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Poincaré recurrences in Hamiltonian systems with a few degrees of freedom

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Hundred twenty years after the fundamental work of Poincaré, the statistics of Poincaré recurrences in Hamiltonian systems with a few degrees of freedom is studied by numerical simulations. The obtained results show that in a regime, where the measure of stability islands is significant, the decay of recurrences is characterized by a power law at asymptotically large times. The exponent of this decay is found to be $\beta \approx 1.3$. This value is smaller compared to the average exponent $\beta \approx 1.5$ found previously for two-dimensional symplectic maps with divided phase space. On the basis of previous and present results a conjecture is put forward that, in a generic case with a finite measure of stability islands, the Poincaré exponent has a universal average value $\beta \approx 1.3$ being independent of number of degrees of freedom and chaos parameter. The detailed mechanisms of this slow algebraic decay are still to be determined. Poincaré recurrences in DNA are also discussed.

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According to the Poincaré recurrence theorem proven in 1890 [1] a dynamical trajectory with a fixed energy and bounded phase space will always return, after a certain time, to a close vicinity of an initial state. This famous result was obtained in relation to the studies of the three-body gravitational problem which fascinating history can be find in [2]. While recurrences will definitely take place a question about their properties, or what is a statistics of Poincaré recurrences, still remains an unsolved problem. The two limiting cases of periodic or fully chaotic motion are well understood: in the first case recurrences are periodic while in the latter case the probability of recurrences $P(t)$ with time being larger than t drops exponentially at $t \to \infty$ (see, e.g., [3,4]). The latter case is analogous to a coin flipping where a probability to drop on one side after t flips decays as $1/2^t$.

However, the statistics of Poincaré recurrences for generic two-dimensional (2D) symplectic maps is much richer. Such systems generally have a divided phase space where islands of stable motion are surrounded by a chaotic component [5,6]. In such a case trajectories are sticking around stability islands and recurrences decay algebraically with time

$$P(t) \propto 1/t^{\beta}, \quad \beta \approx 1.5. \tag{1}$$

The studies and discussions of this behavior can be find in [7–13] and references therein. According to the above studies the Poincaré exponent β has a universal average value for 2D symplectic generic maps.

While the statistics of Poincaré recurrences in 2D maps has been studied in great detail [7–13], the original three-body problem with a few degrees of freedom $N=9$ addressed by Poincaré [1] (effective number of degrees of freedom is $N_{\text{eff}}=6$ if to exclude the center of mass motion) has not been studied yet in great detail. The case of four-dimensional (4D) and six-dimensional (6D) symplectic maps has been considered in [14] and an algebraic decay of type (1) has been found with $1.1 < \beta < 1.5$ and $1.7 < \beta < 2$, respectively. A more detailed study, with up to $N=25$ degrees of freedom, has been performed in [15] with a variation in β found to be in a range $1.3 < \beta < 5.5$ depending on map parameters and values of N. In this work I study the statistics of Poincaré recurrences in a model system for $4 \leq N \leq 8$ going up to two orders of magnitude larger times comparing to [14,15].

To reach a high efficiency of numerical simulations I use a dynamical map

$$\bar{p}_n = p_n + (K/2\pi)\{\sin[2\pi(x_n - x_{n-1})] + \sin[2\pi(x_n - x_{n+1})]\},$$

$$\bar{x}_n = x_n + \bar{p}_n, \tag{2}$$

which was studied numerically in [16–18]. Here bars mark values of dynamical variables after one map iteration. Periodic boundary conditions are used in $x_n(\text{mod } 1)$ and $p_n(\text{mod } 1)$ with $-0.5 \leq p_n \leq 0.5$. The map is symplectic. I use N particles, $1 \leq n \leq N$, with a periodic boundary conditions in $n(\text{mod N})$. For $N=1$ map (2) is equivalent to the Chirikov standard map [5] (assuming that all variables for $n > 1$ are equal to zero). The properties of $P(t)$ for this case can be find at [9,11–13] and references therein. For a number of particles $N > 2$ the total momentum of the whole system is preserved so that one can say that this situation corresponds effectively to $N_{\text{eff}}=N-1/2$ degrees of freedom. In the following I consider $4 \leq N \leq 8$.

