Discrete and mesoscopic regimes of finite-size wave turbulence

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Bounding volume results in discreteness of eigenmodes in wave systems. This leads to a depletion or complete loss of wave resonances (three-wave, four-wave, etc.), which has a strong effect on *wave turbulence* (WT) i.e., on the statistical behavior of broadband sets of weakly nonlinear waves. This paper describes three different regimes of WT realizable for different levels of the wave excitations: *discrete, mesoscopic and kinetic WT*. *Discrete WT* comprises chaotic dynamics of interacting wave "clusters" consisting of discrete (often finite) number of connected resonant wave triads (or quarters). *Kinetic WT* refers to the infinite-box theory, described by well-known wave-kinetic equations. *Mesoscopic WT* is a regime in which either the discrete and the kinetic evolutions alternate or when none of these two types is purely realized. We argue that in mesoscopic systems the wave spectrum experiences a *sandpile* behavior. Importantly, the mesoscopic regime is realized for a *broad range* of wave amplitudes which typically spans over several orders on magnitude, and not just for a particular intermediate level.

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I. INTRODUCTION

Dispersive waves play a crucial role in a vast range of physical applications, from quantum to classical systems, from microscopic to astrophysical scales. For example, Kelvin waves propagating on quantized vortex lines provide an essential mechanism of turbulent energy cascades in quantum turbulence in cryogenic helium [1–6]; water waves aid the momentum and energy transfers from wind to ocean [7]; internal waves on density stratifications and inertial waves due to rotation are important in turbulence behavior and mixing in planetary atmospheres and oceans [8–10]; planetary Rossby waves are important for the weather and climate evolutions [11]; and Alfven waves are ubiquitous in turbulence of solar wind and interstellar medium [12–17]. More often than not, nonlinear interaction of different wave modes is important in these and other applications, and there has been a significant amount of work done in the past to describe evolution of such interacting wave systems. If the number of excited modes is large, they experience random evolutions, which must be described by a statistical theory. *Weak wave turbulence* (WT) refers to such a statistical theory for weakly nonlinear dispersive waves in unbounded domains [18]. This approach was initiated by Peierls in 1929 to describe phonons in anharmonic crystals [19], and it was reinvigorated in 1960s in plasma physics [20–22] and in the theory of water waves [23]. By now, it has been applied to description of a great variety of physical phenomena, from synoptic Rossby waves [24–27] to magneto-hydrodynamic turbulence [14–16], to acoustic waves [28], to waves in stratified [8,9] and rotating fluids [10], and many other physical wave systems.

On the other hand, it has become increasingly clear that in the majority laboratory experiments and numerical simulations of nonlinear dispersive wave systems the discreteness of the wave-number space due to a finite size is a crucially important factor which causes the system behave differently from the predictions of the classical theory of wave turbulence based on the continuous (infinite domain) limit [29–37]. Moreover, similar behavior often occurs in nature when waves are bounded, e.g., for planetary Rossby waves bounded by the finite planet radius [38].

Description of transition from regular to random regimes and characterization of the intermediate states where both regular and random wave motions are present and mutually interconnected, is an intriguing and challenging problem. Such intermediate states where the number of waves is big and yet the discreteness of the wave number space still remains important are called discrete and mesoscopic wave turbulence.

II. WEAKLY INTERACTING WAVES

A. Normal modes of linearized problem

An evolution equation is called *dispersive* if its linear part has wavelike solutions $\psi(\boldsymbol{r},t)$ that depend on the coordinate in the d-dimensional physical space, $\boldsymbol{r}\in\mathbb{R}^d$ and time t as follows:

$$\psi(\boldsymbol{r},t) = A_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \text{c.c.} = |A_{\boldsymbol{k}}|\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\varphi), \quad (1a)$$

where "c.c." means "complex conjugate." Here $A_{\boldsymbol{k}} = |A_{\boldsymbol{k}}|\exp(i\varphi)\in\mathbb{C}$ is a constant wave amplitude, φ is the wave phase, $\boldsymbol{k}\in\mathbb{R}^d$ is a wave vector and wave frequency $\omega \equiv \omega(\boldsymbol{k})\in\mathbb{R}$ is such that $|\partial^2\omega/\partial k_i\partial k_j| \not\equiv 0$, where k_i and k_j are components of $\boldsymbol{k}$. Physically, the latter condition means that wave packets with different wave-numbers propagate at different speeds so that localized initial data would disperse (spread) in space.

In bounded systems, the set of normal wave modes becomes discrete. For waves in a periodic d-dimensional cube with side L, the normal modes are given by (A) with a discrete set of wave numbers $\boldsymbol{k}=\frac{2\pi \boldsymbol{l}}{L}$, where $\boldsymbol{l}\in\mathbb{Z}^d$. For different boundary conditions, normal modes of the linearized problem may differ from the propagating plane waves Eq. (1a). For instance, zero boundary conditions in a rectangular box typically (but not always) lead to standing waves,

$$\psi(\boldsymbol{r},t) = |A_{\boldsymbol{k}}|\sin(\boldsymbol{k}\cdot\boldsymbol{r} + \varphi_{\text{sp}})\sin(\omega t + \varphi_{\text{t}}), \tag{1b}$$

where φ_{sp} and φ_{t} are the space and time phases correspondingly. A more complex form of the normal mode is given by ocean planetary motions in a rectangular domain $[0,L_x]\times[0,L_y]$ with zero boundary conditions, see, e.g. [39]:

$$\psi(\boldsymbol{r},t) = |A_{\boldsymbol{k}}|\sin\left(\pi\frac{mx}{L_x}\right)\sin\left(\pi\frac{ny}{L_y}\right)\sin\left(\frac{\beta}{2\omega}x + \omega t + \varphi_{\text{t}}\right), \tag{1c}$$

where $\underline{m, \quad n\in\mathbb{N}}$ are integers and $\omega = \beta/[2\pi\sqrt{(m/L_x)^2+(n/L_y)^2}]$ with a constant β being the gradient of the Coriolis parameter.

B. Equation of motion

A rather general class of nondissipative nonlinear waves can be described within the framework of the classical Hamiltonian approach. This means that after a proper change in variables the motion equation in natural variables (fluid velocity, electrical field, density variations, etc.) can be presented in the universal form of canonical Hamiltonian equations for canonical variables $b(\boldsymbol{r},t)$, $b^*(\boldsymbol{r},t)$, which characterize the wave amplitudes. Here "*" denotes complex conjugation. The Hamiltonian equations for the space-homogeneous systems are most conveniently written in Fourier space because it is a natural space for describing the wave solutions. Introducing the Fourier transform of $b(\boldsymbol{r},t)$ and calling it $a_{\boldsymbol{k}}\equiv a(\boldsymbol{k},t)$, the Hamiltonian equation can be written as follows [18]:

$$i\frac{da_{\boldsymbol{k}}}{dt} = \frac{\partial\mathcal{H}}{\partial a_{\boldsymbol{k}}^*}. \tag{2}$$

Hamiltonian $\mathcal{H}\equiv\mathcal{H}\{a_{\boldsymbol{k}},a_{\boldsymbol{k}}^*\}$ is usually (but not necessarily) is the energy of the wave system, expressed in the terms of the canonical variables $a_{\boldsymbol{k}}$, $a_{\boldsymbol{k}}^*$ for all allowed by the boundary conditions wave vectors $\boldsymbol{k}$. In the simplest case of a periodical box $\boldsymbol{k}=2\pi\boldsymbol{l}/L$, with wave number $\boldsymbol{l}\in\mathbb{Z}^d$ and L being the box size and d is space dimension.

