Eq. (7)] and the heat flux [Eq. (8)]. For the axial symmetry considered in this work, there are only two nonzero components, i.e., the x component for the heat flux and the x-x component for the stress tensor,

$$\tilde{q}_x = \frac{4\pi}{\sqrt{3}}m^4c^6M_1^{1,0,1}, \tag{27}$$

$$\tilde{\Pi}_{xx} = \frac{8\pi}{3\sqrt{5}}m^4c^5M_2^{0,0,3/2}. \tag{28}$$

Note here, that these coefficients depend on the first and the second anisotropic distribution function. Substituting the expression of $\tilde{f}_1$ [Eq. (A11)] into Eq. (27) and the expression of $\tilde{f}_2$ [Eq. (A12)] into Eq. (28), we obtain

$$\tilde{q}_x(\xi,z_0) = Q_E(\xi,z_0)n_0cT_0\left(\frac{ik}{|k|}\frac{\tilde{n}}{n_0} + \frac{q}{|k|}\frac{\tilde{E}}{T_0}\right) + Q_T(\xi,z_0)n_0cT_0\frac{ik}{|k|}\frac{\tilde{T}}{T_0} + Q_V(\xi,z_0)n_0cT_0\frac{\tilde{V}}{c}, \tag{29}$$

$$\tilde{\Pi}_{xx}(\xi,z_0) = \Pi_E(\xi,z_0)n_0T_0\left(\frac{\tilde{n}}{n_0} - q\frac{ik}{k^2}\frac{\tilde{E}}{T_0}\right) + \Pi_T(\xi,z_0)n_0T_0\frac{\tilde{T}}{T_0} + \Pi_V(\xi,z_0)n_0T_0\frac{ik}{|k|}\frac{\tilde{V}}{c}, \tag{30}$$

where $\xi=\omega/kc$ is the normalized phase velocity. In Eqs. (29) and (30), we have introduced the dimensionless transport coefficients Q_E, Q_T Q_V, Π_E, Π_T, and Π_V. For the sake of clarity, their explicit expressions are given in Appendix B.

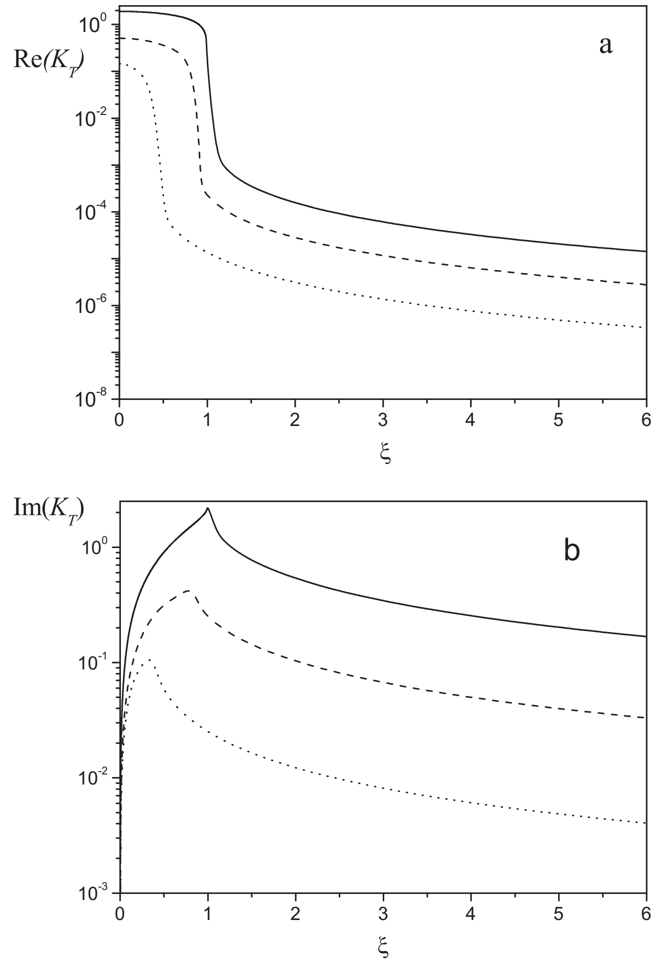

FIG. 1. Real part (panel a) and imaginary part (panel b) of the dimensionless thermal conductivity K_T as a function of the normalized phase velocity $\xi=\frac{\omega}{kc}$ for different values of the relativistic parameter $z_0=mc^2/T_0$, $z_0=10^{-2}$ (solid line), $z_0=10$ (dashed line), and $z_0=100$ (dotted line)

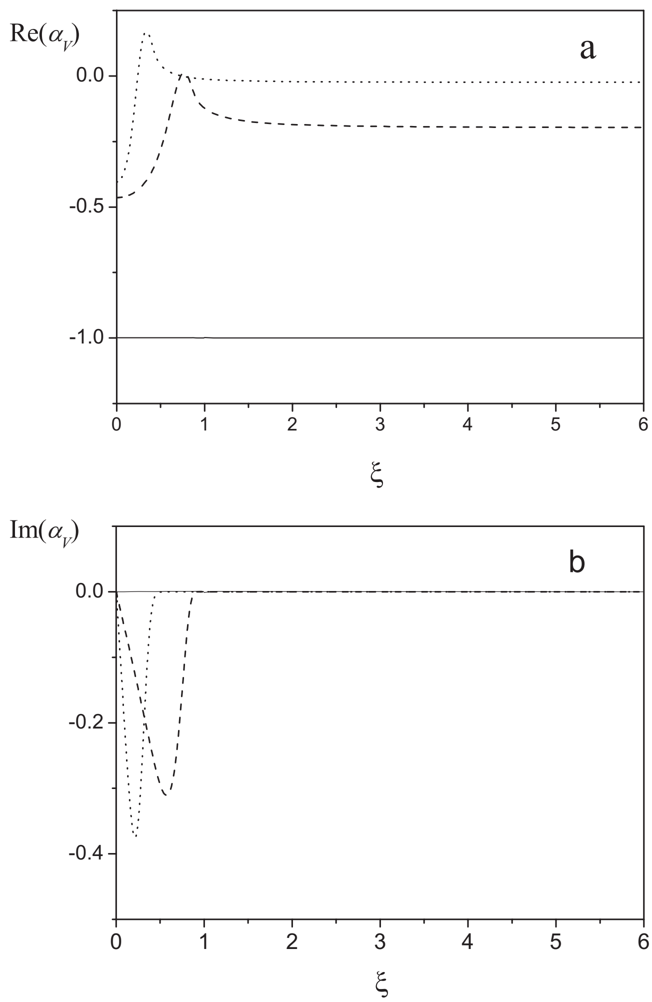

FIG. 2. Real part (panel a) and imaginary part (panel b) of the dimensionless convective heat flux coefficient α_V as a function of the normalized phase velocity $\xi=\frac{\omega}{kc}$ for different values of the relativistic parameter $z_0=mc^2/T_0$, $z_0=10^{-2}$ (solid line), $z_0=10$ (dashed line), and $z_0=100$ (dotted line).