The recurrences are considered on line $p_n=0$ for each particle, the integral probability of recurrences, averaged over all particles, is defined as a total integral probability $P(t)$ of recurrences with time larger than time t, which is measured in number of map iterations. In a more formal way, I count the number of map iterations t_r between the consecutive crossing of line $p_n=0$ for each particle, such an event is called a recurrence. Then the relative number of recurrences with time t_r larger than t ($t_r > t$) is taken to be equal to the recurrence probability $P(t)$ with averaging over all particles.

As in [9,11], to compute $P(t)$ I usually used one trajectory iterated up to time $t_{\text{tot}} \leq 10^{12}$. Special checks with other trajectories or other t_{tot} unsure that $P(t)$ remains unchanged in the limit of statistical fluctuations which appear only when the number of recurrences becomes of the order of a few events. It should be noticed that map (2) is similar, in certain aspects, to the one studied in [15] (e.g., both are built on the

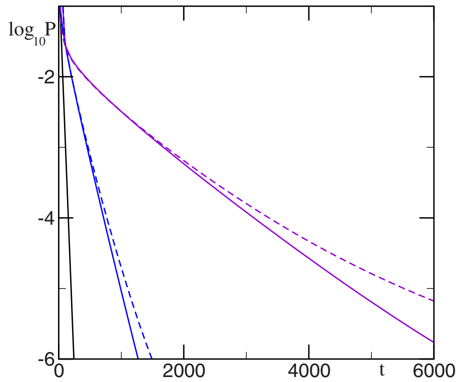

FIG. 1. (Color online) Dependence of statistics of Poincaré recurrences $P(t)$ on time t for $N=8$ and parameter $K=1,0.6,0.4$ (full curves from left to right at $\log_{10} P=-4$) and for $N=6$ and $K=0.6,0.4$ (dashed curves from left to right at $\log_{10} P=-4$). Here $P(t)$ is an integrated probability of recurrences with time larger than t; recurrences are considered on line $p_n=0$, sum is taken over all N degrees of freedom.

basis of the Chirikov standard map), but in the present case the couplings between particles are local, while all particles are coupled in [15].

An example of dependence of $P(t)$ on t is shown in Fig. 1 for relatively short times and large N when the dynamics is mainly fully chaotic. The initial decay drops exponentially $P(t) \propto \exp(-t/t_D)$ with a certain time scale t_D which depends on K. The dependence of t_D on N is relatively weak since up to a certain time $P(t)$ curves are practically independent of N (see Figs. 1 and 2). At large times the exponential decay is replaced by a power-law decay which is well visible for $N=4,6$ in Fig. 2.

The time scale t_D is related to a diffusive spreading in p_n characterized by a diffusion rate $D/4\pi^2 = \langle p_n^2 \rangle / t$. Indeed, such a relaxation diffusive process on an interval $-0.5 \le p_n \le 0.5$ of size $L=1$ is described by the Fokker-Plank equation

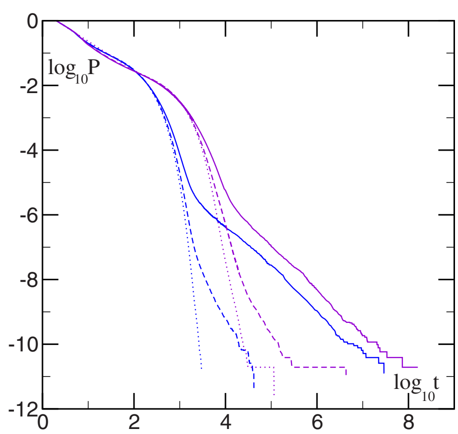

FIG. 2. (Color online) Same as in Fig. 1 for $K=0.6$ and $N=4,6,8$ (left group of blue/black full, dashed and dotted curves from right to left at $\log_{10} P=-8$, respectively) and for $K=0.4$ and $N=4,6,8$ (right group of violet/gray full, dashed and dotted curves from right to left at $\log_{10} P=-8$, respectively). The data are obtained from one trajectory with the total number of iterations $t_{\rm tot} = 10^{12}$ (for $N=8$ I used $t_{\rm tot}=10^{11}$).