For the waves of small amplitudes (for example, when the elevation of the gravity waves on the water surface is smaller then the wavelength) the Hamiltonian can be expanded in powers $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^*$:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}, \tag{3a}$$

$$\mathcal{H}_{\text{int}} = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \cdots, \tag{3b}$$

where $\mathcal{H}_j$ is a term proportional to product of j amplitudes $a_{\boldsymbol{k}}$ and the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ describes the wave coupling, as explained below. We omitted here the independent of $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^*$ part of the Hamiltonian $\mathcal{H}_0$ because it does not contribute to the motion Eq. (2). In this paper we consider only waves exited about steady equilibrium states, i.e., if absent initially, the waves must remain absent for all time, $a_{\boldsymbol{k}}=a_{\boldsymbol{k}}^*\equiv 0$. Thus, the linear Hamiltonian is zero, $\mathcal{H}_1=0$.

Expansion Eq. (3b) utilizes the smallness of the wave amplitudes, therefore, generally speaking,

$$\mathcal{H}_3 > \mathcal{H}_4 > \mathcal{H}_5 > \cdots. \tag{4a}$$

In particular cases, due to specific symmetries of a problem, the odd expansion terms vanish (e.g. for spin waves in magnetics with exchange interactions, Kelvin waves on quantum vortex lines). In these cases, instead of Eq. (4a) one has

$$\mathcal{H}_3 = \mathcal{H}_5 = \mathcal{H}_7 = \cdots = 0, \tag{4b}$$

$$\mathcal{H}_4 > \mathcal{H}_6 > \mathcal{H}_8 > \cdots. \tag{4c}$$

Three-wave interactions often dominate in wave systems with small nonlinearity, e.g., for Rossby waves in the atmosphere and ocean, capillary waves on the water surface, drift waves in plasmas, etc. On the other hand, if $\mathcal{H}_3=0$, or if three-wave resonances are forbidden (in the sense that will be clarified below) the leading nonlinear processes may be four-wave interactions. Further, there are examples of systems where the four-wave interaction is absent and the leading nonlinear process is five-wave, e.g., for one-dimensional gravity water waves [40–42], or even six order, e.g., for Kelvin waves on quantum vortex lines [3,43]. However, such higher-order wave systems are rather rare and, therefore, in this paper we will discuss three- and four-wave interactions only, which describe most of weakly interacting waves.

C. Noninteracting waves

The first physically meaningful expansion term, quadratic Hamiltonian

$$\mathcal{H}_2 = \sum_{n=1}^{\infty}\omega_{\boldsymbol{k}}|a_{\boldsymbol{k}}|^2, \tag{5a}$$

according to Eq. (2) produces a linear equation of motion,

$$i\frac{da_{\boldsymbol{k}}}{dt} = \omega_{\boldsymbol{k}}a_{\boldsymbol{k}}, \tag{5b}$$

and thus describes noninteracting waves with the dispersion relation $\omega_{\boldsymbol{k}}\equiv\omega(\boldsymbol{k})$. For waves, considered in this paper, when $\min_{a_{\boldsymbol{k}},a_{\boldsymbol{k}}^*}\{\mathcal{H}\}=\mathcal{H}_0$, $\omega_{\boldsymbol{k}}\geq 0$. Notice that $\mathcal{H}_2$ in Eq. (5a) does not have $a_{\boldsymbol{k}}a_{-\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^*a_{-\boldsymbol{k}}^*$ terms. They were removed by linear canonical transformation [known as the Bogolubov (u,v) transformation] after which $\mathcal{H}_2$ takes the fully diagonal form (5a).

D. Three-wave interactions

The first contribution to the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ is

$$\mathcal{H}_3 = \frac{1}{2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} V_{23}^1 a_1^* a_2 a_3 \delta_{23}^1 + \text{c.c.}, \tag{6a}$$

which describes the processes of decaying of single wave into two waves ($1\Rightarrow 2$ processes) or confluence of two waves into a single one ($2\Rightarrow 1$ processes). In Eq. (6) for brevity we introduced notations $a_1\equiv a_{\boldsymbol{k}_1}$, etc. and δ_{23}^1 is the Kronecker symbol, i.e., $\delta_{23}^1=1$ if and only if $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3$. Clearly, $V_{23}^1=V_{32}^1$. Generally speaking, $\mathcal{H}_3$ also includes $a_1a_2a_3$ and

$a_1^* a_2^* a_3^*$ terms that describe $3 \Leftrightarrow 0$ processes (confluence of three waves or spontaneous appearance of three waves out of vacuum). However they can be eliminated by corresponding nonlinear transformation [18] that leads to the canonical form of $\mathcal{H}_3$, presented in Eq. (6).

Hamiltonian $\mathcal{H}_2+\mathcal{H}_3$ with Eq. (2) yields the three-wave equation:

$$i\frac{da_k}{dt} = \omega_k a_k + \sum_{k_1,k_2}\left[\frac{1}{2}V_{12}^k a_1 a_2 \delta_{12}^k + V_{k2}^{1*} a_1 a_2^* \delta_{k2}^1\right]. \quad (6b)$$

Two sets of terms in the RHS of this equation have time dependence of the form $\exp[-i(\omega_2+\omega_3)t]$ and $\exp[-i(\omega_2-\omega_3)t]$, correspondingly [we used shorthand notations, $\omega_j \equiv \omega(\boldsymbol{k}_j)$]. They become important if their frequencies are close to the eigenfrequency of a_k, ω_k: $\omega_2+\omega_3 \approx \omega_k$ or $\omega_2-\omega_3 \approx \omega_k$. By relabeling the wave vectors, we can write both of these conditions in the same form as follows:

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3). \quad (7a)$$

This condition of time synchronization should be complemented by the condition of space synchronization that formally originates from the Kronecker symbols in Eq. (6b),

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3. \quad (7b)$$

Both relations (6b) are named the resonance conditions of the three-wave interactions or conditions of the *three-wave resonances*.

There exists a simple conditions for the three-wave resonance conditions to be satisfied for the power-law dispersion relations $\omega \sim k^\alpha$ (α=const). In two-dimensional (2D), it is most easily proved graphically, as suggested in [21]. Thus, it was shown that the three-wave resonance is possible if and only if $\alpha \geq 1$ for the continuous case, $\boldsymbol{k} \in \mathbb{R}^2$. Obviously, this condition becomes a necessary condition if $\boldsymbol{k}$ is restricted to discrete values due to boundary conditions.

