Wave-number dependent current correlation for a harmonic oscillator

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The wave-number k dependent current-correlation function is considered for a harmonic oscillator model. An explicit analytic expression for the Laplace transformed correlation function is derived. It is compared with numerical solutions and results obtained by the recurrence relation method. Several limiting cases such as the long-wavelength limit $k \to 0$ and the deep inelastic limit $k \to \infty$ are discussed in detail. In particular, we show that the deep inelastic limit allows for an explicit summation of the continued fraction. An approximation scheme for the recurrants at intermediate values of k is also considered.

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I. INTRODUCTION

Correlation functions are an important tool in statistical physics, both for equilibrium and nonequilibrium systems [1]. Applications range from structural information in solid-state physics [2] to unraveling collective effects in plasmas [3] and liquids [4]. For many applications, the time and space dependences of correlations are of interest. A prominent example is the current-correlation function which is connected to transport properties such as the optical conductivity and the dynamical structure factor [1]. From the latter, dispersion results for collective modes in condensed matter can be obtained.

A lot of information on correlation functions have been gathered for various systems, e.g., in one-dimensional or quasi-one-dimensional systems. Experimental studies range from colloidal suspensions [5], carbon nanotubes [6], particles in storage rings [7], microelectronic devices [8], microfluid crystals [9], and to ordered structures in complex (dusty) plasmas [10]. In the case of a chain of dust particles in a complex plasma, the wave-number dependent correlation function is easy to determine experimentally due to the direct access to position and velocity of the dust particles at all times [11]. However, it is not easy to determine the time- and space-dependent correlation functions theoretically starting from a microscopic statistical approach. The problem is twofold. First, we have to determine the time evolution of the quantity in question. Second, an ensemble average has to be carried out. Even for one-dimensional systems, exact results are known only for very simple types of correlation functions.

Therefore, we simplify the system under consideration further and study the current-correlation function for a single particle in a harmonic potential. For such a system, the time evolution can be determined exactly and also the statistical average in the canonical ensemble can be performed. Thus, we can derive an exact result for the wave-number dependent current-correlation function. Since this downscaled setup shows a rich complexity, the result will be useful as a building block for more complicated systems such as the one-dimensional chain of harmonic oscillators or the so-called independent oscillator model [12,13].

Time-dependent correlation functions have been extensively studied and a host of methods have been developed to obtain approximate expressions for many-body systems (see, e.g., Ref. [14] for a review). Having an exact result at our disposal, we can examine the power of these methods. As a method of choice, we consider the recurrence relation method (RRM) [15–17]. It has been successfully applied to determine, e.g., the momentum correlation for a harmonic-oscillator chain [18] and the independent oscillator model [19]. Moreover, an extension to harmonic oscillators on the Bethe lattice has been studied [20]. Also anharmonic potentials have been studied in Ref. [21]. The RRM was applied to elaborate the dynamics of spin systems [17,22–30], correlations in the electron gas [31–33], one component [34,35], as well as multicomponent plasmas [36]. Specific features such as a long-time tail are discussed in Refs. [37,38]. For a particular simple application, we refer the reader to Ref. [39]. Very recently, the RRM has been used for investigation of ergodicity in Ref. [40].

In this paper, we derive analytic results for the time evolution of the wave-number k dependent current-correlation function and its Laplace transform. We analyze these expressions in limiting cases and compare them with expressions derived from the RRM. Supplementing the exact expressions with the RRM, simplified analytic results are obtained. The theoretical analysis is carried out in Sec. II. Numerical illustrations as well as some tests for approximate methods are presented in Sec. III. Detailed calculations are postponed to several appendices.

II. THEORETICAL ANALYSIS

Dynamic correlation function of two operators A and B for a single particle is defined as

$$\langle A;B\rangle_z = \int_0^\infty dt e^{-zt}\langle A(t)^* B(0)\rangle, \tag{1}$$

where the statistical average $\langle \cdots \rangle$ is given by

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$$\langle A(t)^* B(0)\rangle = \frac{1}{Z}\int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dP \exp(-\beta H) A(t)^* B(0), \quad (2)$$

with Z being the partition function, $\beta = 1/k_B T$ being the inverse temperature, and $*$ indicating complex conjugation. The canonical coordinate and momentum are Q and P, respectively. The frequency z is a complex quantity and physical relevant expressions are obtained by taking the limit $z = \epsilon - i\omega$ with $\epsilon \to 0$. The time evolution of A is governed by the Liouvillian as $A(t) = e^{i\mathcal{L}t} A(0)$. Note that this correlation function obeys the relation [41]

$$z\langle A;B\rangle_z = \langle A^* B\rangle + \langle i\mathcal{L}A;B\rangle_z \quad (3)$$

for two operators A and B. This relation follows from integration by parts and allows to trade a correlation function involving A for a correlation function containing $\dot{A} = i\mathcal{L}A$.

We consider a one-dimensional oscillator in an external harmonic potential. This assumption is only for simplicity, and the results can be generalized to higher dimensions. Let Q be the elongation and P be the corresponding momentum. We start with the Hamiltonian given as

$$H = \frac{P^2}{2M} + M\Omega^2 \frac{Q^2}{2}, \quad (4)$$

with the eigenfrequency Ω and the mass M. The corresponding Liouvillian is

$$i\mathcal{L} = \frac{P}{M}\frac{\partial}{\partial Q} - M\Omega^2 Q \frac{\partial}{\partial P}. \quad (5)$$

The current is given by the product of momentum and position $J(x,t) = P(t)\delta[x - Q(t)]$ in the system written in Eq. (4). It is convenient to go to Fourier space studying $J_k(t) = P(t)e^{ikQ(t)}$, where k is the wave number. The current-correlation function studied here is defined as

$$\langle J_k;J_k\rangle_z = \langle Pe^{ikQ};Pe^{ikQ}\rangle_z. \quad (6)$$

Note that this expression reduces to the momentum correlation function for $k=0$, which is well known [42] and simply given by

$$\langle P;P\rangle_z / \langle PP\rangle = \frac{z}{z^2 + \Omega^2}. \quad (7)$$

Since the right-hand side of this equation is the Laplace transform of the cosine function, it reflects the periodic behavior of the momentum. Equation (7) can be directly obtained by using Eq. (3) twice exploiting the equation of motion $\ddot{P} = -\Omega^2 P$.

