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Universal shocks in random matrix theory

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We link the appearance of universal kernels in random matrix ensembles to the phenomenon of shock formation in some fluid dynamical equations. Such equations are derived from Dyson's random walks after a proper rescaling of the time. In the case of the Gaussian unitary ensemble, on which we focus in this paper, we show that the *characteristics polynomials and their inverse* evolve according to a viscid Burgers equation with an effective "spectral viscosity" $\nu_s = 1/2N$, where N is the size of the matrices. We relate the edge of the spectrum of eigenvalues to the shock that naturally appears in the Burgers equation for appropriate initial conditions, thereby suggesting a connection between the well-known *microscopic* universality of random matrix theory and the universal properties of the solution of the Burgers equation in the vicinity of a shock.

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In a seminal paper [1], Dyson showed that the distribution of eigenvalues of a random matrix could be interpreted as the result of a random walk performed independently by each of the matrix elements. The equilibrium distribution yields the so-called "Coulomb gas" picture, with the eigenvalues identified to charged point particles repelling each other according to Coulomb law. For matrices of large sizes, this correctly describes the bulk properties of the spectrum [2]. In his original work Dyson introduced a restoring force preventing the eigenvalues to spread forever as time goes. This is what allowed him to find an equilibrium solution corresponding to the considered random ensemble, with a chosen variance (related to the restoring force). In this note, we point out that Dyson's approach yields a richer structure if one performs a rescaling of the time of the random walk. The random walk can then be described by an equation of fluid dynamics, the viscid Burgers equation. In this picture, which we may refer to as "Dyson fluid," the edge of the spectrum appears as the precursor of a shock wave, and its universal properties follow from a simple analysis of the Burgers equation that was developed in other contexts [3]. This enables us to recover familiar results of random matrix theory in a simple way and to point out a connection between the well-known universal characters of these results to universal properties of the solution of the Burgers equation in the vicinity of a shock.

In this paper we discuss only the Gaussian unitary ensemble, although we believe that many of our results can be extended to other ensembles, as we argue in the concluding remarks of this paper. Thus, we consider $N\times N$ Hermitian matrices H with complex entries. We assume that these matrices evolve in time according to the following random walk: in the time step $\delta\tau$, $H_{ij}\to H_{ij}+\delta H_{ij}$, with $\langle \delta H_{ij}\rangle=0$, and $\langle(\delta H_{ij})^2\rangle=(1+\delta_{ij})\delta t$. That is, we assume that at each time step, the increments of the matrix elements are drawn from a Gaussian distribution with a variance proportional to δt. The initial condition on the random walk is that at time $t=0$, all the matrix elements vanish. Alternatively, let x_i denote the eigenvalues of H. The previous random walk translates into a corresponding random walk of the eigenvalues, with the following characteristics [1]:

$$\langle \delta x_i\rangle = E(x_i)\,\delta t, \quad \langle(\delta x_i)^2\rangle = \delta t, \tag{1}$$

where the "Coulomb force"

$$E(x_j) = \sum_{i\neq j}\left(\frac{1}{x_j - x_i}\right) \tag{2}$$

originates from the Jacobian Δ of the transformation from the matrix elements to the eigenvalues, $\Delta=\Pi_{i<j}(x_i-x_j)^2$. The joint probability $P(x_1,\ldots,x_N,t)$ for finding the set of eigenvalues near the values $x_1,\ldots,x_N$ at time t obeys the Smoluchowski-Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{1}{2}\sum_i \frac{\partial^2 P}{\partial x_i^2} - \sum_i \frac{\partial}{\partial x_i}[E(x_i)P]. \tag{3}$$

The average density of eigenvalues, $\tilde{\rho}(x,t)$, may be obtained from P by integrating over $N-1$ variables. Specifically,

$$\tilde{\rho}(x,t) = \int \prod_{k=1}^{N} dx_k P(x_1,\ldots,x_N,t)\sum_{l=1}^{N}\delta(x-x_l), \tag{4}$$

with normalization $\int dx\tilde{\rho}(x,t)=N$. Similarly we define the "two-particle" density $\tilde{\rho}(x,y,t)=\langle\Sigma_{l=1}^{N}\Sigma_{j\neq l}\delta(x-x_l)\delta(y-x_j)\rangle$, with $\int dxdy\tilde{\rho}(x,y,t)=N(N-1)$. These various densities obey an infinite hierarchy of equations obtained form Eq. (3) for P [akin to the well-known Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of statistical mechanics]. Thus, the equation relating the one- and two-particle densities reads

$$\frac{\partial\tilde{\rho}(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2\tilde{\rho}(x,t)}{\partial x^2} - \frac{\partial}{\partial x}⨍ dy\frac{\tilde{\rho}(x,y,t)}{x-y}, \tag{5}$$

where ⨍ denotes the principal value of the integral.