E. Four-wave interactions

When the three-wave resonances are forbidden, one has to account for processes with weaker nonlinearity, the four-wave interactions. The canonical part of the four-wave interaction Hamiltonian,

$$\mathcal{H}_4 = \frac{1}{4}\sum_{k_1,k_2,k_3,k_4} T_{34}^{12} a_1^* a_2^* a_3 a_4 \delta_{34}^{12}, \quad (8a)$$

describes a four-wave scattering processes $2 \Leftrightarrow 2$. Terms $a_1 a_2 a_3 a_4$ and its complex conjugate describing $4 \Leftrightarrow 0$ processes can be eliminated by an appropriate nonlinear canonical transformation [18]. After that the four-wave interaction Hamiltonian takes the canonical form (8a). There also exist $1 \Leftrightarrow 3$ systems with $a_1 a_2 a_3 a_4^*$ and its complex conjugate terms in $\mathcal{H}_4$ [5,6]. They can be treated similarly, but for simplicity we omit them in the present paper.

Note that besides trivial symmetries with respect to the indexes permutations, $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, the interaction coefficient Eq. (8) has the symmetry $T_{12}^{34} = (T_{34}^{12})^*$, because the Hamiltonian has to be real, $\mathcal{H}_4 = \mathcal{H}_4^*$.

The dynamical equation for the four-wave case follows from Eq. (2) with the Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4$:

$$i\frac{da_k}{dt} = \omega_k a_k + \frac{1}{2}\sum_{k_1,k_2,k_3} T_{23}^{k1} a_1^* a_2 a_3 \delta_{23}^{k1}. \quad (8b)$$

Considering this equation similarly to Eq. (6b), one realizes that the terms in the right-hand side (RHS) of Eq. (8b) oscillate with the frequencies $\omega_2+\omega_3-\omega_1$ and becomes resonant if this combination is close to ω_k. in the other words, the condition of time synchronization (after proper renaming of the variables) takes form (9a)

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4), \quad (9a)$$

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4, \quad (9b)$$

while Eq. (9b) represents condition of space synchronization that comes from the Kronecker symbol in Eq. (8b).

F. Physical examples

In the context of the problem at the hand, a choice of physically important and methodologically illustrative Hamiltonian systems is not an easy task. The corresponding wave systems should preferably be well studied, both theoretically and experimentally (or numerically). They should be *simple enough* to be understood by the nonexperts in the area of wave turbulence and at the same time *not too simple* in order to demonstrate the main characteristics of the resonant wave systems described by different nonlinear dispersive partial differential equations (PDEs), with different number of interacting modes and different boundary conditions.

1. Surface water waves

Our first example is the system of surface water waves, with dispersion relation of the general form:

$$\omega_k = \sqrt{gk + \frac{\sigma k^3}{\rho}}, \quad (10a)$$

where g is the gravity acceleration, σ is the surface tension, and ρ is the fluid density. For small k Eq. (10a) turns into dispersion law for the gravity waves:

$$\omega_k = \sqrt{gk}, \quad (10b)$$

while for large k it is simplified to the capillary wave form

$$\omega_k = \sqrt{\frac{\sigma k^3}{\rho}}. \quad (10c)$$

In both limiting cases the dispersion law have scale-invariant form, $\omega_k \propto k^\alpha$. Notice that for the gravity waves $\alpha = \frac{1}{2} < 1$ and therefore the leading nonlinear processes are four-wave scattering $2 \Leftrightarrow 2$ with the quartets as the primary clusters, while for the capillary waves $\alpha = \frac{3}{2}$ and thus the leading nonlinear processes are three-wave interactions of $2 \Leftrightarrow 1$ type. In this case the primary clusters are triads.

Surface water waves with the general dispersion law Eq. (10a) can be described by the Hamiltonian equation of motion in canonical form (2) that turns into Eq. (6b) for the capillary waves and into Eq. (8b) for the gravity waves.

Three-wave interaction coefficient for the capillary waves reads as

$$V_{23}^{1}=\frac{i\sqrt{\omega_1\omega_2\omega_3}}{8\pi\sqrt{2\sigma}}\left[\frac{\mathcal{K}_{k_2,k_3}}{k_1\sqrt{k_2k_3}}-\frac{\mathcal{K}_{k_1,-k_2}}{k_3\sqrt{k_1k_2}}-\frac{\mathcal{K}_{k_1,-k_3}}{k_2\sqrt{k_1k_3}}\right],$$

where

$$\mathcal{K}_{k_2,k_3}=(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)+k_2k_3. \quad (11)$$

The four-wave interaction coefficient for the gravity waves is given by rather long expressions which can be found in [44].

2. Nonlinear Schrödinger model

Probably the simplest known example of the four-wave systems are waves in the nonlinear Schrödinger (NLS) model of nonlinear optical systems and Bose-Einstein condensates [45,46]. *NLS waves* have dispersion function and interaction coefficient as follows:

$$\omega_k=k^2,\quad T_{34}^{12}=1. \quad (12)$$

3. Rossby and drift waves

Another important example of wave system with dominating three-wave interaction, is *Rossby waves*, which are similar to *drift waves* in inhomogeneous plasmas. Their amplitudes can be described by the so-called barotropic vorticity equation which can be presented in the form similar to the canonical three-wave Eq. (6b), but all $\boldsymbol{k}$s taking values only in half of the Fourier space,

$$i\frac{da_{\boldsymbol{k}}}{dt}=\omega_{\boldsymbol{k}}a_{\boldsymbol{k}}+\sum_{k_{1x},k_{2x}\geq 0}\left[\frac{1}{2}V_{12}^{\boldsymbol{k}}a_1a_2\delta_{12}^{\boldsymbol{k}}+V_{\mathbf{k}2}^{1*}a_1a_2^*\delta_{\mathbf{k}2}^{1}\right],$$
$$(k_x>0). \quad (13)$$

The phase space in this case is half of the Fourier space is a result is because the original equation in the x space is for a real variable (barotropic vorticity). The difference in the Hamiltonian structure of the Rossby and capillary waves yields the difference in the form of the conservation laws and therefore in their dynamical behavior. We will discuss this later in greater detail.

Rossby waves on an infinite (or double-periodic) β plane have dispersion function [27,47]

$$\omega_{\boldsymbol{k}}=\frac{\beta\rho^2k_x}{1+\rho^2k^2}, \quad (14a)$$

where $\rho=\sqrt{gH}/f$ is the Rossby deformation radius H is the fluid layer thickness, $f=2\Omega\sin\theta$ is the Coriolis parameter, θ is the latitude angle (β-plane approximates a local region on surface of a rotating planet), Ω is the planet rotation frequency, and β is the gradient of the Coriolis parameter, $\beta=2\Omega\cos\theta/R$, and R is the radius of the planet.

In the case of zero boundary conditions in a plane rectangular domain (*oceanic Rossby waves*), the form of the eigenmode is given by Eq. (1c), corresponding dispersion function has the form

$$\omega_k=\frac{\beta L}{2\pi\sqrt{m^2+n^2}}. \quad (14b)$$

Note that this dispersion relation coincides with relation (14a) in the limit $\rho\to\infty$ taking into account that $(k_x,k_y)=(\beta/2\omega\pm\pi m/L,\pm\pi n/L)$ [which follows from Eq. (1c)]. However, the resonant mode sets are different because the resonance in $\boldsymbol{k}$ is now replaced by the resonance conditions in m an n.