The rest of the paper is devoted to determine $\langle J_k;J_k\rangle_z$ at arbitrary wave number k. To this end we follow two schemes. The first is to calculate the Laplace transform of the time-correlation function by solving the equation of motion. The second is to employ the recurrence relation method in order to obtain the dynamic correlation function $\langle J_k;J_k\rangle_z$ from static data by constructing a special basis f_ν via a recurrence relation (see Ref. [15]). In doing so, we start with an initial basis vector $f_0 = P(0)\exp[ikQ(0)]$ and obtain further basis vectors from

$$f_{\nu+1} = i\mathcal{L}f_\nu + \Delta_\nu f_{\nu-1}, \quad (8)$$

$\nu \geq 0$, and the recurrants $\Delta_\nu = \langle f_\nu^* f_\nu\rangle / \langle f_{\nu-1}^* f_{\nu-1}\rangle$ for $\nu \geq 1$ and $\Delta_0 = 0$. The correlation function itself is obtained as a continued fraction (see again Ref. [15]),

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \cfrac{1}{z + \cfrac{\Delta_1}{z + \cfrac{\Delta_2}{z + \cfrac{\Delta_3}{z + \dots}}}}$$

$$= \frac{1|}{|z} + \frac{\Delta_1|}{|z} + \frac{\Delta_2|}{|z} + \frac{\Delta_3|}{|z} + \dots . \quad (9)$$

Hereafter we use the second notation for the continued fraction throughout this paper. Putting this approach into work, two tasks have to be accomplished. First, the recurrants Δ_ν have to be determined as good as possible. Second, the continued fraction has to be evaluated, preferably by exact summation. In most situations, neither of these tasks can be done exactly and one has to resort to approximation schemes. We study these approximation schemes later. As we shall see later, the recurrence relation analysis allows us to simplify the exact expression in limiting cases. It also gives further insight into the time and space dependences of the correlation function.

A. Direct calculation of time evolution and its Laplace transform

For a harmonic oscillator, the time evolution of Q and P is well known,

$$Q(t) = Q(0)\cos(\Omega t) + \frac{P(0)}{M\Omega}\sin(\Omega t),$$

$$P(t) = P(0)\cos(\Omega t) - M\Omega Q(0)\sin(\Omega t),$$

with $Q(0)$ and $P(0)$ being the initial values at time $t=0$. For simplicity, we use $Q_0 = Q(0)$ and $P_0 = P(0)$ instead. With use of this, the time development of the current-correlation function is given as

$$\begin{aligned}
&\langle P(t)\exp[-ikQ(t)]P(0)\exp[ikQ(0)]\rangle \\
&\quad = \langle [P_0\cos(\Omega t) - M\Omega Q_0 \sin(\Omega t)]P_0 \\
&\qquad \times \exp[-ikP_0\sin(\Omega t)/M\Omega]\exp\{ikQ_0[1-\cos(\Omega t)]\}\rangle \\
&\quad = \langle P_0^2 \exp[-ikP_0\sin(\Omega t)/M\Omega]\rangle\langle\exp\{ikQ_0[1 \\
&\qquad - \cos(\Omega t)]\}\rangle\cos(\Omega t) - \langle P_0\exp[-ikP_0\sin(\Omega t)/M\Omega]\rangle \\
&\qquad \times \langle Q_0\exp\{ikQ_0[1-\cos(\Omega t)]\}\rangle M\Omega\sin(\Omega t). \qquad (10)
\end{aligned}$$

Now, we expand the exponentials

$$\exp\{ikQ_0[1-\cos(\Omega t)]\} = \sum_{n=0}^{\infty}\frac{1}{n!}[ik(1-\cos\Omega t)Q_0]^n,$$

$$\exp[-ikP_0/M\Omega\sin(\Omega t)] = \sum_{n=0}^{\infty}\frac{1}{n!}[-ikP_0\sin(\Omega t)/M\Omega]^n.$$

Crossing terms of P_0 and Q_0 such as $\langle P_0^2 Q_0^3\rangle$ vanish and the integrals for P_0 and Q_0 can be solved,

$$\langle Q_0^{2n}\rangle = (2n-1)!!\left(\frac{1}{\beta M\Omega^2}\right)^n,$$

$$\langle P_0^{2(n+1)}\rangle = (2n+1)!!\left(\frac{M}{\beta}\right)^{n+1}.$$

Using $(2n-1)!!/(2n)! = 1/(2^n n!)$ and introducing the dimensionless parameter A, i.e., the normalized wave number squared k^2,

$$A = k^2/(\beta M\Omega^2), \tag{11}$$

we can simplify Eq. (10) as

$$\begin{aligned}&\langle P(t)\exp[-ikQ(t)]P_0\exp(ikQ_0)\rangle\\ &\quad= \frac{M}{\beta}\exp\left(-\frac{A}{2}\sin^2\Omega t\right)\exp\left[-\frac{A}{2}(1-\cos\Omega t)^2\right]\\ &\qquad\times[(1-A\sin^2\Omega t)\cos\Omega t - A\sin^2\Omega t(1-\cos\Omega t)].\end{aligned} \tag{12}$$

Simplifying Eq. (12) further, we arrive at

$$\frac{\langle J_k^*(t)J_k(0)\rangle}{\langle J_k^* J_k\rangle} = e^{-A(1-\cos\Omega t)}(\cos\Omega t - A\sin^2\Omega t). \tag{13}$$

The k-sum rule has a negative value as

$$\int_0^{\infty} dk\frac{\langle J_k^*(t)J_k(0)\rangle}{\langle J_k^* J_k\rangle} = -\frac{\sqrt{2\pi}}{4}\sqrt{\beta M\Omega^2}\left|\sin\frac{\Omega t}{2}\right|. \tag{14}$$

Next, we calculate the Laplace transform by using the expansion of the exponential of the cosine in terms of the modified Bessel functions (see [43]),

$$e^{A\cos\Omega t} = I_0(A) + 2\sum_{n=1}^{\infty} I_n(A)\cos(n\Omega t), \tag{15}$$

with the modified Bessel function given by

$$I_n(A) = \left(\frac{A}{2}\right)^n\sum_{l=0}^{\infty}\frac{(A/2)^{2l}}{l!\Gamma(n+l+1)}. \tag{16}$$

Proceeding by

$$\int_0^{\infty} dte^{-zt}\cos\omega_1 t\cos\omega_2 t = \frac{1}{2}\left(\frac{z}{z^2+(\omega_1-\omega_2)^2} + \frac{z}{z^2+(\omega_1+\omega_2)^2}\right)$$

and taking advantage of the recurrence relation between modified Bessel functions,

$$I_{n-1}(A) + I_{n+1}(A) = \frac{2n}{A}I_n(A), \tag{17}$$

we finally arrive at

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = 2A^{-1}e^{-A}\sum_{n=1}^{\infty} n^2 I_n(A)\frac{z}{z^2+(n\Omega)^2}. \tag{18}$$

This is an exact result for the Laplace transformed current correlation of Eq. (6). It shows that the correlation function splits into frequency dependent Lorentzian functions for all harmonics of the fundamental eigenfrequency Ω. The weight is given by elementary functions and modified Bessel functions in the parameter A given by Eq. (11), i.e., in the normalized wave number squared k^2. Note that the frequency dependence and the wave-number dependence disentangle in this expression. Equation (15) is the expansion by the cosine of multiples of the angle Ωt and the key equation of this paper. It is more advantageous than the conventional expansion by the power of $A\cos\Omega t$ in that we easily have the position of the poles and can soon evaluate their residues as a function of A. The conventional expansion enables us only to have an approximate Laplace transform for Eq. (13). That approximate form is straightforward to obtain, not shown here, but less helpful to expect Eq. (18) correctly. The exact result is a sum of modified Bessel functions and Lorentzians, which can be done numerically. It is still worthwhile to look for additional simplifications. With this in mind, we study limiting expressions of Eq. (18).