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In the large-N limit, this equation becomes a closed equation for the one-particle density. To show that, we rescale the time so that $\tau=Nt$ [4], and we define

$$\tilde{\rho}(x,t) = N\rho(x,\tau). \tag{6}$$

Similarly, we set

$$\tilde{\rho}(x,y,t) = N^2\rho(x,\tau)\rho(y,\tau) - N\rho_c(x,y,\tau), \tag{7}$$

with $\int dxdy\rho_c(x,y,\tau)=1$. One then obtains

$$\frac{\partial\rho(x,\tau)}{\partial\tau} + \frac{\partial}{\partial x}\rho(x,\tau)\fint dy\frac{\rho(y,\tau)}{x-y} = \frac{1}{2N}\frac{\partial^2\rho(x,\tau)}{\partial x^2} + \frac{1}{N}\fint dy\frac{\rho_c(x,y,\tau)}{x-y}. \tag{8}$$

In the large-N limit, the right-hand side is negligible, leaving as announced a closed equation for $\rho(x,\tau)$. This equation can be further transformed into an equation for the resolvent,

$$G(z,\tau) = \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z-H(\tau)}\right\rangle = \int dy\frac{\rho(y,\tau)}{z-y}, \tag{9}$$

whose imaginary part for $z=x-i\epsilon$, and x real, yields the average spectral density $\rho(x,\tau)$, while the real part is the Hilbert transform of ρ,

$$\mathcal{H}\rho(x,\tau) = \fint dy\frac{\rho(y,\tau)}{(x-y)}. \tag{10}$$

By taking the Hilbert transform of Eq. (8), keeping only the dominant term in the large-N limit, and using well-known properties of the Hilbert transform, one gets the following equation for $G(z,\tau)$:

$$\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0. \tag{11}$$

This is the inviscid (complex) Burgers equation [5]. Note that the Laplacian, which naturally appears in the description of diffusive processes, has disappeared in the large-N limit, as already indicated after Eq. (8). We shall return later to the role of diffusion, focus now on the solution of Eq. (11), and recover from it well-known results of random matrix theory.

The solution of Eq. (11) can be obtained by using the method of (complex) characteristics [6], with the characteristics determined by the implicit equation

$$z = \xi + \tau G_0(\xi), \quad G_0(z) = G(z,\tau=0) = 1/z. \tag{12}$$

Assuming the solution $\xi(z,\tau)$ to be known, the Burgers equation can be solved parametrically as $G(z,\tau) = G_0[\xi(z,\tau)] = G_0[z-\tau G(z,\tau)]$. The solution of this equation that is analytical in the lower half plane is

$$G(z,\tau) = \frac{1}{2\tau}(z - \sqrt{z^2-4\tau}), \tag{13}$$

whose imaginary part yields the familiar Wigner's semicircle for the average density of eigenvalues. Solving the Burgers equation with characteristics is perhaps the simplest way of getting this seminal result. It also exemplifies the role of Burgers equation in dealing with free (in the sense of Voiculescu) matrix-valued variables [5].

In the fluid dynamical picture suggested by the Burgers equation, the edge of the spectrum corresponds to a singularity that is associated with the precursor of a shock wave, sometimes referred to as a "preshock" [3]. This singularity occurs when the mapping between z and ξ ceases to be one to one, a condition required for the validity of the method of characteristics. This takes place when $dz/d\xi=0=1+\tau G_0'(\xi_c)$, defining $\xi_c(\tau)$. Note that the existence of a shock in the solution of the Burgers equation is intimately connected with the form of the initial condition, given here by $G_0(z)=1/z$. Since $G_0'(\xi_c)=-1/\xi_c^2$, $\xi_c(\tau)=\pm\sqrt{\tau}$ and $z_c=\xi_c+\tau G_0(\xi_c) = \pm 2\sqrt{\tau}$. That is, the singularity occurs precisely at the edge of the spectrum. Furthermore, the resulting singularity is of the square-root type. To see what is meant by this, let us expand the characteristic equation around the singular point. We get

$$z - z_c = \frac{\tau}{2}(\xi-\xi_c)^2 G_0''(\xi_c) = \frac{\tau}{\xi_c^3}(\xi-\xi_c)^2. \tag{14}$$