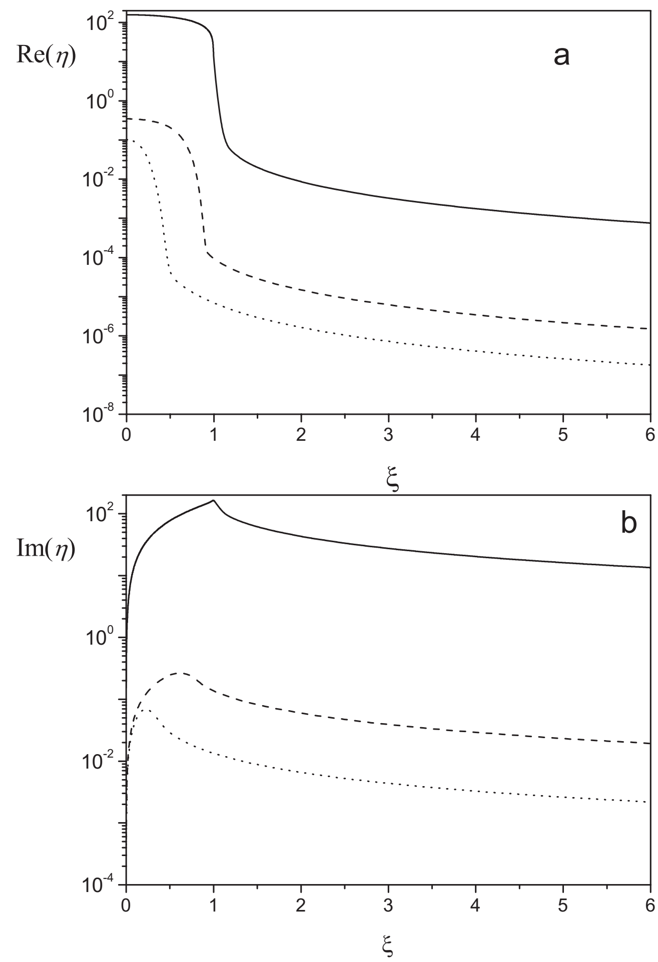

FIG. 3. Real part (panel a) and imaginary part (panel b) of the dimensionless viscosity coefficient η as a function of the normalized phase velocity $\xi=\frac{\omega}{kc}$ for different values of the relativistic parameter $z_0=mc^2/T_0$, $z_0=10^{-2}$ (solid line), $z_0=1$ (dashed line), and $z_0=100$ (dotted line).

To establish the closure relations, a further step is needed which consists to eliminate the electric field in expressions (29) and (30), by using the condition that the mean random velocity is zero, i.e.,

$$\langle \mathrm{v}_x\rangle = \frac{4\pi}{n_0\sqrt{3}}m^3c^4\int_1^\infty(\gamma^2-1)\tilde{f}_1 d\gamma = \frac{4\pi}{n_0\sqrt{3}}m^3c^4M_1^{0,0,1}=0. \tag{31}$$

Using Eq. (A11), Eq. (31) can be rewritten as

$$\langle \mathrm{v}_x\rangle = \Gamma_E(\xi,z_0)c\left(-\frac{ik}{|k|}\frac{\tilde{n}}{n_0}+\frac{q\tilde{E}}{|k|T_0}\right)+\Gamma_T(\xi,z_0)c\frac{ik}{|k|}\left(\frac{\tilde{T}}{T_0}\right) + \Gamma_V(\xi,z_0)c\frac{\tilde{V}}{c}=0, \tag{32}$$

where the coefficients Γ_E, Γ_T, and Γ_V are given in Appendix B [see Eqs. (B7)–(B9)]. As a result, the heat flux and the stress tensor depend only on the temperature and the flow velocity,

$$\tilde{q}_x(\xi,z_0) = -K_T(\xi,z_0)n_0c\frac{ik}{|k|}\tilde{T}+\alpha_V(\xi,z_0)n_0T_0\tilde{V}, \tag{33}$$

$$\tilde{\Pi}_{xx}(\xi,z_0) = -\eta(\xi,z_0)n_0mc\frac{ik}{|k|}\tilde{V}+\alpha_T(\xi,z_0)n_0\tilde{T}, \tag{34}$$