Discussion of limiting cases

Starting from this result, several limiting cases can be considered. In the static limit, the slope of the correlation function is obtained as

$$\Omega^2\lim_{z\to 0}\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle}/z = A^{-1}[1-e^{-A}I_0(A)] \tag{19}$$

due to the relation $e^A = I_0(A) + 2\Sigma_{n=1}^{\infty} I_n(A)$, which is Eq. (15) with $t=0$. Thus, we discover a linear slope of the current-correlation function in the static limit for any value of the parameter A. This expression can be further analyzed for small and large values of A. We obtain the value in the long-wavelength limit $A\to 0$, up to third order in A, as

$$\Omega^2\lim_{z\to 0}\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle}/z = 1 - \frac{3}{4}A + \frac{5}{12}A^2 - \frac{35}{192}A^3 \tag{20}$$

and the asymptotic value for large A as

$$\Omega^2\lim_{z\to 0}\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle}/z = \frac{1}{A}\left(1 - \frac{1}{\sqrt{2\pi A}}\right). \tag{21}$$

For small k, i.e., for small values of the parameter A, the leading contribution in Eq. (18) is due to $n=1$. In particular, we rediscover the well-known result for the momentum correlation function in the limit $k\to 0$,

$$\lim_{k\to 0}\langle J_k;J_k\rangle_z/\langle J_k^* J_k\rangle = \langle P;P\rangle_z/\langle PP\rangle = \frac{z}{z^2+\Omega^2}. \tag{22}$$

At $k\neq 0$, higher harmonics of the eigenmode Ω contribute and the weight of each harmonic is given by the modified Bessel function $I_n(A)$. To quantify the k dependence of the long-wavelength limit, we use Eq. (16) and obtain

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = 2A^{-1}e^{-A}\sum_{n=1}^{\infty} n^2 (A/2)^n/n!\frac{z}{z^2+(n\Omega)^2}. \tag{23}$$

Using the series expansion [44]

$$\Gamma(a)\gamma^*(a,x) = \sum_{n=0}^{\infty}\frac{(-x)^n}{(a+n)n!} \tag{24}$$

for the incomplete gamma function $\gamma^*(a,x)$ defined as $\Gamma(a)\gamma^*(a,x)=x^{-a}[\Gamma(a)-\Gamma(a,x)]$ with the upper incomplete gamma function $\Gamma(a,x)$, the summation in Eq. (23) can be performed, and a representation in terms of the gamma function and the upper incomplete gamma function is found,

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = e^{-A}\frac{z}{\Omega^2}\mathrm{Re}\Bigg[\left(-\frac{A}{2}\right)^{-(1+iz/\omega)} \times\{\Gamma(1+iz/\Omega)-\Gamma(1+iz/\Omega,-A/2)\}\Bigg]. \tag{25}$$

For large k, i.e., for large values of the parameter A, we start with the identity, Eq. (15) with $t=0$,

$$e^A = I_0(A) + 2\sum_{n=1}^{\infty} I_n(A). \tag{26}$$

Noting the asymptotic n-independent behavior, $e^{-A}I_n(A) = (2\pi A)^{-1/2}$ (see [43]), Eq. (26) must hold true and thus we have asymptotically

$$\sum_{n=1}^{\infty} 1 = \frac{\sqrt{2\pi A}-1}{2}. \tag{27}$$

Equation (27) seems to be unlikely; however, we have to face the constraint of Eq. (26) for the analytical treatment. Furthermore, we must substitute $\Sigma_{n=1}^{N}$ with $N=(\sqrt{2\pi A}-1)/2$ for $\Sigma_{n=1}^{\infty}$ in evaluating Eq. (18). Thus, we obtain the asymptotic behavior of the Laplace transform in the leading order of large A as

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1}{A}\frac{z}{\Omega^2}\left[1-\frac{1}{\sqrt{2\pi A}}-\frac{2}{\sqrt{2\pi A}}\sum_{n=1}^{N}\frac{(z/\Omega)^2}{n^2+(z/\Omega)^2}\right]. \tag{28}$$

Taking finite summation is important when numerical application of Eq. (28) can be done. We can reproduce Eq. (21) from Eq. (28).

B. Recurrence relation analysis

Although an exact analytic result has been obtained, i.e., Eq. (18), evaluating the sum involves some complicated mathematics. Therefore, it is desirable to supplement the solution of the equation of motion performed above with a different approach. This will further elucidate the findings, especially in the limiting cases. The recurrence relation method provides a general scheme to obtain the dynamic correlation function $\langle J_k;J_k\rangle_z$ as a continued fraction of Eq. (9),

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1|}{|z} + \frac{\Delta_1|}{|z} + \frac{\Delta_2|}{|z} + \frac{\Delta_3|}{|z} + \cdots .$$

We start our analysis by exactly determining the lowest-order recurrants Δ_ν and $\nu\leq 5$. Setting $f_0=P(0)\exp[ikQ(0)]$ and proceeding en route Eq. (8), we obtain the following results:

$$\Delta_1 = \Omega^2(3A+1), \tag{29}$$

$$\Delta_2 = \Omega^2 A\frac{6A+9}{3A+1}, \tag{30}$$

$$\Delta_3 = \Omega^2\frac{30A^3+95A^2+48A+12}{(2A+3)(3A+1)}, \tag{31}$$

$$\Delta_4 = 20\Omega^2 A\frac{(3A+1)(4A^3+26A^2+45A+30)}{(2A+3)(30A^3+95A^2+48A+12)}, \tag{32}$$

$$\Delta_5 = 3\Omega^2\frac{(2A+3)(140A^6+1470A^5+4977A^4+7398A^3+4470A^2+1800A+360)}{(30A^3+95A^2+48A+12)(4A^3+26A^2+45A+30)}. \tag{33}$$

Needless to say, it is clear that higher-order recurrants have a very involved dependence on the wave number k. A summation of the corresponding continued fraction seems to be unlikely, but an expansion for the long-wavelength limit ($k\to 0$) as well as the deep inelastic limit ($k\to\infty$) is possible.