It follows that, in the vicinity of the positive edge of the spectrum $z\simeq z_c=2\sqrt{\tau}$, $\xi-\xi_c=\pm\tau^{1/4}\sqrt{z-z_c}$. Thus, as z moves toward z_c and is bigger than z_c, ξ moves to ξ_c on the real axis. When z becomes smaller than z_c, ξ moves away from ξ_c along the imaginary axis. The imaginary part therefore exists for $z<z_c$ and yields a spectral density $\rho(z)\sim\sqrt{z_c-z}$, in agreement with Eq. (13). This square-root behavior of the spectral density implies that in the vicinity of the edge of the spectrum, the number of eigenvalues in an interval of width s scales as $Ns^{3/2}$, implying that the interlevel spacing goes as $N^{-2/3}$.

In order to capture the fine structure of the distribution of eigenvalues in the vicinity of the edge, we need to take into account the $1/N$ corrections. We would have liked to do that within the Dyson fluid picture, that is, we proceed along the lines suggested by our initial derivation and start form Eq. (8). This, however, would require the correct handling of the connected two-point function $\rho_c(x,y,\tau)$, and we know of no simple way to do so. We shall therefore proceed differently and adapt standard finite-N techniques of random matrix theory in order to obtain simple equations that generalize Eq. (8), albeit not for the one-particle density, but for different objects that are directly related to the one-particle density only in the large-N limit. The resulting equations take the form of viscid Burgers equations and are valid for any N. The procedure allows, in principle, the calculation of all the correlation functions for finite N and t.

The technique that we shall use is that of orthogonal polynomials. For the considered matrix ensemble, the relevant polynomials are the Hermite polynomials, defined as [7]

$$h_k(x) = (-1)^k e^{Nx^2/2}\frac{d^k}{dx^k}e^{-Nx^2/2}. \tag{15}$$

These admit the following useful integral representation:

$$h_k(x) = (-iN)^k \sqrt{\frac{N}{2\pi}} e^{Nx^2/2} \int_{-\infty}^{\infty} dq\, q^k e^{-N/2q^2 + ixqN}. \quad (16)$$

We shall also use the so-called monic polynomials, i.e., polynomials where the coefficient of the highest-order term is equal to unity: $\pi_k(x) \equiv h_k(x)/N^k = \Pi_{i=1}^k (x - \bar{x}_i)$, with $\bar{x}_i$ denoting the (real) zeros of the Hermite polynomials. In the random walk described above, the probability distribution retains its form at all instants of time. In fact, after the rescaling $t \to \tau/N$, the solution of Eq. (3) reads simply

$$P(x_1, \ldots, x_N, \tau) = C(\tau) \prod_{i<j} (x_i - x_j)^2 \exp\left(-\sum_i \frac{N x_i^2}{2\tau} \right), \quad (17)$$

with $C(\tau)$ fixed by normalization. This suggests that the relevant polynomials are those which remain orthogonal with respect to the time-dependent measure $\exp[-Nx^2/(2\tau)]$. To get these, all that one needs to do is to replace N with N/τ in Eq. (16). One obtains then

$$\pi_k(x, \tau) = (-i)^k \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} dq\, q^k e^{-(N/2\tau)(q - ix)^2}, \quad (18)$$

which satisfies

$$\int_{-\infty}^{\infty} dx e^{-Nx^2/2\tau} \pi_n(x, \tau) \pi_m(x, \tau) = \delta_{nm} c_n^2, \quad (19)$$

with $c_n^2 = n! \sqrt{2\pi} (\tau/N)^{n+1/2}$. The monic character of π_n's is not affected by the time dependence.

By using the integral representation (18), it is easy to show that $\pi_n(x,\tau)$'s satisfy the following equation:

$$\partial_\tau \pi_n(x, \tau) = -\nu_s \partial_x^2 \pi_n(x, \tau), \quad (20)$$

with $\nu_s = 1/2N$. This is like a diffusion equation with, however, a negative diffusion constant. This negative sign, as well as that of the corresponding "viscous term" in Eq. (22) below, somewhat obscures the direct physical interpretation that one may be tempted to give to these equations. From a technical point of view, we may argue that π_n is an analytical function of x, and $\pi_n(-iy, \tau)$ (with y real) satisfies a diffusion equation with a positive constant. But we believe that the negativity of the diffusion constant allows for the following intuitive interpretation: a positive diffusion constant would act as a friction that smooths the shocks. Conversely, a negative diffusion constant should act just in the opposite way and produce oscillations accompanying the shock. We shall verify below that this is indeed what happens.