where

$$K_T = Q_T - \frac{Q_E\Gamma_T}{\Gamma_E} \tag{35}$$

is the thermal conductivity,

$$\alpha_V = Q_V - \frac{\Gamma_V Q_E}{\Gamma_E} \tag{36}$$

is the convective heat flux coefficient,

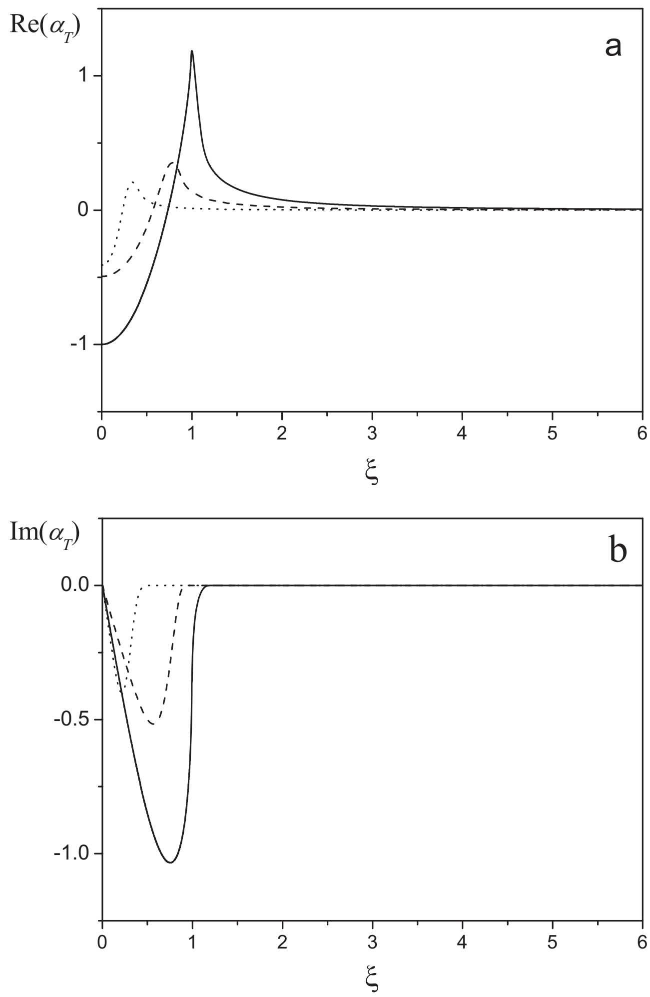

FIG. 4. Real part (panel a) and imaginary part (panel b) of the dimensionless temperature anisotropy α_T as a function of the normalized phase velocity $\xi=\frac{\omega}{kc}$ for different values of the relativistic parameter $z_0=mc^2/T_0$, $z_0=10^{-2}$ (solid line), $z_0=10$ (dashed line), and $z_0=100$ (dotted line).

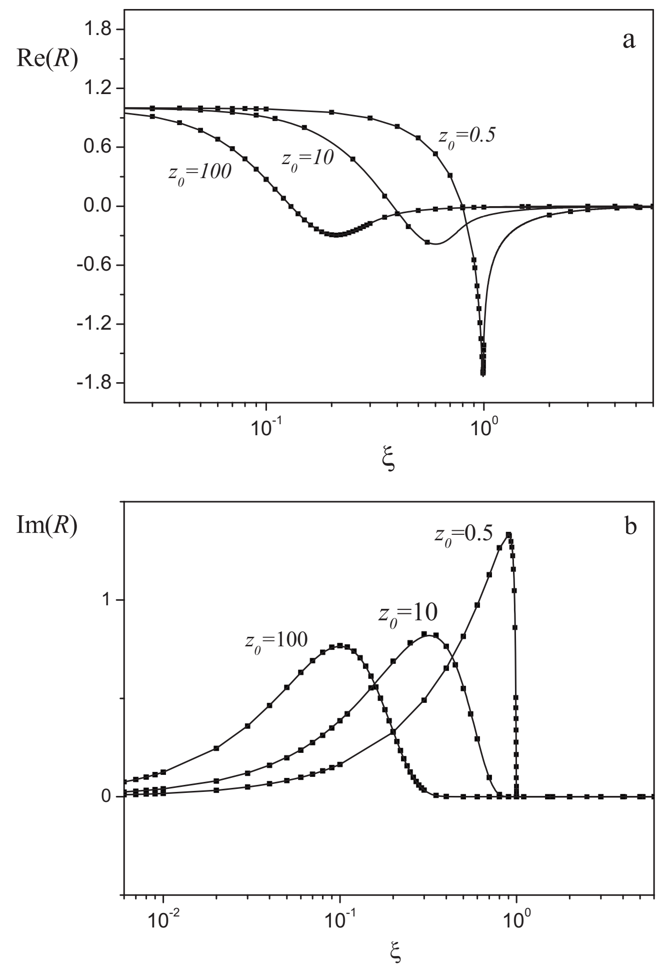

FIG. 5. Real part (panel a) and imaginary part (panel b) of the response function $R=ikT_0\tilde{n}/qn_0\tilde{E}$ as a function of the normalized phase velocity $\xi=\frac{\omega}{kc}$ for different values of the relativistic parameter $z_0=mc^2/T_0$. The solid lines correspond to the kinetic response function $R_{kin}(\xi)$ and, the dark squares correspond to the fluid response function $R_f(\xi)$.

$$\eta=\Pi_V-\Pi_E\frac{\Gamma_V}{\Gamma_E} \tag{37}$$

is the viscosity coefficient and

$$\alpha_T=\Pi_T+\Pi_E\frac{\Gamma_T}{\Gamma_E} \tag{38}$$

is the temperature anisotropy coefficient. We should note that for numerical purposes, in the Onsager relations (33) and (34), the transport coefficients are defined as dimensionless quantities. Their standard definitions should be, $K_{Ts}(k,\omega,z_0)=\frac{n_0c}{|k|}K_T(\xi,z_0)$, $\alpha_{Vs}(\xi,z_0)=n_0T_0\alpha_V(\xi,z_0)$, $\eta_s(k,\omega,z_0)=\frac{n_0mc}{|k|}\eta(\xi,z_0)$, and $\alpha_{Ts}(\xi,z_0)=n_0\alpha_T(\xi,z_0)$ where the subscript "s" stands for the standard definitions.

We now present the numerical results for the transport coefficients on Figs. 1–4 where we give their real and imaginary parts as a function of the normalized phase velocity ξ, for different values of the relativistic parameter, ($z_0=10^{-2}$, $z_0=10$, and $z_0=100$, corresponding to the ultrarelativistic, mildly relativistic and non relativistic ranges, respectively). The analysis of these results shows that the stationary transport coefficients [18], i.e., $K_T(\xi=0,z_0)$, $\alpha_V(\xi=0,z_0)$, $\alpha_T(\xi=0,z_0)$ and η $(\xi=0,z_0)$ are recovered and this whatever the value of the parameter z_0. Furthermore, we note that the coefficients $K_T(\xi,z_0)$, $\alpha_V(\xi,z_0)$, $\alpha_T(\xi,z_0)$, and $\eta(\xi,z_0)$ are complex quantities in contrast to those established in the stationary approximation [18], which are real coefficients. We recall that these coefficients are calculated in the Fourier space (ω,k) and their imaginary parts introduce a phase shift between the transport terms and the thermodynamic forces. We can observe that the imaginary parts are comparable in magnitude to the real parts into the intermediate phase velocity range. We have also checked that for $\xi<10^{-3}$, the transport coefficients are very close to the stationary coefficients calculated in Ref. [18].

In plasmas, the dissipative effects are described by the real part of the thermal conductivity and the viscosity coefficient. We have shown above that these two coefficients are proportional to the factor $\frac{1}{|k|}$ which represents the collisionless dissipative effects as pointed out by Hammett and Perkins in Ref. [2]. Let us recall, that in collisionless plasmas, the dissipation mechanisms are due to the resonant wave-particle interactions. From Figs. 1(a) and 3(a), it appears that $\mathrm{Re}(K_T)$ and $\mathrm{Re}(\eta)$ decrease as ξ increases. This can be explained by the number of resonant particles, which depends on the distribution function of the background plasma given here by the global *MBJ* distribution function. Indeed, for a given value of z_0, the number of the resonant particles decreases when ξ increases. Thus, the energy transfer from waves to plasma particles is less efficient and the dissipative effects are therefore less important. On the other hand, these dissipative coefficients increase continuously with the relativistic effects. This can be explained by the modification of the MBJ distribution function with respect to the temperature. The number of resonant particles is strongly affected by the change of the distribution function. In particular, in the ultrarelativistic limit ($z_0 \ll 1$), the plasma particles have velocities that are close to the speed of light. As a consequence the number of resonant particles is particularly important.