1. Long-wavelength limit

In the long-wavelength limit $k\to 0$, the above given recurrants Δ_ν can be expanded to read

$$(\Delta_1,\ldots,\Delta_5)_{k\to 0} = \left[\Omega^2(3A+1), 9\Omega^2 A, \Omega^2\left(\tfrac{4}{3}A+4\right), \tfrac{50}{3}\Omega^2 A, \Omega^2\left(\tfrac{3}{2}A+9\right)\right]. \tag{34}$$

Explicit results for further recurrants are given in Appendix B. A general representation for the recurrants in the long-wavelength limit can be obtained. For the even recurrants it is proportional to A,

$$\Delta_{2\nu,k\to 0} = 2\frac{(2\nu+1)^2}{\nu+1}\Omega^2 A. \quad (35)$$

As a consequence, even recurrants disappear in the limit $k \to 0$. Contrary, odd recurrants have finite values in the long-wavelength limit due to a contribution from the frequency,

$$\Delta_{2\nu-1,k\to 0} = \Omega^2(a_{2\nu-1}A + \nu^2). \quad (36)$$

The coefficients $a_{2\nu-1}$ follow from

$$a_{4\nu-1} = \frac{4\nu}{2\nu+1},$$

$$a_{4\nu+1} = \frac{2\nu+1}{\nu+1}.$$

Thus, in the limit $k \to 0$, a summation of the continued fraction given by the recurrants

$$(\Delta_1, \ldots, \Delta_8, \ldots) = \left(\Omega^2, 9\Omega^2 A, 4\Omega^2, \tfrac{50}{3}\Omega^2 A, 9\Omega^2, \tfrac{49}{2}\Omega^2 A, 16\Omega^2, \tfrac{162}{5}\Omega^2 A, \ldots\right) \quad (37)$$

is sought after. The sequence in Eq. (37) already shows some important features which will be confirmed by numerical evaluation in Sec. III A. For all $k \neq 0$, the correlation function will also show features at the harmonics of the eigenfrequency Ω. However, each harmonic is, including a numerical factor, suppressed by Eq. (11), i.e., $A = k^2/(\beta M \Omega^2)$, compared to the preceding one. The larger k will become, the more harmonics will contribute.

2. Deep inelastic limit

In this limit, we consider the expansion $k \to \infty$. Asymptotically, the lowest recurrants Δ_ν, $\nu \le 5$, are obtained as

$$(\Delta_1, \ldots, \Delta_5)_{k\to\infty} = (3\Omega^2 A, 2\Omega^2 A, 5\Omega^2 A, 4\Omega^2 A, 7\Omega^2 A). \quad (38)$$

A more extensive list including additional contribution in A^0 and A^{-1} can be found in Appendix C. Again, the leading contribution shows a regular behavior,

$$\Delta_{2\nu} = (2\nu)\Omega^2 A,$$

$$\Delta_{2\nu-1} = (2\nu+1)\Omega^2 A. \quad (39)$$

Appendix D outlines the summation of the corresponding continued fraction. Consequently, we obtain a representation in terms of the upper incomplete gamma function with $\tilde{z} = z/(\sqrt{2A}\Omega)$ as

$$\sqrt{2A}\Omega \frac{\langle J_k; J_k\rangle_z}{\langle J_k^* J_k\rangle} = \tilde{z}^2 e^{\tilde{z}^2}\Gamma(-1/2, \tilde{z}^2). \quad (40)$$

This is equivalently written by the generalized error function $w(x)$ (see Ref. [43]),

$$\sqrt{2A}\Omega \frac{\langle J_k; J_k\rangle_z}{\langle J_k^* J_k\rangle} = 2\tilde{z}[1 - \sqrt{\pi}\tilde{z} w(i\tilde{z})]. \quad (41)$$

C. Analysis of small-frequency behavior

We here confront our results with conclusions drawn earlier by Kim on the asymptotic long-time limit, i.e., the behavior at small frequencies as shown in Ref. [37]. If a Laplace transform $\bar{F}(z) = \int_0^\infty dt e^{-zt} F(t)$ is written as a continued fraction,

$$\bar{F}(z) = \frac{1|}{|z} + \frac{d_1|}{|z} + \frac{d_2|}{|z} + \frac{d_3|}{|z} + \ldots , \quad (42)$$

then we have

$$\lim_{z\to 0} z\bar{F}(z) = \left(1 + \frac{d_1}{d_2} + \frac{d_1 d_3}{d_2 d_4} + \frac{d_1 d_3 d_5}{d_2 d_4 d_6} + \cdots\right)^{-1}. \quad (43)$$

Equation (43) reminds us of the K parameter to discuss on $\lim_{t\to\infty} F(t)$ in Ref. [40]. Here, we apply this to study the small-frequency behavior of $\langle J_k; J_k\rangle_z / \langle J_k^* J_k\rangle$ and prove that $\lim_{z\to 0}\langle J_k; J_k\rangle_z / (z\langle J_k^* J_k\rangle)$ is a finite quantity. For this aim, we have to show that

$$h(z) = \lim_{z\to 0} \frac{z|}{|z} + \frac{\Delta_2|}{|z} + \frac{\Delta_3|}{|z} + \frac{\Delta_4|}{|z} + \ldots \quad (44)$$

does exist. Then, the limit itself is given by

$$\lim_{z\to 0}\langle J_k; J_k\rangle_z / (z\langle J_k^* J_k\rangle) = \frac{1}{\Delta_1}\left(1 + \frac{\Delta_2}{\Delta_3} + \frac{\Delta_2\Delta_4}{\Delta_3\Delta_5} + \frac{\Delta_2\Delta_4\Delta_6}{\Delta_3\Delta_5\Delta_7} + \cdots\right). \quad (45)$$

Now, we consider the long-wavelength limit $k \to 0$ and the deep inelastic limit $k \to \infty$. The corresponding recurrants are derived above in Eqs. (37)–(39). Equation (45) goes to $(\Omega^2)^{-1}$ as $k \to 0$ due to $\Delta_{2n} \propto A$, which is consistent with Eq. (20). While Eq. (45) goes to $(\Omega^2 A)^{-1}$ as $k \to \infty$ since the parenthesis in Eq. (45) converges to be 3 after paying careful attention to Wallis' product. This result is consistent with Eq. (21) and the slope obtained from Eq. (D8).