One more example is *atmospheric Rossby waves*, propagated on a rotating (with angular velocity Ω) sphere. Eigenmodes in this case, $Y_\ell^m(\sin\varphi,\lambda)\exp[2im/\ell(\ell+1)t]$, are proportional to the spherical functions Y_ℓ^m, where $\ell\leq 1$ and $|m|\leq\ell$ are integers and φ and λ are latitude and longitude, correspondingly. In this case dispersion function is of the form

$$\omega_{\ell,m}=\frac{2m\Omega}{[\ell(\ell+1)]}. \quad (14c)$$

Notice that difference in the dispersion relations (14) leads to essential difference in the topology of resonant clusters and consequently to essential difference in the dynamical and statistical behavior of the systems.

For concreteness we present here the interaction coefficients of the Rossby waves in the (infinite or double-periodic) β plane [48]:

$$V_{23}^{1}=\frac{\beta\sqrt{|k_{1x}k_{2x}k_{3x}|}}{4\pi i}\left[\frac{k_{1y}}{1+\rho^2k_1^2}-\frac{k_{2y}}{1+\rho^2k_2^2}-\frac{k_{3y}}{1+\rho^2k_3^2}\right].$$

The interaction coefficients for the atmospheric Rossby waves can be found in [38,39,49] and for oceanic Rossby waves can be found in [50].

III. REGIMES OF FINITE-SIZE WAVE TURBULENCE

What happens when, due to the finite size, the number of exact resonances and active quasiresonances is depleted or absent? The finite-size effects in WT can be characterized by considering the nonlinear frequency broadening Γ (i.e., the inverse the characteristic time of nonlinear evolution) and comparing it to the frequency spacing Δ_ω between the finite-box eigenmodes. For simplicity, we will restrict our attention to the periodic boundary conditions, in which case

$$\Delta_\omega=\left|\frac{\partial\omega_k}{\partial\boldsymbol{k}}\right|\frac{2\pi}{L}\sim\frac{\omega_k}{kL}. \quad (15)$$

"Twiddle" here means that this is an order of magnitude relationship, which corresponds the approximate character of the physical estimates given below.

The kinetic equation is applicable when $\Gamma\gg\Delta_\omega$, this is the *kinetic regime.* A qualitative different behavior can be expected in the opposite limit $\Gamma\ll\Delta_\omega$: this is a regime of *discrete wave turbulence.* These two regimes are realized when WT forcing is rather high (but not too high so that the nonlinearity is still weak) and low, respectively. However, we will also see that there is also a rather wide intermediate range of forcing for which there is a regime with $\Gamma\sim\Delta_\omega$, which we will call *mesoscopic wave turbulence.*

Name *mesoscopic* refers to an observation made in [34,33] that in existing numerical simulations of the gravity water waves there may be regimes where the statistical properties of the infinite-box systems coexist with effects due to the $\boldsymbol{k}$-space discreteness associated with a finite computational box. In was further argued in [36] [in the context of magnetohydrodynamic (MHD) turbulence] that such a mesoscopic regime is active in a *wide intermediate range of wave amplitudes.* The key reason for such a wide mesoscopic range is the fact that the typical values of Γ for the discrete (dynamical) and the kinetic (statistical) regimes are typically strongly separated.

A. Discrete turbulence (small box, weak waves)

In the discrete WT regime, when $\Gamma \ll \Delta_\omega$, only the terms in the dynamical equations which corresponding to *exact* wave number and frequency resonances contribute to the nonlinear wave dynamics. All the other terms rapidly oscillate and their net long-term effect is null. The most clear example here is the case when there is no exact resonances, like in the system of the capillary water surface waves. In this case, the averaged (over the fast linear oscillations) nonlinearity is negligible and the turbulent cascade over scales is arrested. One can see an analogy with Kolmogorov-Arnold-Moser (KAM) theory which says that trajectories of a perturbed (in our case by nonlinearity) Hamiltonian system remain close to the trajectories of the unperturbed integrable system (in our case the linear wave system whose trajectories are just harmonic oscillations of the individual modes) if there is no resonances. Of course, this analogy should be taken with caution because even in absence of the lower-order resonances (e.g., triad resonances for the capillary waves) higher-order resonances may be important.

Thus, for the discrete WT regime we have the following reduced dynamical equations:

$$i\frac{da_k}{dt} = \sum_{1,2}\left(\frac{1}{2}V^k_{12}a_1a_2R^k_{12} + V^{2*}_{1k}a_1^*a_2R^2_{1k}\right), \tag{16a}$$

for the three-wave case [Eq. (6b) in which we retain only exact wave resonances] and

$$i\frac{da_k}{dt} = \frac{1}{2}\sum_{1,2,3} W^{12}_{3k}a_1a_2a_3^*R^{12}_{3k} \tag{16b}$$

for the four-wave case [Eq. (8b) with only exact resonances left].

In Eq. (16a), factor R^3_{12} is equal to one when modes $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and $\boldsymbol{k}_3$ are in exact wave number and frequency resonance, and it is zero otherwise. Respectively, in Eq. (16b), R^{12}_{34} is equal to one when modes $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, and $\boldsymbol{k}_4$ are in exact wave number and frequency resonance, and it is zero otherwise.

Some resonant triads or quartets (if at all present) may be isolated, in which case their dynamics is integrable, and the respective nonlinear oscillations can be expressed in terms of the elliptic functions. Some triads or quartets may be linked and form clusters of various sizes, whose dynamics is more complicated and to some extent may be chaotic, especially for larger clusters. Study and classification of such exact resonances and their clusters was initiated in [38,50–52] developed further in many papers including [31,33,53–57]. Examples of small and large clusters for the Rossby waves (three-wave system) can be found in [53–55] and for the gravity water waves (four-wave system) in [57].

Frequency broadening Γ for the discrete WT can be estimated from the dynamical Eqs. (16a) and (16b),

$$\Gamma = \Gamma_D^{(3w)} \simeq |Va_k|\mathcal{N}, \tag{17a}$$

$$\Gamma = \Gamma_D^{(4w)} \simeq |Wa_k^2|\mathcal{N}, \tag{17b}$$

where $V=V^k_{12}$ and $W=W^{12}_{3k}$ are the interaction coefficients in (16a) and (16b) respectively. Subscript D indicates that this is a discrete-regime estimate, and superscripts $3W$ and $4W$ stand for "three-wave" and "four-wave," respectively. Here $\mathcal{N}$ is the number of exact resonances which are dynamically important at a fixed $\boldsymbol{k}$, which less or equal to the number of modes connected to $\boldsymbol{k}$ in the resonant cluster. For simplicity we assumed that all the dynamically important resonances are local, i.e., $k_1 \sim k_2 \sim k_3 \sim k$. Strictly speaking, estimates (17a) and (17b) are only valid if $\mathcal{N}$ is not too large, because when $\mathcal{N} \gg 1$ one should expect statistical cancellations of the effect of different triads or quartets, and our estimates would have to be modified. This is the case, for example, of MHD turbulence considered in [36]. Also, our estimates would have to be modified for systems with nonlocal in $\boldsymbol{k}$ interactions.