For the off-diagonal coefficients, α_V and α_T, (Figs. 2 and 4), we also note that the Onsager symmetry verified in the stationary case [18] is no longer valid in the nonstationary case. This can be explained by the fact that the fluid velocity that defines the coefficient of convective flow is not a thermodynamic force. In addition, both the real and the imaginary parts of the transport coefficients vanish for large ξ values, except $\mathrm{Re}(\alpha_V)$ which tends to a constant negative value when the relativistic effects are not negligible.

B. Collisionless hydrodynamic equations

The transport coefficients given by Eqs. (33) and (34) constitute the closure relations for the linear fluid equations. Considering the same assumptions ($\vec{B}=0$ and $V \ll c$) and the same geometry of the problem [the plasma inhomogeneity is along the x axis, $\vec{V}=V(x,t)\vec{x}$ and $\vec{E}=E(x,t)\vec{x}$] as in Sec. II, the linear forms of the hydrodynamic Eqs. (3)–(5) in the Fourier space (ω,k) read,

$$-i\omega\tilde{n} + n_0 ik\tilde{V} = 0, \tag{39}$$

$$-i\omega n_0 m G(z_0)\tilde{V} = -n_0 ik\tilde{T} - T_0 ik\tilde{n} - ik\tilde{\Pi}_{xx} + n_0 q\tilde{E} + \frac{i\omega}{c^2}\tilde{q}_x, \tag{40}$$

$$-i\omega\{z_0^2[1-G^2(z_0)] + 5z_0G(z_0) - 1\}n_0\tilde{T} + n_0T_0ik\tilde{V} = ik\tilde{q}_x, \tag{41}$$

where we recall that the equilibrium state is defined by the density n_0, the temperature T_0, a plasma at rest ($\vec{V}_0=0$), and a zero electric field ($\vec{E}_0=0$). In Eqs. (40) and (41) the moments $\tilde{q}_x$ and $\tilde{\Pi}_{xx}$ are given by Eqs. (33) and (34). To be self-consistent, the set of Eqs. (39)–(41) have to be coupled with the Maxwell equations which provide the wave equation for the electric field $\tilde{E}$.

Benchmarking the model against linear kinetic theory is an important step in verifying the accuracy and reliability of the closure relations for the hydrodynamic equations. At this end, we use the plasma response function $R = ikT_0\tilde{n}/qn_0\tilde{E}$. We compare the response function obtained from the relativistic Vlasov equation with that obtained from the hydrodynamic Eqs. (39)–(41).

Calculating the linearized form of Eq. (1) with respect to the global equilibrium, performing the Fourier transforms $(x \leftrightarrow k, t \leftrightarrow \omega)$ and integrating upon the momentum space, we readily deduce the kinetic response function,

$$R_{kin}(\xi,z_0) = 1 - \frac{z_0\xi^2}{K_2(z_0)}\frac{\partial^2}{\partial z_0^2}\frac{1}{z_0}I_{kin}(\xi,z_0).$$

$$\text{where } I_{kin}(\xi,z_0) = \int_1^\infty \frac{\exp(-z_0\gamma)d\gamma}{(\gamma^2-1)^{1/2}[1+\gamma^2(\xi^2-1)]}. \tag{42}$$

Following the Landau prescription [22], the integral $I_{kin}(\xi,z_0)$ must be continued analytically by considering $\xi=\frac{\omega}{kc}$ as a complex variable with a complex frequency ω. The path of integration along the real γ axis must be deformed around the pole by passing below. For $\xi<1$, the integrand exhibits a simple pole and applying the Plemelj integrals, it results

$$I_{kin}(\xi,z_0) = \frac{1}{(1-\xi^2)^{1/2}} PP\int_1^\infty \frac{\exp(-z_0\gamma)d\gamma}{(\gamma^2-1)^{1/2}[1+\gamma(1-\xi^2)^{1/2}][\gamma - 1/(1-\xi^2)^{1/2}]} - i\frac{\pi}{2\xi}\exp[-z_0/(1-\xi^2)^{1/2}], \tag{43}$$

where PP means the Cauchy principal value. On the other hand, from Eqs. (33), (34), and (39)–(41), we calculate the following expression for the fluid response function:

$$\begin{aligned} R_f(\xi,z_0) = \{&K_T + i\xi[1 - z_0G(z_0)]\}/[\{K_T + i\xi[1 - z_0G(z_0)]\} \\ &\times\{1 - \xi^2[z_0G(z_0) + \alpha_V] + i\xi z_0\eta\} \\ &- i\xi(1+\alpha_V)(1+\alpha_T + i\xi K_T)]. \end{aligned} \tag{44}$$

We note that expression (44) should be a mathematical rewriting of expression (42). To check this, in Fig. 5, we display the real and the imaginary parts of $R_{kin}(\xi,z_0)$ and $R_f(\xi,z_0)$ as a function of ξ for three values of the relativistic parameter z_0. It appears that the two functions coincide very accurately in the whole ξ-range and this whatever the value of z_0. To our knowledge the relativistic collisionless fluid response function (44) is a new result in the literature. It is important to note that Eq. (44) does not involve integrals with poles, as it is the case in the kinetic response function (42), which are not easy to calculate numerically.

IV. SUMMARY

In this work the linear fluid transport equations for relativistic and collisionless plasmas are exactly derived. At this end, the relativistic Vlasov equation perturbed with respect to the global equilibrium has been analytically solved. The mathematical method for solving the Vlasov equation is based on the expansion of the distribution function on the Legendre polynomial basis and on the use of the continued fractions to incorporate the contributions from all the Legendre modes. In addition, the invariance properties of the Vlasov equation have been ensured by the use of the projection operators. The nonstationary distribution function has been explicitly computed in the Fourier space. The transport coefficients corresponding to the generalized heat flux and stress tensor have been deduced. These coefficients constitute exact closure relations for the fluid transport equations to describe small perturbation in relativistic and collisionless plasmas. In addition the fluid response function was calculated and it is shown that it corresponds exactly to the kinetic one. The computation of the fluid response function is an application of the present theory to the dielectric properties of the relativistic plasmas and it can be used to calculate the dispersion relations of eigenmodes in relativistic plasmas (see, e.g., Refs. [14,26]). This application will be the subject of a future work.