D. Analysis of high-frequency behavior

Finally, we consider the high-frequency behavior of the current-correlation function $\langle J_k; J_k\rangle_z / \langle J_k^* J_k\rangle$. With the help of the continued fraction representation, the asymptotic behavior is given as

$$\frac{\langle J_k; J_k\rangle_z}{\langle J_k^* J_k\rangle} = \sum_{n=0}^{\infty} (-1)^n \Delta_n' z^{-2n-1} \quad (46)$$

(see, e.g., Ref. [19]). The coefficients Δ_n' are recursively given as $\Delta_0' = 1, \Delta_1' = \Delta_1, \Delta_2' = \Delta_1(\Delta_1 + \Delta_2), \ldots$. Using the specific values for the recurrants Δ_1 and Δ_2, we arrive at

$$\Omega\frac{\langle J_k; J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{\Omega}{z} - (3A+1)\frac{\Omega^3}{z^3} + (15A^2 + 15A + 1)\frac{\Omega^5}{z^5} - \cdots. \quad (47)$$

If we compare this with the high-frequency expansion of Eq. (18),

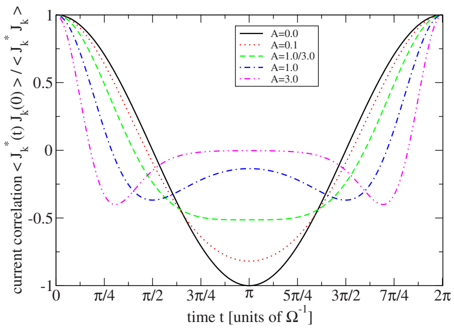

FIG. 1. (Color online) The normalized current-correlation function as a function of the time t in units of the inverse eigenfrequency Ω^{-1}. Several values of the parameter A are considered.

$$\Omega\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{2\Omega}{Az}e^{-A}\left(\sum_{n=1}^{\infty} n^2 I_n(A) - \sum_{n=1}^{\infty} n^4 I_n(A)\frac{\Omega^2}{z^2} + \sum_{n=1}^{\infty} n^6 I_n(A)\frac{\Omega^4}{z^4} - \cdots\right), \tag{48}$$

then we can establish a sequence of identities for modified Bessel functions,

$$2\sum_{n=1}^{\infty} n^2 I_n(A) = Ae^A, \tag{49}$$

$$2\sum_{n=1}^{\infty} n^4 I_n(A) = A(3A+1)e^A, \tag{50}$$

$$2\sum_{n=1}^{\infty} n^6 I_n(A) = A(15A^2+15A+1)e^A, \tag{51}$$

which can also be proven directly using Eq. (26) and the recursion relations obtained by Eq. (17).

III. COMPARISON OF CONTINUED FRACTION WITH NUMERICAL RESULTS

In this section, we present a numerical analysis of the above findings. First, we briefly illustrate the time evolution of the current-correlation function. Next, we discuss the Laplace transformed current-correlation function as a function of the frequency z and the parameter A. Finally, we check the limiting cases obtained above by numerical means.

A. Numerical analysis of the Laplace transform

In a first step, we show in Fig. 1 the time dependent current-correlation function $\langle J_k^*(t)J_k(0)\rangle/\langle J_k^* J_k\rangle$ as obtained in Eq. (13). It is a periodic function with period $T=2\pi/\Omega$. Several values for the parameter A have been explored. The momentum correlation function $\cos\Omega t$ for $A=0$ changes into a bowl-like shape for $0<A\leq 1/3$ with a minimum value at $t=\pi/\Omega$. For $1/3<A$, the correlation function behaves like a double well with minima located at $t=\cos^{-1}\{(\sqrt{4A^2+5}-3)/2A\}/\Omega$. The geometrical transition occurs at $A=1/3$.

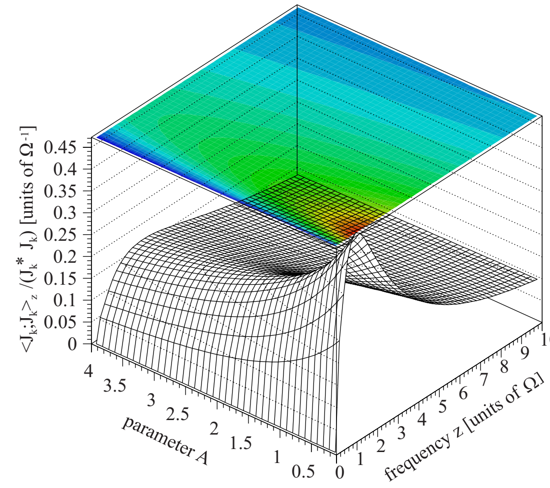

FIG. 2. (Color online) The normalized current-correlation function as a function of the parameter A and the frequency z. Besides the surface plot of the correlation function, also a contour plot is shown.

Note that the k-sum rule has negative values with the oscillation of $\Omega/2$ as in Eq. (14).

Next, we inspect the Laplace transform of the correlation function given in Eq. (13). The analysis can be carried out by numerical integration. Since the integrand of the Laplace transform as given by Eq. (13) is a periodic function, the Laplace transform can be determined as

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = (1-e^{-zT})^{-1}\int_0^T dt e^{-zt}\frac{\langle J_k^*(t)J_k(0)\rangle}{\langle J_k^* J_k\rangle}, \tag{52}$$

which is numerically advantageous because it does not involve integration to infinity. We have compared the numerical integration with performing the finite sum over Bessel functions given in Eq. (18) and found excellent agreement up to $n=100$. For this aim, we have used a continued fraction representation of the ratio of two successive Bessel functions (see Ref. [45]) to obtain a numerical approximation for the modified Bessel functions $I_n(A)$. We do not further elaborate on the numerical expense of either the numerical integration or the summation of Eq. (18). However, the advantage of Eq. (18) is apparent once we want to determine the physical interesting values given by $z=\epsilon-i\omega$. Here, numerical integration at large ω is challenging due to the rapid oscillations of the integrand, while Eq. (18) directly shows the positions of the fundamental mode and its harmonics.

We illustrate the normalized current-correlation function as given by Eq. (52) in Fig. 2. The current correlation is shown as a function of the parameter A and the frequency z in a surface plot combined with contour lines. As expected from our analysis above, the limit for $A\to 0$ is given by Eq. (22) and a maximum appears for small values of k at $z/\Omega=1$. This is flattened out for higher values of k or A.

In order to analyze this in more detail, we show the slope of the current correlation at frequency $z=0$ in Fig. 3 as a function of the parameter A [see Eq. (19)]. The slope decreases from an initial value of unity. The numerical evaluation confirms the small A dependence as obtained by Eq.

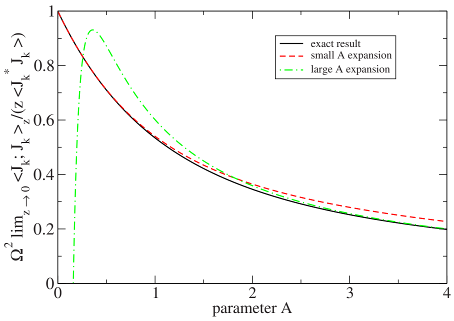

FIG. 3. (Color online) The slope of the current-correlation function as given by Eq. (19). The expansions for small A [Eq. (20)] and large A [Eq. (21)] are also shown.