Thus, the condition of the discrete turbulence regime, $\Gamma_D \ll \Delta_\omega$, becomes

$$|Va_k| \ll \frac{\omega_k}{kL\mathcal{N}}, \quad \text{for three-wave systems}, \tag{18a}$$

$$|Wa_k^2| \ll \frac{\omega_k}{kL\mathcal{N}}, \quad \text{for four-wave systems}. \tag{18b}$$

B. Kinetic wave turbulence (infinite-box limit)

The kinetic regime comprises the classical infinite-box weak WT theory, which is reviewed in the Appendix to this paper, including recent theory extensions to description of the higher-order wave moments and probability density function (PDF) and finding solutions corresponding to turbulence intermittency [58–60]. In this regime, the frequency resonance broadening, denoted Γ_K, is determined by the kinetic Eq. (A10) for three-wave systems and Eq. (A11) for the four-wave systems. This gives for Γ_K,

$$\Gamma_K^{(3w)} \simeq |V|^2 n_k k^d/\omega_k \simeq |V|^2|a_k|^2(kL)^d/\omega_k, \tag{19a}$$

$$\Gamma_K^{(4w)} \simeq |W|^2 n_k^2 k^{2d}/\omega_k \simeq |W|^2|a_k|^4(kL)^{2d}/\omega_k, \tag{19b}$$

where, for simplicity, we have assumed that the wave spectrum is not too narrow and the range of wave numbers interacting with $\boldsymbol{k}$ is of width $\sim k$.

The upper bound for applicability of the wave kinetic equations follows from the condition of weak nonlinearity

$\Gamma_K \ll \delta\omega_k$, where $\delta\omega_k$ is the width of the spectrum in the frequency space. For narrow spectra, $\delta\omega_k \ll \omega_k$, condition $\Gamma_K \gtrsim \delta\omega_k$ signifies onset of so-called phase locking phenomenon [61,62]. Hereafter for simplicity we restrict ourselves to broad spectra, $\delta\omega_k \sim \omega_k$, in which case the upper bound for applicability of the wave kinetic equations becomes

$$|Va_k|(kL)^{d/2} \ll \omega_k, \quad \text{(three-wave)}, \tag{20a}$$

$$|W||a_k|^2(kL)^d \ll \omega_k, \quad \text{(four-wave)}. \tag{20b}$$

Also, the wave amplitudes should be large enough for the broadening Γ_k to be much greater than the frequency spacing Δ_ω. Together with Eq. (15), this condition gives

$$|Va_k| \gg \frac{\omega_k}{(kL)^{(d+1)/2}}, \quad \text{for three-wave systems}, \tag{21a}$$

$$|W||a_k|^2 \gg \frac{\omega_k}{(kL)^{d+1/2}}, \quad \text{for four-wave systems}. \tag{21b}$$

C. Mesoscopic turbulence and sandpile dynamics

Consider first the case when the number of connections of mode $\boldsymbol{k}$ in its discrete resonant cluster is relatively small, $\mathcal{N} \gtrsim 1$, as it is the case, e.g., for the case of the gravity water waves. Comparing the range of kinetic WT [Eqs. (21a) and (21b)] and the one of discrete WT [Eqs. (18a) and (18b)] one can see that there exists a gap,

$$\frac{1}{kL\mathcal{N}} \gg \frac{|Va_k|}{\omega_k} \gg \frac{1}{(kL)^{(d+1)/2}}, \quad \text{(three-wave)}, \tag{22a}$$

$$\frac{1}{kL\mathcal{N}} \gg \frac{|W||a_k|^2}{\omega_k} \gg \frac{1}{(kL)^{d+1/2}}, \quad \text{(four-wave)}, \tag{22b}$$

in which both the conditions for the kinetic WT and for the discrete WT are satisfied. This means that in the region [Eqs. (22a) and (22b)] the wave behavior is neither pure discrete nor pure kinetic WT. Existence of such a gap was first pointed out in [36] in the context of MHD wave turbulence. Regions (22a) and (22b) possess the features of both types of turbulent behavior described above. In the other words, in this region both types of WT may exist and the system may oscillate in time (or parts of the $\boldsymbol{k}$-space) between the two regimes giving rise to a qualitatively new type of WT: mesoscopic wave turbulence. It was suggested in [33] (in the context of the surface gravity waves) that in forced wave systems the discrete and the kinetic regimes may alternate in time, see Fig. 1. Namely, let us consider WT with initially very weak or zero intensity so that initially WT is in the discrete regime, and let us permanently supply more wave energy via a weak source at small ks. During the discrete phase (with fully or partially arrested cascade) the wave energy accumulates until when the resonance broadening Γ_D becomes of order of the frequency spacing Δ_ω. After that the turbulence cascade is released to higher ks in the form of an "avalanche" characterized by predominantly kinetic interactions. At the moment of triggering the avalanche, the broadening Γ jumps up from $\Gamma = \Gamma_D$ to $\Gamma = \Gamma_K \gg \Gamma_D$, see the upper Fig. 1. In the process of the avalanche release, the mean wave amplitude lowers so that the value of broadening $\Gamma = \Gamma_K$ becomes of order of the frequency spacing Δ_ω. Remember, for not too large $\mathcal{N}$ in this intermediate range $\Gamma_K \gg \Gamma_D$. Thus, at this point the system returns to the energy accumulation stage in the discrete WT regime, and the cycle repeats, see Fig. 1. Because of the obvious analogy, this scenario was called *sandpile behavior* in [33].

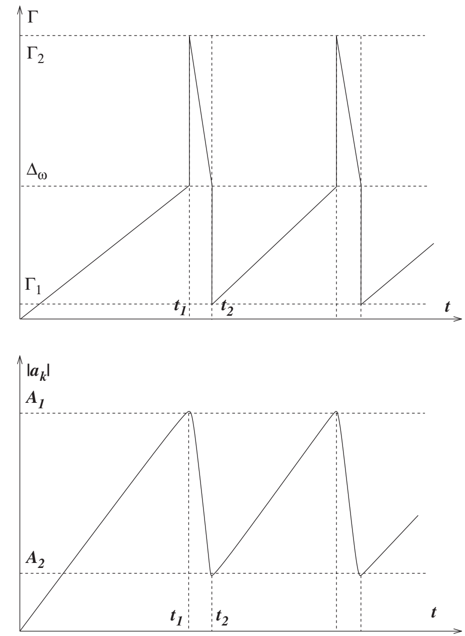

FIG. 1. "Sandpile" behavior in wave turbulence. Upper graph: the frequency broadening Γ follows the discrete turbulence dependence $\Gamma = \Gamma_D$ until reaching the value $\Gamma = \Delta_\omega$ at time $t = t_1$, at which point it jumps to the kinetic branch $\Gamma = \Gamma_K \gg \Gamma_D$ and rapidly drops in the kinetic regime to the value $\Gamma = \Delta_\omega$ at time $t = t_2$. Then it jumps back to the discrete branch $\Gamma = \Gamma_D \ll \Gamma_K$, after which the cycle repeats. Lower graph: the amplitude gradually grows to $a_k \sim A_1$ for $t < t_1$ and then quickly drops to A_2 for $t_1 < t < t_2$, after which the cycle repeats. For the three-wave systems $A_1 \sim (\omega/V)(kL)^{-(d+1)/2}$ and $A_2 \sim \omega/(kLV\mathcal{N})$ and for the four-wave systems $A_1 \sim (\omega/W)^{1/2}(kL)^{-(2d+1)/4}$ and $A_2 \sim \sqrt{\omega/(kLW\mathcal{N})}$.