APPENDIX A: COMPUTATION OF THE ISOTROPIC, THE FIRST AND THE SECOND ANISTROPIC DISTRIUBUTION FUNCTIONS

This appendix is devoted to the computation of the components $\tilde{f}_0$, $\tilde{f}_1$, and $\tilde{f}_2$ in terms of the electric field $\tilde{E}$, the gradients of the density $\tilde{n}$, the temperature $\tilde{T}$ and the flow velocity $\tilde{V}$. The first step is to express $\tilde{f}_1$ and $\tilde{f}_2$ with respect to $\tilde{f}_0$ and to the source terms $\tilde{E}$, $\tilde{n}$, $\tilde{T}$, and $\tilde{V}$. From Eqs. (15), (16), (23), and (24) we deduce

$$\tilde{f}_1 = -\frac{ikc}{\sqrt{3}}\frac{(\gamma^2-1)^{1/2}}{\gamma}F_1\tilde{f}_0 + \frac{1}{\sqrt{3}}\frac{q}{m}z_0\mu_0\frac{(\gamma^2-1)^{1/2}}{\gamma}F_1\exp(-z_0\gamma)\frac{\tilde{E}}{c} - \frac{4}{15\sqrt{3}}z_0\mu_0k^2c^2\frac{(\gamma^2-1)^{3/2}}{\gamma^2}F_1F_2\exp(-z_0\gamma)\frac{\tilde{V}}{c} + \frac{i\omega}{\sqrt{3}}z_0\mu_0(\gamma^2-1)^{1/2}F_1\exp(-z_0\gamma)\frac{\tilde{V}}{c} \tag{A1}$$

and

$$\tilde{f}_2 = -\frac{2}{3\sqrt{5}}k^2c^2\frac{(\gamma^2-1)}{\gamma^2}F_1F_2\tilde{f}_0 - \frac{2}{3\sqrt{5}}\frac{q}{m}z_0\mu_0\frac{(\gamma^2-1)}{\gamma^2}F_1F_2\exp(-z_0\gamma)ik\tilde{E}. \tag{A2}$$

In addition, from Eqs. (22), (24), (A1), and (A2), we obtain the following expression for $\tilde{f}_0$:

$$\tilde{f}_0 = -i\omega F_0\tilde{f}_{MBJ} - \frac{1}{3}\frac{q}{m}z_0\mu_0\frac{(\gamma^2-1)}{\gamma^2}F_0F_1\exp(-z_0\gamma)ik\tilde{E} - \frac{i\omega}{3}z_0\mu_0\frac{(\gamma^2-1)}{\gamma}F_0F_1\exp(-z_0\gamma)ik\tilde{V} - \frac{z_0\mu_0}{3}\frac{(\gamma^2-1)}{\gamma}F_0\exp(-z_0\gamma)ik\tilde{V} + \frac{4}{45}z_0\mu_0k^2c^2\frac{(\gamma^2-1)^2}{\gamma^3}F_0F_1F_2\exp(-z_0\gamma)ik\tilde{V} + \frac{F_0}{I_1I_3-I_2I_4}[(I_3-\gamma I_4)+(1-G(z_0))(I_2-\gamma I_1)]I_1\mu_0\exp(-z_0\gamma)ik\tilde{V} - \frac{ikc}{\sqrt{3}}\frac{F_0}{I_1I_3-I_2I_4}(I_2-\gamma I_1)\exp(-z_0\gamma)M_1^{0,1,1} + \frac{ikc}{\sqrt{3}}\frac{F_0}{I_1I_3-I_2I_4}(I_3-\gamma I_4)\exp(-z_0\gamma)M_1^{0,0,1}. \tag{A3}$$

The solution of Eq. (A3) needs to compute the moments $M_1^{0,0,1}$ and $M_1^{0,1,1}$ as a functions of $\tilde{E}$, $\tilde{n}$, $\tilde{T}$, and $\tilde{V}$. At this end, we consider the conservative properties of the collision Krook operator, i.e., the conservation of the number of particles and the energy given respectively by the following relations:

$$\int \nu(\tilde{f}_{MBJ}-\tilde{f})d\vec{p} = 0 \tag{A4}$$

and

$$\int mc^2(\gamma-1)\nu(\tilde{f}_{MBJ}-\tilde{f})d\vec{p} = 0. \tag{A5}$$

These conditions impose to the distribution function $\tilde{f}$ to have the same density and the same kinetic energy, as the perturbed *MBJ* distribution function. Using the expansion [Eq. (13)] for $\tilde{f}$, the conservative laws [Eqs. (A4) and (A5)] can be rewritten as

$$M_0^{1,0,1/2} = M_{MBJ}^{1,0,1/2} \tag{A6}$$

and

$$M_0^{1,1,1/2} = M_{MBJ}^{1,1,1/2}. \tag{A7}$$

Multiplying Eq. (A3) by $\gamma(\gamma^2-1)$ and $\gamma(\gamma-1)(\gamma^2-1)$, respectively, integrating over γ and using the relations (A6) and (A7) we obtain after some algebra,

$$\begin{aligned} M_1^{0,0,1} = {} & \frac{-3i}{\sqrt{3}kcD}\mu_0[i\omega D + 2I_1I_2I^{2,1/2} - (I_1)^2I^{3,1/2} - (I_2)^2I^{1,1/2}]\frac{\tilde{n}}{n_0} - \frac{3i}{\sqrt{3}kcD}\mu_0[i\omega((1-z_0G(z_0))I_1 - z_0I_2)D + (I_4(1-z_0G(z_0)) \\ & - z_0I_3)(I_1I^{2,1/2} - I_2I^{1,1/2})]\frac{\tilde{T}}{T_0} + \frac{z_0\mu_0 q}{\sqrt{3}mD}[I_1(I^{2,1/2}J^{0,3/2} - I^{3,1/2}J^{-1,3/2}) + I_2(I^{2,1/2}J^{-1,3/2} - I^{1,1/2}J^{0,3/2})]\frac{\tilde{E}}{c} \\ & - \frac{iz_0\mu_0}{\sqrt{3}kcD}\Big\{3I_1D + i\omega[I_1(I^{2,1/2}J^{1,3/2} - I^{3,1/2}J^{0,3/2}) + I_2(I^{2,1/2}J^{0,3/2} - I^{1,1/2}J^{1,3/2})] + I_1(I^{2,1/2}I^{1,3/2} - I^{3,1/2}I^{0,3/2}) + I_2(I^{2,1/2}I^{0,3/2} \\ & - I^{1,1/2}I^{1,3/2}) - \frac{4}{15}kc^2[I_1(I^{2,1/2}L^{-1,5/2} - I^{3,1/2}L^{-2,5/2}) + I_2(-I^{2,1/2}L^{-2,5/2} + I^{1,1/2}L^{-1,5/2})]\Big\}ik\tilde{V} \end{aligned} \tag{A8}$$