(20), while the asymptotic value at large A is given by Eq. (21). The numerical study also reveals that these limiting cases can serve as reliable approximations over a large extend of A. Instead of obtaining the numerical result from an analysis of the modified Bessel function $I_0(A)$ as in Eq. (19), we can consider numerical integration,

$$\Omega^2 \lim_{z\to 0} \frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle}/z = \frac{\Omega^2}{2\pi}\int_0^T dt t^2 \frac{\langle J_k^*(t)J_k(0)\rangle}{\langle J_k^* J_k\rangle}. \tag{53}$$

Equation (53) is obtained by expanding Eq. (52) with respect to z and noting that $\int_0^T dt t^0 \langle J_k(t)^* J_k(0)\rangle = 0$ and $\int_0^T dt t^1 \langle J_k(t)^* J_k(0)\rangle = 0$. Again, Eq. (53) has excellent agreement with the slope obtained by the numerical integration with performing the finite sum given in Eq. (16) due to Eq. (19).

Now, we focus on the behavior at small values of A as a function of the frequency z; i.e., we study the deviation from the behavior given by Eq. (22). Figure 4 visualizes this, where the exact result is shown by solid lines and the approximate expression given by Eq. (25) is shown by circles. For $A=0.1$, both expressions agree well within the resolution of the graph. For $A=1.0$, notable deviations occur which increase for $A=2.0$. Note that Eq. (25) offers us the possibility to extend our results to physical values of the frequency without performing the infinite summation present in Eq. (18).

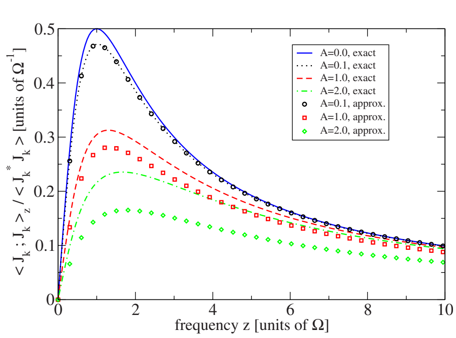

FIG. 4. (Color online) Normalized current-correlation function as a function of the frequency z for small values of the parameter A. We compare the exact expression Eq. (18) with the A expansion Eq. (25) and the $A \to 0$ limit [Eq. (22)].

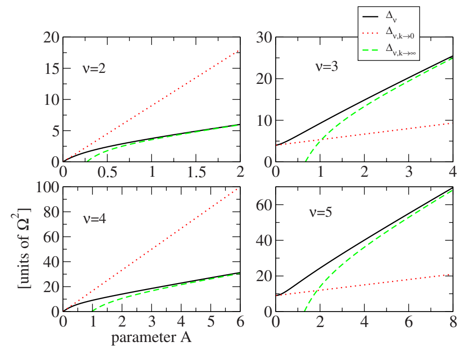

FIG. 5. (Color online) The recurrants $\Delta_1, \ldots, \Delta_5$ as a function of the parameter A. Also shown are the approximate expressions for $\Delta_1\Delta_1, \ldots, \Delta_5$ in the limits $k \to 0$ and $k \to \infty$ [see Eqs. (34) and (38), respectively].

B. Numerical analysis of recurrants and continued fractions

We also illustrate the results obtained by the recurrence relation method. As mentioned above, an exact evaluation of all recurrants Δ_ν, as well as the summation of the corresponding continued fraction, seems unlikely. However, the Δ_ν's in the deep inelastic limit are very regular and allow for an explicit summation of the continued fraction leading to an incomplete gamma function. We support this result by a numerical evaluation of the deep inelastic limit. Before doing this, we show the full expressions for the recurrants Δ_ν in Fig. 5 [see Eqs. (29)–(33)] together with its limiting cases, i.e., Eq. (34) and Eq. (38). Being a rational function in A with a strictly positive denominator, the Δ_ν's show a very smooth behavior. The deep inelastic asymptote (dashed-dotted line) diverges for small A. From this comparison, we expect the deep inelastic limit to be a reasonable approximation for $A>5$, while the long-wavelength limit (dashed line) is restricted to $A<0.1$.

We confirm this by evaluating the Laplace transform for large k. We compare the numerically obtained results from Eq. (52) with the summed continued fraction given above and in Appendix D. As can be seen in Fig. 6, the continued fraction indeed is the correct limit for large k since the numerical expression becomes identical to the analytical expression. Thus, we can study the deep inelastic limit suitably.

Finally, we study the high-frequency limit $z\to\infty$ for a fixed value of k or A. As shown above, the leading order in z

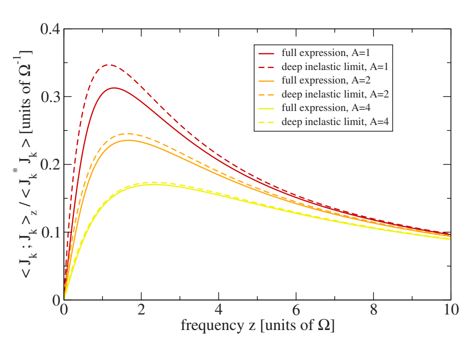

FIG. 6. (Color online) Laplace transform as given by Eq. (52). Various values of A are considered. We also show the continued fraction for the deep inelastic limit [see Eqs. (39) and (40)].

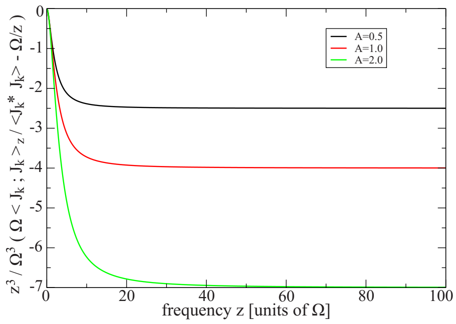

FIG. 7. (Color online) Laplace transform as given by Eq. (52) for large values of the frequency z. The parameters are $A=0.5,1,2$. The leading order is indeed given by $-(3A+1)z^3/\Omega^3$ [cf. Eq. (47)].