As we see, the sandpile behavior is characterized by a *hysteresis* where in the same range of amplitudes, from A_1 to A_2 in the lower Fig. 1, the WT intensity increases in the discrete regime and decreases in the kinetic regime.

For the small-amplitude part of the sandpile cycle, the system will be close to the critical spectrum, where resonance broadening Γ_K is of order of the omega spacing Δ_ω. This gives the frequency spectrum ω^{-6}, which was predicted in [33] and experimentally confirmed in [29] [cf. ω^{-4} for the Kolmogorov-Zakharov (KZ) spectrum in this case [23]]. Finding spectrum close to the large-amplitude part of the cycle is not so straightforward because we do not know the dependence of $\mathcal{N}$ on ω.

So far, we only considered the case when $\mathcal{N}$ is not too large. Case $\mathcal{N} \gg 1$ can be very different. Namely, instead of the range where *both* conditions satisfied simultaneously, the one for the kinetic WT [Eqs. (21a) and (21b)] and the one for the discrete WT [(18a) and (18b)] one gets a range where *none* of these two conditions are satisfied. This kind of mesoscopic turbulence was considered using the MHD example in [33]. We will see that in this case the frequency broadening Γ remains of the order of the omega spacing Δ_ω in a broad (mesoscopic) range of wave amplitudes. Remembering that Γ is a characteristic nonlinear evolution time, we note that constancy of Γ points at a possibility that the energy transfer in such a mesoscopic regime is driven by a hidden effectively linear process, which is yet to be understood.

D. Possible coexistence of different regimes

Strength of WT typically varies in along the turbulent cascade in the $\boldsymbol{k}$ space and, therefore, one may expect different wave turbulence regimes present in the different parts of the $\boldsymbol{k}$ space at the same instant in time. For example, the nonlinearity increases along the cascade toward high wave numbers in WT of surface gravity waves and of MHD Alfvén waves. Thus we can expect WT in these systems to be discrete at low ks and kinetic at high ks. Moreover, on the crossover regions one can expect nontrivial gradual transition which involves blending and interaction of different dynamical and statistical mechanisms. This effect is expected to be more pronounced if the interaction of scales is nonlocal, so that some wave number(s) from a particular resonant triad (or quartet) could be in the discrete range whereas the other wave number(s) from the same triad (or quartet) could be in the kinetic range. As a result, in the crossover range a continuous spectrum described by the kinetic equation (e.g., KZ) could coexist with selected few modes belonging to isolated resonant clusters which would evolve coherently at deterministic time scales. Moreover, the same set of modes might randomly alternate in time from being discrete to kinetic and back, as we described above in the sandpile scenario.

Some basic consequences of variability of the finite-size effects in the $\boldsymbol{k}$ space can be seen in an very simple kinematic cascade model suggested in [35]. This model builds a "cascade tree" in the following three steps:

(i) Let us put some energy into a small collection of initial modes. We denote this initial collection of excited modes by S_0 (e.g., in within a circle or a ring at small ks which corresponds to forcing at large scales). One can view set S_0 as the cascade tree's "trunk."

(ii) Next, find the modes which can interact with the initial ones at the given level of nonlinear broadening Γ. Namely, we define a new set of modes S_1 as the union of all ks satisfying the quasiresonance conditions,

$$|\omega_3-\omega_2-\omega_1|<\Gamma, \quad \boldsymbol{k}_3-\boldsymbol{k}_2-\boldsymbol{k}_1=0 \tag{23a}$$

for the three-wave case and

$$|\omega_4+\omega_3-\omega_2-\omega_1|<\Gamma,$$

$$\boldsymbol{k}_4+\boldsymbol{k}_3-\boldsymbol{k}_1-\boldsymbol{k}_2=0, \tag{23b}$$

for the four-wave case, with all but one wave numbers in S_0 and the remaining wave number outside of S_0. Provided that Γ is large enough, the set S_1 will be greater than S_0. Set S_1 comprises the cascade tree's "biggest branches."

(iii) Now iterate this procedure to generate a series of cascade generations $S_0, S_1, \ldots, S_N$ which will mark the sets of active modes as the system evolves. The union of these sets constitutes the whole of the cascade tree with all of its bigger and smaller branches included.

This model is purely kinematic. It does not say anything about how energy might be exchanged dynamically among the active modes or how rapidly a certain cascade generation is reached. However, the kinematics alone allows one to make some interesting observations about the systems with variable in k finite-size effects.

Let us consider the example of the gravity waves on deep water, for which the following results were obtained in [33]. If one starts with a set of low-k modes, with broadening Γ below a critical value $\Gamma_{\text{crit}}=1.4\times10^{-5}$, a finite number of modes outside the initial region get excited (generation 2) but there will be no quasiresonances to carry energy to outer regions in further generations. If the broadening is larger than Γ_{crit}, the energy cascades infinitely. Further, such the kinematic cascades were shown to have the fractal snowflake structure with the active modes being rather sparse in the front of the cascade propagating to higher k, with pronounced anisotropic and intermittent character.

Similar picture of intermittent cascades was also observed for the capillary wave system [35]. However, because there is no exact resonances for this system, the generation 1 an higher appear only if Γ is greater than some minimal value $\Gamma_{\text{crit }1}$. Further, there exists a second critical value $\Gamma_{\text{crit }2}>\Gamma_{\text{crit }1}$: the number of generations is finite for $\Gamma_{\text{crit }2}>\Gamma>\Gamma_{\text{crit }1}$ and the cascade process dies out not reaching infinite ks, whereas for $\Gamma>\Gamma_{\text{crit }2}$ the number of generations is infinite and the cascade propagates to arbitrarily high ks. Note that the later property makes the capillary wave system different from the gravity waves for which the cascade always spread through the wave number space infinitely provided $\Gamma>\Gamma_{\text{crit}}$.

Another example where the (three-wave) quasiresonances and the kinematic energy cascades were studied is the system of inertial waves in rotating three-dimensional (3D) fluid volumes [37]. This system is anisotropic and the study of the kinematic cascades allows to find differences between the 2D modes, with wave vectors perpendicular to the rotation axis, and the 3D modes. It appears that the "catalytic" interactions which involve triads including simultaneously 2D and 3D wave vectors dominate over the triads which involve 3D wave vectors only.