At this point, one may perform an inverse Cole-Hopf transform, i.e., define the new function

$$f_k(z, \tau) \equiv 2\nu_s \partial_z \ln \pi_k(z, \tau) = \frac{1}{N} \sum_{i=1}^k \frac{1}{z - \bar{x}_i(\tau)}. \quad (21)$$

The resulting equation for f_k is the viscid Burgers [8] equation

$$\partial_\tau f_k(z, \tau) + f_k(z, \tau) \partial_z f_k(z, \tau) = -\nu_s \partial_z^2 f_k(z, \tau), \quad (22)$$

with $-\nu_s$ playing the role of a (negative) viscosity.

Equation (22) is satisfied by all the functions f_k. We shall focus now on the function f_N associated with $\pi_N(z,\tau)$. The reason is that $\pi_N(z,\tau)$ is known to be equal to the average characteristic polynomial [9], i.e.,

$$\langle \det[z - H(\tau)] \rangle = \pi_N(z, \tau). \quad (23)$$

Since in the large-N limit, $\partial_z \ln\langle \det[z - H(\tau)]\rangle \approx \partial_z \langle \ln \det[z - H(\tau)]\rangle = NG(z)$ [10], $f_N(z,\tau)$ coincides in this limit with the average resolvent $G(z,\tau)$. In fact the structure of f_N, as clear from Eq. (21), is very close to that of the resolvent, with its poles given by the zeros of the average characteristic polynomial. Equation (22) for $f_N(z,\tau)$ is exact. The initial condition $f_N(z, \tau=0) = 1/z$ does not depend on N, so that all the finite-N corrections are taken into account by the viscous term. Note however that this exact equation does not give directly the finite-N corrections to the resolvent or the spectral density.

We turn now to the study of the viscid Burgers equation for $f_N(z,t)$, i.e., Eq. (22) for $k=N$, in the vicinity of the (moving) preshock, which is for x near $z_c(\tau)$. The Cole-Hopf transformation used above provides a solution in terms of $\pi_N(x,\tau)$, which allows us to study the effects of a small viscosity by performing a saddle-point approximation on integral (18). The saddle-point equation coincides with the characteristic equation discussed earlier (with the identification $\xi \to -iq$). However, the method of characteristics breaks down in the vicinity of the shock because of the merging of the two saddle points associated with the square-root singularity. A better analysis is then called for. One could still proceed by a careful saddle-point analysis, as done, for instance, in order to get the asymptotic behavior of the Hermite polynomials (see, e.g., [7]). However, we shall choose a more direct approach, relying in what follows on tools borrowed from the theory of turbulence [11], more in line with the present discussion. Let us recall that in the vicinity of the edge of the spectrum, and in the inviscid limit,

$$f_N(z, \tau) \simeq \pm \frac{1}{\sqrt{\tau}} \mp \frac{1}{\tau^{3/4}} \sqrt{z - z_c}. \quad (24)$$

We set

$$x = z_c(\tau) + \nu_s^{2/3} s, \quad f_N(x, \tau) = \dot{z}_c(\tau) + \nu_s^{1/3} \chi_N(s, \tau), \quad (25)$$

with $\dot{z}_c \equiv \partial_\tau z_c = \pm 1/\sqrt{\tau}$. The particular scaling of the coordinate is motivated by the fact that near the square-root singularity the spacing between the eigenvalues scales as $N^{-2/3}$. A simple calculation then yields the following equation for $\chi(s,\tau)$ in the vicinity of $z_c(\tau) = 2\sqrt{\tau}$:

$$\partial_\tau^2 z_c + \nu_s^{1/3} \frac{\partial \chi}{\partial \tau} + \chi \frac{\partial \chi}{\partial s} = -\frac{\partial^2 \chi}{\partial s^2}, \quad (26)$$

which, ignoring the small term of order $\nu_s^{1/3}$, we can write as

$$\frac{\partial}{\partial s}\left[-\frac{s}{2\tau^{3/2}} + \frac{1}{2}\chi^2 + \frac{\partial \chi}{\partial s} \right] = 0, \quad (27)$$

or, setting $\chi = 2\partial_s \ln \phi(s)$, as

$$\partial_s^2\phi - as\phi = 0, \quad a \equiv \frac{1}{4\tau^{3/2}}. \tag{28}$$