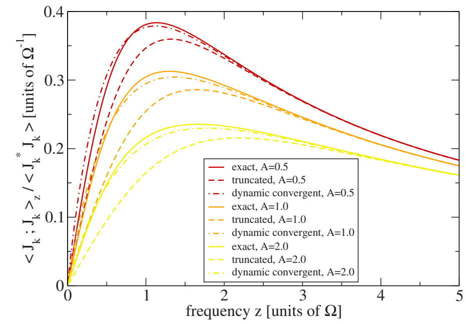

FIG. 8. (Color online) Laplace transform as given by Eq. (52) for large values of the frequency z. The parameters are $A=0.5,1.0,2.0$. Compared is the approximate expression [Eq. (55)] with the exact expression [Eq. (18)] and a simple truncation of the continued fraction beyond the third order.

is given by z^{-1} and we are interested in the next leading order by inspecting $(z/\Omega)^3(\Omega\langle J_k;J_k\rangle_z/\langle J_k^*J_k\rangle - z/\Omega)$. The results for $A=0.5,1,2$ are shown in Fig. 7. For large values of z, the limit $-(3A+1)$ is approached as expected from Eq. (47).

C. Approximation strategy in summation of continued fractions

As it appears to be impossible to sum up the continued fraction given by the recurrants Δ_ν exactly, approximation strategies have been devised [17]. In particular, Hong and Lee [46] introduced a so-called dynamic convergent calculation by using a few exact recurrants and supplementing these with approximate expressions for all other recurrants. A similar procedure was discussed in Ref. [47]. Here, we follow this approach combining the exact expressions for $\Delta_1,\Delta_2,\Delta_3$ as given by Eqs. (29)–(31) with the $\Delta_{\nu,k\to\infty}$ for $\nu\geq 4$ from the deep inelastic limit. Having the summed continued fraction $F_{k\to\infty}(z)$ for the deep inelastic limit at our disposal [see Eq. (40)] we can also determine the analytic representation for the continued fraction $G_{k\to\infty}(z)$ starting from $\Delta_{4,k\to\infty}$ by partial inversion,

$$G_{k\to\infty}(z) = \frac{1}{\Delta_{3,k\to\infty}}\left[\Delta_{2,k\to\infty}\left\{\Delta_{1,k\to\infty}\left[\frac{1}{F_{k\to\infty}(z)}-z\right]^{-1}-z\right\}^{-1}-z\right]. \tag{54}$$

Therefore, our approximate expression reads

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^*J_k\rangle} \approx \frac{1|}{|z} + \frac{\Delta_1|}{|z} + \frac{\Delta_2|}{|z+\Delta_3 G_{k\to\infty}(z)}. \tag{55}$$

We examine this approximation scheme for three different values of the parameter A in Fig. 8. Here, the approximate expression given by Eq. (55) is compared with the exact result given by Eq. (18). As a simple approximation, the truncation of the continued fraction beyond the third order, i.e.,

$$\frac{1|}{|z} + \frac{\Delta_1|}{|z} + \frac{\Delta_2|}{|z} + \frac{\Delta_3|}{|z}$$

is also shown. Clearly, the proposed approximate expression is superior to a simple truncation of the continued fraction for all values of the frequency z and all parameters A considered here. As expected from the very way of constructing the approximation, the agreement is better for larger values of A. For $A=0.5$, disagreement in the slope of the correlation function for small frequencies z is obvious. In conclusion, Eq. (55) leads to a reasonable approximation but high precision is restricted to large values of A.

IV. SUMMARY AND CONCLUSIONS

Although simple in setup, the k-dependent current-correlation function of a harmonic oscillator shows a rich variety at finite wave number k. The Laplace transformed current-correlation function has Lorentzian contributions from all harmonics of the fundamental eigenmode. The strength of each harmonic n is controlled by a modified Bessel function $I_n(A)$ for the normalized wave number squared A. The analytic expressions derived above are particularly helpful in taking the physical limit $z=\epsilon-i\omega$ where numerical calculations are challenging. The recurrence relation method (RRM) allows insight into the analytical structure of the correlation function especially in the limits $k\to 0$ and $k\to\infty$. For the latter deep inelastic limit, the RRM generates analytical expressions in terms of the upper incomplete gamma function.

A wide range of options extending this study is available. It is straightforward to generalize the treatment to a multidimensional single harmonic oscillator. The results presented here can be extended also to a harmonic oscillator in a thermal bath, which is a work in progress. In particular, analytical expressions for the k-dependent current-correlation function of this important system can be obtained. Furthermore, one can consider the current-correlation function for a quantal harmonic oscillator. Anharmonicity such as a Duffing potential is also of interest. These will be lines of future research.

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APPENDIX A: RECURRENCE METHOD AT WORK—A SIMPLE EXAMPLE

In order to illustrate the RRM (see Ref. [15]) we consider the example of the correlation function $\langle PQ^2;PQ^2\rangle_z$ which arises as one contribution in expanding the current correlation. We show that the RRM results lead to a finite continued fraction. The result is reproduced by a straightforward solution of the equation of motion for the correlation functions with the help of Eq. (3).

Let $f_0=PQ^2$ be the initial basis vector. Applying the RRM, we have for the other basis vectors

$$f_1 = Q(2P^2 - \Omega^2 Q^2), \tag{A1}$$

$$f_2 = 2P^3 - 2\Omega^2 PQ^2, \tag{A2}$$

$$f_3 = -\tfrac{6}{5}(3\Omega^2 P^2 Q + \Omega^4 Q^3), \tag{A3}$$

$$f_4 = 0. \tag{A4}$$

Note that we obtain a four-dimensional Hilbert space for the problem considered here. For the connection between the RRM and the dimensionality of the realized Hilbert space, see, e.g., [48]. In the current context, it is of importance to realize that the dimensionality increases with increasing power of Q; e.g., for $f_0=PQ^4$, a six-dimensional Hilbert space is generated. If $f_0=PQ^n$, then the Hilbert space is $(n+2)$ dimensional. In consequence, an infinite-dimensional Hilbert space is obtained for the exponential expression. The recurrants corresponding to $f_0,\ldots,f_3$ are given by

$$(\Delta_1,\Delta_2,\Delta_3) = \left(\tfrac{25}{5}\Omega^2, \tfrac{16}{5}\Omega^2, \tfrac{9}{5}\Omega^2\right). \tag{A5}$$

For $\nu \geq 4$, Δ_ν vanish. Thus, the correlation function is given by a finite continued fraction as (see Ref. [15])

$$\begin{aligned}\frac{\langle PQ^2;PQ^2\rangle_z}{\langle PQ^2 PQ^2\rangle} &= \frac{1|}{|z} + \frac{\Delta_1|}{|z} + \frac{\Delta_2|}{|z} + \frac{\Delta_3|}{|z}\\ &= \frac{z^3+5z\Omega^2}{9\Omega^4+10\Omega^2 z^2+z^4}\\ &= \frac{1}{2}\left(\frac{z}{z^2+(3\Omega)^2}+\frac{z}{z^2+\Omega^2}\right).\end{aligned} \tag{A6}$$

The same result is obtained by applying Eq. (3) directly.