and

$$\begin{aligned} M_1^{0,1,1} = {} & \frac{3i}{\sqrt{3}kcD}\mu_0[I_3(I_2I^{1,1/2} - I_1I^{2,1/2}) + I_4(I_1I^{3,1/2} - I_2I^{2,1/2}) - i\omega I_4D]\frac{\tilde{n}}{n_0} + \frac{3i\mu_0}{\sqrt{3}kcD}[(I_4(1-z_0G(z_0)) + z_0I_3)(I_3I^{1,1/2} - I_4I^{2,1/2}) \\ & - i\omega(I_4(1-z_0G(z_0)) + z_0I_3)D]\frac{\tilde{T}}{T_0} + \frac{q}{m}\frac{z_0\mu_0}{\sqrt{3}D}[I_3(I^{2,1/2}J^{-1,3/2} - I^{1,1/2}J^{0,3/2}) + I_4(I^{2,1/2}J^{0,3/2} - I^{3,1/2}J^{-1,3/2})]\frac{\tilde{E}}{c} \\ & + \frac{z_0\mu_0 i}{\sqrt{3}kcD}\Big\{-(3/z_0)I_1(1-G(z_0))D + I_3(I^{1,1/2}I^{1,3/2} - I^{2,1/2}I^{0,3/2}) + I_4(I^{3,1/2}I^{0,3/2} - I^{2,1/2}I^{1,3/2}) + i\omega[J^{1,3/2}(I_3I^{1,1/2} - I_4I^{2,1/2}) \\ & - J^{0,3/2}(I_3I^{2,1/2} - I_4I^{3,1/2})] - \frac{4}{15}kc^2[I_3(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2}) + I_4(I^{3,1/2}L^{-2,5/2} - I^{2,1/2}L^{-1,5/2})]\Big\}ik\tilde{V} \end{aligned} \tag{A9}$$

where we used the notation:$D=[(I^{2,1/2})^2 - I^{3,1/2}I^{1,1/2}]$, $I^{i,j}=\int_1^\infty F_0\gamma^i(\gamma^2-1)^j\exp(-z_0\gamma)d\gamma$, $J^{i,j}=\int_1^\infty F_0F_1\gamma^i(\gamma^2-1)^j\exp(-z_0\gamma)d\gamma$ and $L^{i,j}=\int_1^\infty F_0F_1F_2\gamma^i(\gamma^2-1)^j\exp(-z_0\gamma)d\gamma$. Substituting Eqs. (A8) and (A9) into Eq. (A3) we find the desired expression for $\tilde{f}_0$ in terms of, $\tilde{n}$, $\tilde{T}$, $\tilde{E}$ and $\tilde{V}$,

$$\begin{aligned} \tilde{f}_0 = {} & \frac{F_0\mu_0\exp(-z_0\gamma)}{D}[(I_2I^{2,1/2} - I_1I^{3,1/2}) + \gamma(I_1I^{2,1/2} - I_2I^{1,1/2})]\frac{\tilde{n}}{n_0} + \frac{F_0\mu_0\exp(-z_0\gamma)}{D}[((1-z_0G(z_0))I_4 + z_0I_3)(I^{2,1/2} - \gamma I^{1,1/2})]\frac{\tilde{T}}{T_0} \\ & + \frac{q}{m}\frac{z_0\mu_0}{3}\frac{F_0\exp(-z_0\gamma)}{D}\left[-\left(\frac{\gamma^2-1}{\gamma^2}\right)F_1D + (I^{2,1/2}J^{0,3/2} - I^{3,1/2}J^{-1,3/2}) + \gamma(I^{2,1/2}J^{-1,3/2} - I^{1,1/2}J^{0,3/2})\right]ik\tilde{E} \\ & + \frac{i\omega z_0\mu_0}{3}\frac{F_0\exp(-z_0\gamma)}{D}\left[-\left(\frac{\gamma^2-1}{\gamma}\right)F_1D + (I^{2,1/2}J^{1,3/2} - I^{3,1/2}J^{0,3/2}) + \gamma(I^{2,1/2}J^{0,3/2} - I^{1,1/2}J^{1,3/2})\right]ik\tilde{V} \\ & + \frac{z_0\mu_0}{3}\frac{F_0\exp(-z_0\gamma)}{D}\left[-\left(\frac{\gamma^2-1}{\gamma}\right)D + (I^{2,1/2}I^{1,3/2} - I^{3,1/2}I^{0,3/2}) + \gamma(I^{2,1/2}I^{0,3/2} - I^{1,1/2}I^{1,3/2})\right]ik\tilde{V} \\ & + \frac{4k^2c^2z_0\mu_0}{45}\frac{F_0\exp(-z_0\gamma)}{D}\left[\left(\frac{(\gamma^2-1)^2}{\gamma^3}\right)F_1F_2D - (I^{2,1/2}L^{-1,5/2} - I^{3,1/2}L^{-2,5/2}) + \gamma(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2})\right]ik\tilde{V}. \end{aligned} \tag{A10}$$

The first and the second anisotropic distribution functions can be readily deduced from Eqs. (A1), (A2), and (A10),