The general solution of this equation is given by Airy functions [12] $\phi_i(s)=\epsilon_i Ai(\epsilon_i s a^{1/3})$, where ϵ_i for $i=0,1,2$ are cubic roots of 1 and $\Sigma_0^2\phi_i(s)=0$. In the case of the characteristic polynomial, the solution corresponds to $\epsilon_0=1$. Finally,

$$\chi(s,\tau) = 2\frac{Ai'(a^{1/3}s)}{Ai(a^{1/3}s)}. \tag{29}$$

This function captures the oscillatory behavior mentioned earlier. It coincides with what would be obtained from the asymptotic behavior of the Hermite polynomials which yields the well-known universal Airy kernel of random matrix theory [13]. In our interpretation, the oscillatory Airy function is a signal of the preshock phenomenon. In the context of random matrix theory, the universal character of this preshock comes from the fact that the rescaling of the variables x with a fractional power of viscosity (26) has the effect, in the large-N limit, to dwarf and make therefore irrelevant all the terms that make the random matrix probability distribution deviate from the Gaussian distribution that we explicitly considered here.

We complete our description of the Dyson fluid by considering the Cauchy transforms of the monic orthogonal polynomials,

$$p_k(z,\tau) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx \frac{\pi_k(x,\tau)e^{-Nx^2/2\tau}}{x-z}. \tag{30}$$

The motivation for doing so is that the average of the inverse characteristic polynomial is related to p_{N-1} [14]:

$$\left\langle \frac{1}{\det[z-H(\tau)]} \right\rangle = -\frac{2\pi i}{c_{N-1}^2} p_{N-1}(z,\tau), \tag{31}$$

where c_{N-1}^2 is given after Eq. (19). To get the time-dependent version of p_k's, we plug into Eq. (30) the integral representation of the time-dependent monic polynomials $\pi_k(x,\tau)$. A simple calculation then yields, for Im $z>0$,

$$p_k(z) = (-i)^k \sqrt{\frac{N}{2\pi\tau}} \int_0^{\infty} dq\, q^k e^{-(N/2)\tau(q^2-2iqz)}, \tag{32}$$

and $p_k(-z,\tau)=(-1)^{k+1}p_k(z,\tau)$. It can be shown by a direct calculation that the functions

$$\tilde{p}_k(z,\tau) = e^{(N/2\tau)z^2} p_k(z,\tau) \tag{33}$$

satisfy the same recurrence relations and differential equations as the polynomials $\pi_k(z,\tau)$, in particular Eq. (20). One may also verify that the function

$$g_k(z,\tau) = 2\nu_s \partial_z \ln \tilde{p}_k(z,\tau), \tag{34}$$

analogous to $f_k(z,\tau)$ in Eq. (21), satisfies a viscid Burgers equation:

$$\partial_\tau g_k + g_k \partial_z g_k = -\nu_s \partial_z^2 g_k. \tag{35}$$

Also this equation exhibits the phenomenon of universal preshock, but this time the scaling solution involves the two other solutions of the Airy equation that were mentioned earlier, namely, ϕ_1 or ϕ_2, depending on the sign of the imaginary part of z, in agreement with [15]. The fact that two such different objects as the monic orthogonal polynomials and their Cauchy transforms fulfill similar, and relatively simple, equations is quite remarkable. It may reflect deeper relations at the level of Riemann-Hilbert construction [16].

We note that since $\pi_N(z)$ and $p_N(z)$, and their first derivatives, are the building blocks of all relevant multipoint correlators (products of characteristic polynomials, products of inverse characteristic polynomials, and mixed products [17]), all universal kernels have an implicit memory of the dynamics of shocks in the viscid Dyson fluid. It is interesting to speculate and wonder to what extent the phenomenon of shock waves and their universal predecessors is a generic phenomenon of random matrix theory and is not restricted to the simplest Gaussian dynamics representing additive matrix diffusion. We have in fact reasons to suspect that it is generic.