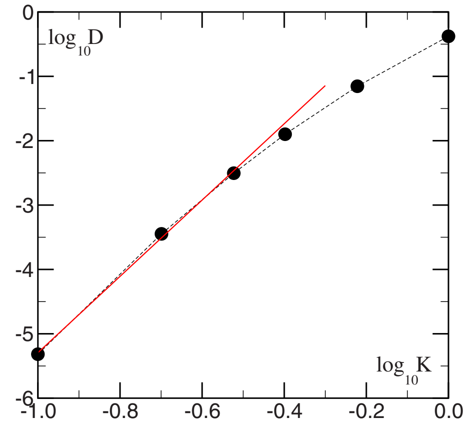

FIG. 3. (Color online) Dependence of the diffusion rate D on chaos parameter K (points). The dashed curve is drawn to adapt an eye, the full straight line shows the fit of last points with $D=aK^b$ and $\log_{10} a=0.587$, $b=5.93 \pm 0.22$.

$$\partial\rho/\partial t = D/(8\pi^2)\partial^2\rho/\partial^2 p. \tag{3}$$

This equation with zero boundary conditions $\rho(p=\pm 0.5)=0$ gives the exponential relaxation of probability to stay inside the interval at large times: $P(t) \sim \exp(-t/t_D)$ with $1/t_D = \pi^2(D/4\pi^2)/(2L^2) = D/8$ [see, e.g., Equation (2.2.4) in [19], in our case the interval size is $L=1$]. Thus with this relation one can extract from the initial exponential drop of $P(t)$ the relaxation time t_D and from it the diffusion rate D. In such a way I obtain the dependence of D on K and N. As discussed above the dependence on N is very weak and can be neglected. On the contrary the dependence of t_D and D on K is very strong as it is shown in Fig. 3.

The dependence $D(K)$ has a few interesting features. For $K=1$ I find $D \approx 1/2$ that corresponds to a random phase approximation valid in a regime of strong chaos. With a decrease in K the diffusion drops rapidly, at small values of K one has approximately algebraic decay $D \propto K^b$ with the exponent $b=5.93 \pm 0.22$. This value of the exponent is in a good agreement with the values obtained in [16,18] which are $b=6.6$ and $b=6.3$, respectively. It should be stressed that the methods of computation of D in [16,18] were rather different compared to those used here.

In fact an enormously powerful numerical method has been used by Chirikov and Vecheslavov [18] to compute an extremely small rate of the fast Arnold diffusion (down to $D \sim 10^{-44}$ at $K \approx 8 \times 10^{-7}$ and $N=16$). This diffusion appears in very tiny chaotic layers around multidimensional resonances. By its structure, the method used in [18] determines the diffusion in a local domain of phase space, while the method used here gives the global diffusion. The agreement between two methods shows that these two diffusion coefficient are approximately the same.

In these studies I want to analyze how this chaotic web influence the statistics of Poincaré recurrences. Of course one is not able to go to so small values of K but also in a certain sense one does not need this. The algebraic decay of $P(t)$ appears due to sticking of trajectories around stability islands so that one simply needs to have a significant measure of stability islands.