IV. DISCUSSION

In this paper we have considered the three different regimes which can be observed in wave turbulence (WT) bounded by a finite box—discrete, mesoscopic, and kinetic. For very low amplitudes and small boxes, we expect the discrete WT, whose dynamics is driven by the exact resonances. In the opposite infinite-box limit, we expect the kinetic WT, which is driven by quasiresonances and for which the exact resonances do not play a role as they are hugely outnumbered by the quasiresonances. This is the classical and the most studied WT regime, and it is summarized in our Appendix. In the middle, there is a regime of the mesoscopic WT. We have shown that this regime is characterized by sandpilelike oscillations between the discrete and the kinematic regimes (if the size of the active resonant clusters is small) or it settles to an intermediate (critical) state in which the nonlinear frequency broadening is of order of the frequency spacing between the discrete modes for a wide range of wave numbers (if the size of the active resonant clusters is large).

The key fact that has led us to the observation that the mesoscopic regime should be realized in a *wide* range of wave intensities is that the dependence of the frequency broadening on the wave intensity is very different for the dynamical and the kinetic equations; cf. Γ_D given by Eqs. (17a) and (17b) and Γ_K given by Eqs. (19a) and (19b). Thus, for the same wave intensities in which Γ_D and Γ_K are typically very different in size, and there exist a wide mesoscopic range where either *both* the discrete and the kinetic regimes can exist, or *none* of them is realizable—hence the two types of the mesoscopic behavior described above.

Signs of bursty behavior typical of the sandpile behavior suggested in this paper has already seen in laboratory and numerical experiments [29,33]. In future, one should aim to perform more direct diagnostics of the quantities allowing to identify and to distinguish the different WT regimes describe in the present paper, including the nonlinear frequency broadening and character of its evolution in time.

In conclusion, we would like to emphasize the main message of this paper: the WT processes are much reacher and their range is much broader than it was previously believed. Great care should be taken when one aims to test WT predictions in laboratory and numerical experiments, as well as in field observations. Namely, one has to make sure that the applicability conditions assumed the theory are satisfied. In particular, the great majority of the numerical and laboratory tests aimed at verifying the predictions of the *kinetic* WT theory via confirming the KZ spectrum. However, applying the estimates obtained in the present paper reveals that in many of such tests the wave systems are not in the kinetic regime but rather in the mesoscopic or even in the discrete ones. Apart from the qualitative picture discussed in the present paper, such as the sandpile and hysteresis behavior, the discrete and the mesoscopic regimes remain largely unstudied, and much of theoretical work remains done in this direction in future.

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APPENDIX: KINETIC WAVE TURBULENCE

Classical wave turbulence theory provides a statistical description of weakly nonlinear waves with random phases. As discussed above, theory of wave turbulence is valid in a range of wave-field strengths such that

$$1 > \frac{\Gamma}{\omega_k} > \frac{\Delta_\omega}{\omega_k} \sim \frac{1}{kL}, \tag{A1}$$

where Γ is given by Eq. (19a) or Eq. (19b) for the three- or four-wave processes, respectively.

The most popular statistical object in the theory of wave turbulence is the wave action spectrum although theory of wave turbulence has been recently extended to description of higher moments and PDFs in [58–60]. This allowed to deal with non-Gaussian wave fields, as well as to study validity of the underlying statistical assumptions such as, e.g., random phases. We will now briefly describe these results.

Let us represent the complex amplitudes as $a_k = \sqrt{J_k}\psi_k$ with wave intensity $J_k \in \mathbb{R}^+$ (positive real number) and phase factor $\psi_k \in \mathbb{S}^1$ (complex number of length 1). Let us define the M-mode joint PDF $\mathcal{P}^{(M)}$ so that the probability for the wave intensities of the selected M modes, J_k, to be in the range $(s_k, s_k + ds_k)$ and for their phase factors ψ_k to be on the unit-circle segment between ξ_k and $\xi_k + d\xi_k$ is $\mathcal{P}^{(M)}\Pi_{k=1}^{M} ds_k |d\xi_k|$. (Therefore $\mathcal{P}^{(M)}$ is a function of $2M+1$ variables: M amplitudes, M phases, and time).

Notion of random phases refers to the cases where all factor ψ_k are statistically independent and uniformly distributed on $\mathbb{S}^1$, i.e.,

$$\mathcal{P}^{(M)} = \frac{1}{(2\pi)^M}\mathcal{P}_a^{(M)} \tag{A2}$$

for any $M \le N$, where N is the total number of dynamically active modes. Here $\mathcal{P}_a^{(M)}$ is the joint PDF of the amplitudes only. Kinetic WT considers wave fields with random phases at some initial time and with intensities satisfying condition (A1). This leads to the following equation for the joint PDF for the three-wave case:

$$\frac{\partial \mathcal{P}^{(N)}}{\partial t} = 16\pi \int |V_{23}^1|^2 \delta(\omega_1 - \omega_2 - \omega_3)\,\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \times \left[\frac{\delta}{\delta s}\right]_3 \left(s_1 s_2 s_3 \left[\frac{\delta}{\delta s}\right]_3 \mathcal{P}^{(N)} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3, \tag{A3}$$

where $[\delta/\delta s]_3 = \delta/\delta s_1 - \delta/\delta s_2 - \delta/\delta s_3$. This equation was first derived for a specific example of waves in anharmonic crystals by Peierls [19] and for general three-wave systems in [58,60,63]. It was also extended to the four wave systems in [59]. Note that the phase variables are not involved in these equations. Therefore, the random phase assumption is consistent with these equations, namely the system which has random phases initially will remain random phased over the typical nonlinear time (i.e., its PDF will remain independent of ξs). Thus, equations for the joint PDF Eq. (A3) allows an *a posteriori* justification of the random phase assumption underlying their derivations.

However, as we already mentioned, the most frequently considered object in the theory of wave turbulence is the spectrum which is defined as

$$n_k = \left(\frac{2\pi}{L}\right)^d \langle J_k \rangle, \tag{A4}$$

where d is the dimension of the space and the angular brackets mean the ensemble averaging over the wave statistics. The spectrum is a one-mode statistical object, and it is the first in the series of one-mode moments,

$$M_k^{(p)} = \left(\frac{2\pi}{L}\right)^{pd} \langle J_k^p \rangle = \left(\frac{2\pi}{L}\right)^{pd} \int_0^\infty s_k^p \mathcal{P}^{(1)}(s_k) ds_k.$$

Note that for deriving closures for the one-mode objects the random phase property is insufficient and one has to assume additionally that the amplitudes J_k are also statistically independent of each other at different ks. Statistical independent of the amplitude can also be justified based on the equation for the joint PDF Eq. (A3), although this issue is more subtle than the phase randomness because variables s_k do not separate in Eq. (A3) and, therefore any product factorization of the joint PDF in terms of the one-mode PDFs would not generally be preserved by the nonlinear evolution. However, this situation seems to be typical for many systems, e.g., for the relation between the multiparticle and one-particle distribution functions described by the Louisville and Boltzmann equations, respectively. In these situations, a sufficient for the closures property is that the *low-order* PDFs, $\mathcal{P}^{(M)}$ with $M \ll N$, can be product factorized. It can be seen from Eq. (A3) that it is the case for the weakly nonlinear wave systems, i.e., that factorization, $\mathcal{P}^{(M)} = \prod_{k=1}^M \mathcal{P}_k^{(1)} + O(M/N)$, survives over the characteristic nonlinear time.