We start by recalling that the fluid dynamical picture presented here holds also for other matrix-valued diffusion processes, in particular for the multiplicative diffusions, corresponding to random walks involving products of complex matrices [18,19]. We note here that the relation between additive and multiplicative matrix-valued random walks is not a trivial one since $\exp H_1 \exp H_2 \neq \exp(H_1+H_2)$, where H_1, H_2 are noncommuting entries. Besides, the product of Hermitian matrices is in general not a Hermitian matrix, so the closest matrix analog of the multiplicative random walk is perhaps the product of unitary matrices [20,21]. This case has an intriguing relation to two-dimensional Yang-Mills theory with a large number of colors N_c [22] and the associated universality conjectured by Narayanan and Neuberger [23]. We have earlier emphasized the relevance of shock waves in this context [6], as providing a simple mechanism for the Durhuus-Olesen [24] order-disorder transition. Recently, Neuberger used the explicit expressions for the averages of the characteristic polynomial and its inverse that he obtained through a character expansion to prove that closely related functions satisfy Burgers equations with a spectral viscosity $1/2N_c$ [25,26]. This result is remarkable and corroborates the "hydrodynamic" picture of universality put forward in the present work. Recall that, despite the fact that the explicit solution for the time-dependent joint probability is known in the case of Smoluchowski-Fokker-Planck equation for multiplicative random walks of unitary matrices, the explicit construction in terms of the orthogonal polynomials (belonging here to the Schur class) is missing. As in the case discussed in this paper, the edges of the evolving spectrum of a Wilson loop (the role of the time is played in this case by the area of the loop, and the eigenvalues are restricted to lie on a unit circle) develop universal preshocks of the Airy type. However, a different phenomenon emerges from the fact that, due to the compactness of the circle, left and right "front waves" can collide at some critical area, developing a different kind of universality, the Pearcey preshock, also encoded in the structure of the underlying Burgers equation [27,28]. Mathematically, this corresponds to the vanishing of the second derivative in the expansion of the complex characteristics in the vicinity of the shock [see Eq. (14)], leaving the third derivative as the leading term, and a resulting singularity of a cubic nature, rather than quadratic as in Eq. (14). Further details are given in our earlier papers [6,28] and the works of Neuberger [25–27]. The universal character seems to be confirmed by the lattice studies of $d=3$ Yang-Mills theory, reproducing correctly all critical exponents of the emerging preshock. A similar Pearcey preshock appears when two intervals supporting the Hermitian spectrum merge as a function of an external parameter [29,30].

It is also tempting to reanalyze from the perspective of the fluid dynamics of eigenvalues the well-known phenomenon of chiral phase transition in Euclidean quantum chromodynamics with massless fermions. Spontaneous breakdown of the chiral symmetry corresponds to a dramatic rearrangement of the eigenvalues of the Dirac operator near zero, as Bank-Casher argument tells us. In the narrow gap around the zero eigenvalue, the universal class of the chiral phase transition is defined by chiral matrix models [31] and is well studied on the lattice [32]. In this case the universal preshock appears when the wave of eigenvalues approaches the "hard wall" at zero, and the oscillating pattern is attributed to Bessel rather than to Airy functions, a phenomenon known as "hard edge" universality.

Finally, one could contemplate the possibility that similar shock phenomena could be identified for noncompact random matrix models, e.g., in mesoscopic systems where the role of the time is played by the length of the wire [33]. At first sight this seems not to be the case, because the corresponding Smoluchowski-Fokker-Planck (SFP) equation, known as the Dorokhov-Mello-Pereyra-Kumar (DKMP) equation looks different. Surprisingly, the reason why this equation was solved exactly is attributed to the fact that, by nontrivial changes of variables, the DKMP equation can be mapped to a complex Burgers equation [34]. Taking into account the underlying diffusive mechanism of conductance in mesoscopic wires it seems quite unlikely that the emergence of this Burgers structure in this context is purely accidental.

Last but not least, let us recall that this picture of shock formation finds analogies in other branches of physics, e.g., in the description of merging singularities in optics, leading to the so-called diffraction catastrophes (see, e.g., [35]), in the study of growth processes of the Kardar-Parisi-Zhang universality class and statistical properties of the equilibrium shapes of crystals [36] and finally, in mathematics, in the study of so-called "vicious walker models" [37]. On could also mention the recent studies of general random matrix models under the influence of an external environment that depends on a single parameter playing the role of time [38]. All these examples lead us to believe that pushing Dyson's original concept of temporal dynamics for random matrices may allow for a better understanding of the above analogies.

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