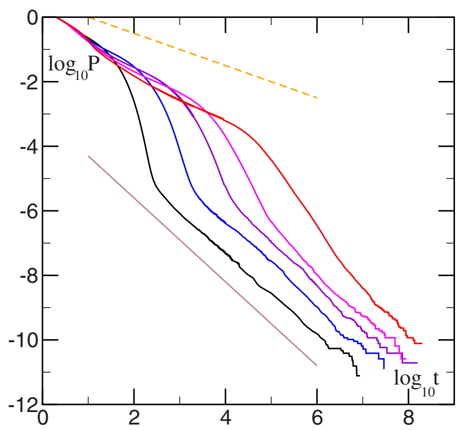

FIG. 4. (Color online) Statistics of Poincaré recurrences for map (1) shown by curves for parameters $N=4$, $K=1,0.6,0.4,0.3,0.2$ (curves from left to right at $\log_{10} P=-8$, respectively). The exponents β for the power-law decay $P(t) \propto 1/t^{\beta}$ are 1.243 ± 0.001, 1.292 ± 0.002, 1.385 ± 0.003, 1.427 ± 0.007, and 1.476 ± 0.005, respectively. The full straight line shows the dependence $P(t) \propto 1/t^{\beta}$ with $\beta=1.30 \pm 0.003$ corresponding to the average of above five values of β. The dashed straight line shows the diffusive decay $P(t) \propto 1/\sqrt{t}$. For each K the data are obtained from one trajectory with the total number of iterations $t_{\rm tot}=10^{12}$ that allows to reach the maximum recurrence time t up to approximately $t \approx 2 \times 10^{8}$.

The data of Fig. 2 show that for $N=8$ one has practically only an exponential decay of $P(t)$ indicating that the measure of stable component is of the order of $\mu_s \sim tP(t) < 10^{-8}$ for $K=0.6$ and $\mu_s < 10^{-5}$ for $K=0.4$ (I use the relation between μ and $P(t)$ discussed in [9,11]). For $N=6$ the algebraic decay becomes to be visible at large t showing that the measure of stability islands starts to be reachable only for $t_{\rm tot}=10^{12}$.

The power-law decay of $P(t)$ is most visible for $N=4$ case shown in Fig. 4. Initially there is a slow decay of $P(t)$ which is compatible with a diffusive spreading on a semi-infinite line with $P(t) \propto 1/\sqrt{t}$ (see, e.g., discussion at [7]). Since t_D grows significantly with the decrease in K the range of this diffusive decay of $P(t)$ increases when $K \to 0$. However, already for $K \leq 0.07$ the measure of chaotic component becomes rather small and one needs to use special methods described in [18] to be able to place initial conditions inside tiny chaotic layers. Due to these reasons I stop at values of $K \geq 0.1$. In any case for small K the time t_D becomes very large and a lot of computational time becomes lost for not very interesting diffusive decay.

After the time scale t_D a trajectory starts to feel a finite width of the chaotic layer with $-1/2 \leq p_n \leq 1/2$ and an algebraic decay due to sticking around islands starts to be dominant. In this regime I find the exponent $\beta=1.3$. The statistical error of this value is rather small but certain oscillations in logarithmic scale of time are visible for $K=0.6,0.4,0.3$ so that the real uncertainty of β can be larger. At the same time the amplitude of these oscillations is significantly smaller compared to the case of 2D symplectic maps discussed in [8,9,11,12]. The fit for β is done for times $t_{\rm dr} < t < t_{\rm tot}$ where $t_{\rm dr}$ marks the end of the drop transition from diffusive spreading to sticking in a vicinity of islands.

The values of β, given in the caption of Fig. 4, have a certain tendency to increase with a decrease in K. However, this increase is rather small (about 19% while K is changed by factor 5). I attribute this to a decrease in fit interval at small values of K, where the diffusion time t_D becomes larger and larger, that gives a reduction in the fit interval between $t_{\rm dr}$ and $t_{\rm tot}$. It is clear that the fit interval $t_{\rm dr} < t < t_{\rm tot}$ for asymptotic algebraic decay should be sufficiently large to determine β reliably. This is clearly not so for $N=6$ case shown in Fig. 2, where the transition from exponential diffusive decay only starts to be replaced by an asymptotic algebraic decay. In my opinion a fit in such a small interval would artificially increase the value of β, since a sharp drop of $P(t)$ visible at $t < t_{\rm dr} \sim t_D$ and being characteristic of diffusive exponential decay, is not yet terminated completely. The data of Fig. 4 clearly show that the scale $t_{\rm dr}$ grows significantly with a decrease in K and D.