$$f_1 = -\frac{ikc}{\sqrt{3}}\frac{F_0F_1}{D}\mu_0 \exp(-z_0\gamma)\frac{(\gamma^2-1)^{1/2}}{\gamma}[I_2I^{2,1/2} - I_1I^{3,1/2} + \gamma(I_1I^{2,1/2} - I_2I^{1,1/2})]\frac{\tilde{n}}{n_0}$$
$$+\frac{1}{\sqrt{3}}\frac{q}{m}\frac{z_0\mu_0\exp(-z_0\gamma)}{D}\left\{\frac{\gamma}{(\gamma^2-1)^{1/2}}(1+i\omega F_0)[(I^{2,1/2}J^{0,3/2} - I^{3,1/2}J^{-1,3/2}) + \gamma(I^{2,1/2}J^{-1,3/2} - I^{1,1/2}J^{0,3/2})]\right.$$
$$\left.-i\omega F_1F_0\frac{(\gamma^2-1)^{1/2}}{\gamma}D\right\}\frac{\tilde{E}}{c} - \frac{ikc}{\sqrt{3}}\frac{F_0F_1}{D}\mu_0\exp(-z_0\gamma)[((1-z_0G(z_0))I_4 + z_0I_3)(I^{2,1/2} - \gamma I^{1,1/2})]\frac{\tilde{T}}{T_0}$$
$$-\frac{ikc}{3\sqrt{3}}\frac{F_0F_1}{D}z_0\mu_0\exp(-z_0\gamma)\frac{(\gamma^2-1)^{1/2}}{\gamma}\left\{-\left(\frac{\gamma^2-1}{\gamma}\right)D + \gamma(I^{2,1/2}I^{0,3/2} - I^{1,1/2}I^{1,3/2}) + I^{2,1/2}I^{1,3/2} - I^{3,1/2}I^{0,3/2}\right.$$
$$-i\omega\left[\left(\frac{\gamma^2-1}{\gamma}\right)DF_1 - I^{2,1/2}J^{1,3/2} + I^{3,1/2}J^{0,3/2} - \gamma(I^{2,1/2}J^{0,3/2} - I^{1,1/2}J^{1,3/2})\right]$$
$$\left.+\frac{4k^2c^2}{15}\left[\frac{1}{\gamma}\left(\frac{\gamma^2-1}{\gamma}\right)^2DF_1F_2 + \gamma(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2}) - I^{2,1/2}L^{-1,5/2} + I^{3,1/2}L^{-2,5/2}\right]\right\}ik\tilde{V}$$
$$-\frac{4k^2c^2}{15\sqrt{3}}F_1F_2\frac{(\gamma^2-1)^{3/2}}{\gamma^2}z_0\mu_0\exp(-z_0\gamma)\frac{\tilde{V}}{c} + \frac{i\omega}{\sqrt{3}}F_1(\gamma^2-1)^{1/2}z_0\mu_0\exp(-z_0\gamma)\frac{\tilde{V}}{c} \quad \text{(A11)}$$

and

$$\tilde{f}_2 = -\frac{2k^2c^2}{3\sqrt{5}}\frac{F_0F_1F_2\mu_0\exp(-z_0\gamma)}{D}\left(\frac{\gamma^2-1}{\gamma^2}\right)[I_2I^{2,1/2} - I_1I^{3,1/2} + \gamma(I_1I^{2,1/2} - I_2I^{1,1/2})]\frac{\tilde{n}}{n_0} - \frac{2k^2c^2}{3\sqrt{5}}\frac{F_0F_1F_2}{D}\mu_0\exp(-z_0\gamma)\frac{(\gamma^2-1)}{\gamma^2}[((1$$
$$-z_0G(z_0))I_4 + z_0I_3)(I^{2,1/2} - \gamma I^{1,1/2})]\frac{\tilde{T}}{T_0} - \frac{2}{3\sqrt{5}}\frac{q}{m}z_0\mu_0\frac{(\gamma^2-1)}{\gamma^2}\exp(-z_0\gamma)\left[F_1F_2 - \frac{1}{3}\frac{k^2c^2}{D}F_0F_1F_2\left[\frac{(\gamma^2-1)}{\gamma^2}DF_1 + (I^{3,1/2}J^{-1,3/2}\right.\right.$$
$$\left.\left.-I^{2,1/2}J^{0,3/2}) + \gamma(I^{1,1/2}J^{0,3/2} - I^{2,1/2}J^{-1,3/2})\right]\right]ik\tilde{E} - \frac{2k^2c^2}{9\sqrt{5}}\frac{F_0F_1F_2}{D}z_0\mu_0\exp(-z_0\gamma)\frac{(\gamma^2-1)}{\gamma^2}i\omega[(\mathrm{I}^{2,1/2}J^{1,3/2} - \mathrm{I}^{3,1/2}J^{0,3/2})$$
$$+\gamma(\mathrm{I}^{2,1/2}J^{0,3/2} - \mathrm{I}^{1,1/2}J^{1,3/2})]ik\tilde{V} - \frac{2k^2c^2}{9\sqrt{5}}\frac{F_0F_1F_2}{D}z_0\mu_0\exp(-z_0\gamma)\frac{(\gamma^2-1)}{\gamma^2}[(I^{2,1/2}I^{1,3/2} - I^{3,1/2}I^{0,3/2}) + \gamma(I^{2,1/2}I^{0,3/2} - I^{1,1/2}I^{1,3/2})$$
$$+\frac{4}{15}[(I^{3,1/2}L^{-2,5/2} - I^{2,1/2}L^{-1,5/2}) + \gamma(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2})]ik\tilde{V} \quad \text{(A12)}$$

APPENDIX B: EXPRESSIONS OF THE DIMENSIONLESS TRANSPORT COEFFICIENTS

In this appendix we present the expressions of the dimensionless generalized transport coefficients. First, we give the coefficients Q_E, Q_T Q_V, Π_E, Π_T, and Π_V defined from Eqs. (29) and (30) of the components of the heat flux $\tilde{q}_x$ and of the stress tensor $\tilde{\Pi}_{xx}$, respectively. After some algebra we obtain from Eqs. (29) and (A11),

$$Q_E = \frac{z_0^2}{3K_2(z_0)}\frac{kc}{D}[I_1(J^{1,3/2}I^{2,1/2} - J^{0,3/2}I^{3,1/2}) + I_2(J^{0,3/2}I^{2,1/2} - J^{1,3/2}I^{1,1/2})], \quad \text{(B1)}$$

$$Q_T = -\frac{z_0^2}{3K_2(z_0)}\frac{kc}{D}[((1-z_0G(z_0))I_4 + z_0I_3)(J^{0,3/2}I^{2,1/2} - J^{1,3/2}I^{1,1/2})], \quad \text{(B2)}$$

$$Q_V = \frac{z_0^3}{9K_2(z_0)}\frac{k^2c^2}{D}\left\{(I^{0,3/2} + i\xi J^{0,3/2})(I^{2,1/2}J^{1,3/2} - I^{3,1/2}J^{0,3/2}) + (I^{1,3/2} + i\xi J^{1,3/2})(I^{2,1/2}J^{0,3/2} - I^{1,1/2}J^{1,3/2}) + \frac{4}{15}k^2c^2[J^{0,3/2}(I^{3,1/2}L^{-2,5/2} - I^{2,1/2}L^{-1,5/2}) + J^{1,3/2}(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2})] - \frac{9K_2(z_0)G(z_0)}{z_0^3}D\right\}, \quad \text{(B3)}$$