Importantly, the distribution of wave fields in the theory of kinetic WT does not need to be Gaussian or close to Gaussian, and one can consider evolution of the one-mode PDFs $\mathcal{P}^{(1)}$ that correspond to strongly non-Gaussian fields (Gaussian fields would mean $\mathcal{P}^{(1)} \sim e^{-s/\langle J \rangle}$). Integrating the joint PDF Eq. (A3) we get

$$\frac{\partial \mathcal{P}_k^{(1)}}{\partial t} + \frac{\partial F_k}{\partial s_k} = 0, \tag{A5}$$

with F is a probability flux in the s space,

$$F_k = - s_k \left(\gamma_k \mathcal{P}_k^{(1)} + \eta_k \frac{\delta \mathcal{P}_k^{(1)}}{\delta s_k} \right), \tag{A6}$$

where for the three-wave case we have

$$\eta_k = 4\pi \int (|V_{12}^k|^2 \delta(\omega_k - \omega_1 - \omega_2) \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) + 2|V_{k1}^2|^2 \delta(\omega_2 - \omega_k - \omega_1) \delta(\boldsymbol{k}_2 - \boldsymbol{k} - \boldsymbol{k}_1)) n_1 n_2 d\boldsymbol{k}_1 d\boldsymbol{k}_2,$$

$$\gamma_k = 8\pi \int (|V_{12}^k|^2 \delta(\omega_k - \omega_1 - \omega_2) \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) n_1 + |V_{k1}^2|^2 \Delta_{k1}^2 \delta(\omega_2 - \omega_k - \omega_1) \delta(\boldsymbol{k}_2 - \boldsymbol{k} - \boldsymbol{k}_1)(n_1 - n_2)) d\boldsymbol{k}_1 d\boldsymbol{k}_2. \tag{A7}$$

Equation (A5) has an obvious exponential solution which corresponds to a zero flux F:

$$\mathcal{P}_k^{(1)} = \frac{1}{\langle J_k \rangle} e^{-s_k/\langle J_k \rangle},$$

which corresponds to Gaussian statistics of the wave field a_k. However, there are also solutions corresponding to $F = \text{const} \neq 0$ which for $s_k \gg \langle J_k \rangle$ has a power-law asymptotic [58,59],

$$\mathcal{P}_k^{(1)} = - \frac{F}{\gamma_k s_k}.$$

These solution corresponds to enhanced probability (with respect to Gaussian) of strong waves which is called intermittency of WT. Here, the constant flux in the amplitude space F can be associated with a wave breaking process the exact form of which depends on the physical system. For example, for the gravity water surface waves the wave breaking process takes form of whitecapping, and for the focusing NLS system the wave breaking is represented by filamentation or collapsing events. Obviously, this power-law tail of the PDF cannot extend to infinity because the integral of the PDF must converge. Thus, there exists a cutoff which can also be associated with the wave breaking, which can simply be understood that the probability of waves with amplitude greater than some critical value must be zero. Such critical value roughly corresponds to the amplitude for which the nonlinear term becomes of the order of the nonlinear one so that the WT description breaks.

Multiplying Eq. (A5) by s_k^p and integrating over s_k, we have the following equation for the moments $M_j^{(p)} = \langle J_j^p \rangle$:

$$\frac{d}{dt} M_k^{(p)} = - p \gamma_k M_k^{(p)} + p^2 \eta_k M_k^{(p-1)}, \tag{A8}$$

which, for $p=1$ gives the kinetic equation for the wave action spectrum,

$$\frac{d}{dt} n_k = - \gamma_k n_k + \eta_k. \tag{A9}$$

Substituting into this equation expressions for γ_k and η_k, we obtain more familiar forms of the kinetic equations:

$$\frac{d}{dt} n_k = 4\pi \int |V_{12}^k|^2 \delta(\omega_k - \omega_1 - \omega_2) \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)(n_1 n_2 - n_1 n_k - n_2 n_k) d\boldsymbol{k}_1 d\boldsymbol{k}_2 + 8\pi \int |V_{k1}^2|^2 \delta(\omega_2 - \omega_k - \omega_1) \times \delta(\boldsymbol{k}_2 - \boldsymbol{k} - \boldsymbol{k}_1)(n_1 n_2 - n_1 n_k + n_2 n_k) d\boldsymbol{k}_1 d\boldsymbol{k}_2, \tag{A10}$$

and for the four-wave case

$$\frac{d}{dt} n_k = 4\pi \int |T_{23}^{k1}|^2 \delta(\omega_k + \omega_1 - \omega_2 - \omega_3) n_k n_1 n_2 n_3 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3. \tag{A11}$$

Based on Eqs. (A10) and (A11) [or Eqs. (A5) and (A8)] one can obtain the estimate for the nonlinear frequency broadening in the WT regime, i.e., inverse characteristic time of the nonlinear evolution as in Eq. (19a) or Eq. (19b). This leads to the WT applicability condition (21a) or condition (21b).

Classical statistical approach allows to obtain some interesting and physically relevant solutions, such as KZ spectra corresponding to the energy and wave action cascades through scales. Such solutions can be obtained analytically using so-called Kraichnan-Zakharov transformation, as well as from the scalings of the frequency and the interaction coefficients based on the dimensional analysis. Discussion of these issues is beyond the scope of our review, and the interested reader is referred for details to book [18]. Here, it suffices to say that in most systems there exists a shortcut way to obtain KZ spectra. It works for the systems with only one relevant dimensional parameter, for example the gravity constant g for the water surface gravity waves, surface tension constant σ for the capillary waves, speed of sound c_s for acoustic turbulence, quantum of circulation κ for Kelvin waves on quantized vortex lines, etc. In this case the 1D energy spectrum $E_k \sim k^{\nu}$ can be immediately obtained from the physical dimension of this constant which gives for the direct cascade [35]:

$$\nu = 2\alpha + d - 6 + \frac{5 - 3\alpha - d}{N - 1}, \tag{A12}$$

where α is the power of the dispersion relation $\omega \sim k^{\alpha}$ (which is uniquely determined by the above dimensional constant), d is the dimension of the system, and N is the number of waves involved in the resonance interaction. For example, for the water surface gravity waves we have $E_k \sim k^{-5/2}$, for the capillary waves $E_k \sim k^{-7/4}$ (both of these spectra are called Zakharov-Filonenko spectra [23]), and for acoustic turbulence $E_k \sim k^{-3/2}$ (Zakharov-Sagdeev spectrum [28]). For Kelvin waves on quantized vortex lines, considering them as a local six-wave process, one formally gets $E_k \sim k^{-7/5}$ (Kozik-Svistunov spectrum [3]). However, this spectrum was recently shown in to be nonlocal

Similar approach one can use for finding the inverse cascade spectra, e.g., for the water surface gravity waves or Kelvin waves [35].

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