This view, obtained on the basis of my results for rather long $t_{\rm tot}$, leads me to another interpretation of previous results [14,15] which claimed the growth of β with growth of N and chaos parameter (see, e.g., Fig. 2 in [15]). Thus, on a first glance, in Fig. 2(c) of [15] β increases from $\beta \approx 1.4$ to 2.8 for $N=4$ when the chaos parameter ξ is changed from 0.03 to 0.1. This is in drastic contrast to the results presented here in Fig. 4 clearly showing that $\beta \approx 1.3 \approx \text{const}$ when the chaos parameter is changed by a factor 5. I think that such an increase of β with ξ in [15] should be attributed to shorter times considered there in comparison with the present studies.

In view of that I make a conjecture that in a generic case, when the islands of stability have nonzero measure, the asymptotic decay of Poincaré recurrences has form (1) with a universal average Poincaré exponent $\beta \approx 1.3-1.4$ being independent of chaos parameter and number of degrees of freedom N (at least for moderate and large but finite values of N).

The data of present studies confirm the approximate independence of β of chaos parameter K (see Fig. 4). At the same time the data of Fig. 2(c) of [15] at moderate values of chaos parameter $\xi=0.03$ clearly show that β is approximately 1.4–1.5 for $2 \leq N \leq 10$. This confirms the above conjecture. In my opinion, a further increase in β for $11 \leq N \leq 25$, visible in Fig. 2(c) of [15] for $\xi=0.03$, should be attributed to a significant reduction in the available fit interval $t_{\rm dr} < t < t_{\rm tot}$ which is clearly seen in Figs. 2(a) and 2(b) of [15]. It is also clear that for the model of [15] the growth of N gives an effective increase of the chaos parameter due to long-range interactions present in the model. The data of [14] for 4D map give approximately the same universal value of β, while for 6D I expect that the time interval was not so long to see the asymptotic behavior.

It is now well established that generic 2D symplectic maps have Poincaré recurrences with a universal average Poincaré exponent $\beta \approx 1.5$ [7–10,12]. This slow decay is linked to sticking in a vicinity of stability islands. It is naturally to expect that for larger number of degrees of freedom N the structure of such sticking regions is more complicated giving more possibilities for sticking with slow Arnold diffusion processes. Hence, intuitively it is natural to expect that for a few degrees of freedom the average value of β will be smaller. The universal average value $\beta \approx 1.3-1.4$ found here and in [15] is in agreement with such expectations.

It is interesting to note that recent extensive numerical simulations of DNA dynamics [20] show an algebraic decay of survival probability which is proportional to Poincaré recurrences $P(t)$. The Poincaré exponent there is approximately $\beta \approx 2$ for times t being in the range $10^{-6} < t < 10^{-4}$ s (Fig. 4(d) in [20]) but certain cases, e.g., W122, show $\beta \approx 1$ at the very large times reached numerically. Thus it is possible that a universal value of $\beta \approx 1.3$ can appear on times $t > 10^{-4}$ s.

In conclusion, the studies of the statistics of Poincaré recurrences in Hamiltonian systems with a few degrees of freedom show that at large times it is characterized by power-law decay (1) with the universal average exponent $\beta \approx 1.3$. This value is not so far from the average exponent $\beta \approx 1.5$ found for the 2D symplectic maps. It is possible that the physical mechanisms of this slow decay have similar grounds related to sticking of trajectories in a vicinity of small islands of stability for enormously long times. Further extensive studies are required to understand in a deeper way the detailed mechanisms of this slow decay. Even more than hundred twenty years after the work of Poincaré [1] this fundamental problem of dynamical chaos remains unsolved.

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