and from Eqs. (30) and (A12)

$$\Pi_E = -\frac{4z_0^2}{45K_2(z_0)}\frac{k^2c^2}{D}[I_1(L^{-1,5/2}I^{2,1/2} - L^{-2,5/2}I^{3,1/2}) + I_2(L^{-2,5/2}I^{2,1/2} - L^{-1,5/2}I^{1,1/2})], \quad \text{(B4)}$$

$$\Pi_T = -\frac{4z_0^2}{45K_2(z_0)}\frac{k^2c^2}{D}[((1-z_0G(z_0))I_4 + z_0I_3)(L^{-2,5/2}I^{2,1/2} - L^{-1,5/2}I^{1,1/2})], \quad \text{(B5)}$$

$$\Pi_V = -\frac{4z_0^3}{135K_2(z_0)}\frac{k^3c^3}{D}\Bigg\{ i\xi[L^{-2,5/2}(I^{2,1/2}J^{1,3/2} - I^{3,1/2}J^{0,3/2}) + L^{-1,5/2}(I^{2,1/2}J^{0,3/2} - I^{1,1/2}J^{1,3/2})] + L^{-2,5/2}(I^{2,1/2}I^{1,3/2} - I^{3,1/2}I^{0,3/2}) + L^{-1,5/2}(I^{2,1/2}I^{0,3/2} - I^{1,1/2}I^{1,3/2}) + \frac{4}{15}k^2c^2[L^{-2,5/2}(I^{3,1/2}L^{-2,5/2} - I^{2,1/2}L^{-1,5/2}) + L^{-1,5/2}(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2})]\Bigg\}. \quad \text{(B6)}$$

Now, we give the expressions of the coefficients Γ_E, Γ_T et Γ_V defined from the definition of the mean value of the random velocity [Eq. (32)]. Substituting Eq. (A11) into Eq. (32), we obtain

$$\Gamma_E = \frac{z_0}{3K_2(z_0)}\frac{kc}{D}[I_1(J^{0,3/2}I^{2,1/2} - J^{-1,3/2}I^{3,1/2}) + I_2(J^{-1,3/2}I^{2,1/2} - J^{0,3/2}I^{1,1/2})], \quad \text{(B7)}$$

$$\Gamma_T = -\frac{z_0}{3K_2(z_0)}\frac{kc}{D}[((1 - z_0G(z_0))I_4 + z_0I_3)(J^{-1,3/2}I^{2,1/2} - J^{0,3/2}I^{1,1/2})], \quad \text{(B8)}$$

and

$$\Gamma_V = \frac{z_0}{9K_2(z_0)}\frac{k^2c^2}{D}\Bigg\{ (I^{0,3/2} + i\xi J^{0,3/2})(J^{0,3/2}I^{2,1/2} - J^{-1,3/2}I^{3,1/2}) + (I^{1,3/2} + i\xi J^{1,3/2})(J^{-1,3/2}I^{2,1/2} - J^{0,3/2}I^{1,1/2}) + \frac{4}{15}k^2c^2[J^{-1,3/2}(I^{3,1/2}L^{-2,5/2} - I^{2,1/2}L^{-1,5/2}) + J^{0,3/2}(I^{1,1/2}L^{-1,5/2} - I^{2,1/2}L^{-2,5/2})] - \frac{9K_2(z_0)D}{z_0}\Bigg\}. \quad \text{(B9)}$$

[1] H. Grad, Phys. Fluids **6**, 147 (1963).
[2] G. W. Hammett and F. W. Perkins, Phys. Rev. Lett. **64**, 3019 (1990).
[3] A. Bendib, G. Matthieusent, and F. Bouzid, Phys. Plasmas **9**, 35 (2002).
[4] D. I. Dzhavakhishvili and N. L. Tsintsadze, Zh. Eksp. Teor. Fiz **64**, 1314 (1973) [Sov. Phys. JETP **37**, 666 (1973)].
[5] S. I. Braginskii, in *Transport Processes in a Plasma*, edited by M. A. Leontovich, Review of Plasma Physics Vol. 1, (Consultants Bureau, New York, 1965), p. 205.
[6] V. Yu. Bychenkov, W. Rozmus, V. T. Tikhonchuk, and A. V. Brantov, Phys. Rev. Lett. **75**, 4405 (1995).
[7] B. A. Shadwick, G. M. Tarkenton, and E. Esarey, Phys. Rev. Lett. **93**, 175002 (2004).
[8] B. A. Shadwick, G. M. Tarkenton, E. Esarey, and C. B. Schroeder, Phys. Plasmas **12**, 056710 (2005).
[9] R. D. Hazeltine and S. M. Mahajan, Phys. Plasmas **9**, 3341 (2002).
[10] J. M. TenBarge, R. D. Hazeltine, and S. M. Mahajan, Phys. Plasmas **15**, 062112 (2008).
[11] J. J. Podesta, Phys. Plasmas **15**, 122902 (2008).
[12] A. B. Langdon and D. E. Hinkel, Phys. Rev. Lett. **89**, 015003 (2002).
[13] D. J. Strozzi, E. A. Williams, A. B. Langdon, and A. Bers, Phys. Plasmas **14**, 013104 (2007).
[14] A. Bers, I. P. Shkarosfsky, and M. Shoucri, Phys. Plasmas **16**, 022104 (2009).
[15] K. C. Tzeng and W. B. Mori, Phys. Rev. Lett. **81**, 104 (1998).
[16] A. Pukhov, Z.-M. Sheng, and J. Meyer-Ter-Vehn, Phys. Plasmas **6**, 2847 (1999).
[17] Z.-M. Sheng, K. Mima, Y. Sotenku, and K. Nishihara, Phys. Rev. E **61**, 4362 (2000).
[18] K. Bendib-Kalache, A. Bendib, and G. Matthieussent, Phys. Plasmas **11**, 4254 (2004).
[19] V. S. Tsypin, R. M. O. Galvao, I. C. Nascimento, N. L. Tsintsadze, L. N. Tsintsadze, M. Tendler, and J. P. Net, Phys. Rev. E **60**, 4754 (1999).
[20] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, London, 1962).
[21] F. Jüttner, Ann. Phys. **34**, 856 (1911).
[22] L. D. Landau, J. Phys. (Moscow) **10**, 25 (1946).
[23] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon, New York, 1981).
[24] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).
[25] K. Bendib and A. Bendib, Phys. Plasmas **6**, 1500 (1999).
[26] R. Schlickeiser, Phys. Plasmas **11**, 5532 (2004).