APPENDIX B: DETAILS ON THE LONG-WAVELENGTH LIMIT

Here, we list the long-wavelength expansion of the recurrants Δ_ν, $6 \leq \nu \leq 14$, and give results up to second order in A given by Eq. (11),

$$\Delta_{6,k\to0} = \Omega^2\left(\tfrac{49}{2}A - \tfrac{49}{20}A^2\right),$$

$$\Delta_{7,k\to0} = \Omega^2\left(16 + \tfrac{8}{5}A + \tfrac{4}{25}A^2\right),$$

$$\Delta_{8,k\to0} = \Omega^2\left(\tfrac{162}{5}A - \tfrac{54}{25}A^2\right),$$

$$\Delta_{9,k\to0} = \Omega^2\left(25 + \tfrac{5}{3}A - \tfrac{5}{63}A^2\right),$$

$$\Delta_{10,k\to0} = \Omega^2\left(\tfrac{121}{3}A - \tfrac{121}{63}A^2\right),$$

$$\Delta_{11,k\to0} = \Omega^2\left(36 + \tfrac{12}{7}A - \tfrac{27}{98}A^2\right),$$

$$\Delta_{12,k\to0} = \Omega^2\left(\tfrac{338}{7}A - \tfrac{169}{98}A^2\right),$$

$$\Delta_{13,k\to0} = \Omega^2\left(49 + \tfrac{7}{4}A - \tfrac{7}{16}A^2\right),$$

$$\Delta_{14,k\to0} = \Omega^2\left(\tfrac{225}{4}A - \tfrac{25}{16}A^2\right).$$

APPENDIX C: DETAILS FOR THE DEEP INELASTIC LIMIT

Here, we list the deep inelastic expansion of the recurrants Δ_ν, $\nu \leq 15$, up to order A^{-1},

$$\Delta_{1,k\to\infty} = \Omega^2(1+3A),$$

$$\Delta_{2,k\to\infty} = \Omega^2\left(\tfrac{7}{3} + 2A - \tfrac{7}{9}A^{-1}\right),$$

$$\Delta_{3,k\to\infty} = \Omega^2\left(\tfrac{20}{3} + 5A - \tfrac{121}{18}A^{-1}\right),$$

$$\Delta_{4,k\to\infty} = \Omega^2\left(\tfrac{26}{3} + 4A - \tfrac{548}{45}A^{-1}\right),$$

$$\Delta_{5,k\to\infty} = \Omega^2\left(\tfrac{49}{3} + 7A - \tfrac{1477}{45}A^{-1}\right),$$

$$\Delta_{6,k\to\infty} = \Omega^2\left(19 + 6A - \tfrac{231}{5}A^{-1}\right),$$

$$\Delta_{7,k\to\infty} = \Omega^2\left(30 + 9A - \tfrac{903}{10}A^{-1}\right),$$

$$\Delta_{8,k\to\infty} = \Omega^2\Big(\tfrac{100}{3} + 8A - \tfrac{5168}{45}A^{-1}\Big),$$

$$\Delta_{9,k\to\infty} = \Omega^2\Big(\tfrac{143}{3} + 11A - \tfrac{8602}{45}A^{-1}\Big),$$

$$\Delta_{10,k\to\infty} = \Omega^2\Big(\tfrac{155}{3} + 10A - 2071A^{-1}\Big),$$

$$\Delta_{11,k\to\infty} = \Omega^2\Big(\tfrac{208}{3} + 13A - \tfrac{6253}{18}A^{-1}\Big),$$

$$\Delta_{12,k\to\infty} = \Omega^2(74 + 12A - 404A^{-1}),$$

$$\Delta_{13,k\to\infty} = \Omega^2(95 + 15A - 571A^{-1}),$$

$$\Delta_{14,k\to\infty} = \Omega^2\Big(\tfrac{301}{3} + 14A + \tfrac{29\,183}{45}A^{-1}\Big),$$

$$\Delta_{15,k\to\infty} = \Omega^2\Big(\tfrac{374}{3} + 17A - \tfrac{78\,659}{90}A^{-1}\Big).$$

APPENDIX D: SUMMATION OF THE CONTINUED FRACTION IN THE DEEP INELASTIC LIMIT

We show that the continued fraction given by the approximate recurrants of the deep inelastic limit can be summed up, leading to an analytic result. We start from the recurrants in Eq. (39) to yield the corresponding continued fraction,

$$\frac{\langle J_k; J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1|}{|z} + \frac{3\Omega^2 A|}{|z} + \frac{2\Omega^2 A|}{|z} + \frac{5\Omega^2 A|}{|z} + \frac{4\Omega^2 A|}{|z} + \dots . \tag{D1}$$

A more general expression can be considered by taking the recurrants as

$$\Delta_{2n-1}(k) = (n-1+s)Q,$$

$$\Delta_{2n}(k) = nQ. \tag{D2}$$

This pattern for the recurrants has already been considered by Hong and Lee in Ref. [31] in the context of the deep inelastic limit for interacting electron gas. Our case is included by taking $Q=2\Omega^2 A$ and $s=3/2$.

We give here a generalized treatment. By equivalence transformation [49], we obtain

$$\frac{\langle J_k; J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1}{z}\Bigg(\frac{1|}{|1} - \frac{-sQ/z^2|}{|1} - \frac{-1Q/z^2|}{|1} - \frac{-(1+s)Q/z^2|}{|1} - \frac{-2Q/z^2|}{|1} - \dots\Bigg). \tag{D3}$$

The term in brackets can be identified as a continued fraction representation of the confluent hypergeometric function ${}_2F_0$,

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1}{z}\frac{{}_2F_0(s,1;-Q/z^2)}{{}_2F_0(s,0;-Q/z^2)} \tag{D4}$$

(see Refs. [50,51] and for the definition of ${}_2F_0$ also Ref. [43]). Using ${}_2F_0(a,b,x)=(-1/x)^a U(a,1+a-b,-1/x)$ and taking advantage of the link between Kummer's function of another type U and the upper incomplete gamma function,

$$U(1-a,1-a;x) = e^x\Gamma(a,x), \tag{D5}$$

we arrive at

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1}{z}\left(\frac{z^2}{Q}\right)^s e^{z^2/Q}\Gamma(1-s,z^2/Q). \tag{D6}$$

This expression is particularly useful due to the known limiting properties of the upper incomplete gamma functions; e.g., the high-frequency behavior results from the asymptotic expansion (see Ref. [43]),

$$\Gamma(a,x) = x^{a-1}e^{-x}\left[1 + \frac{a-1}{x} + \frac{(a-1)(a-2)}{x^2} + \cdots\right],$$

to obtain

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{1}{z}\left(1 - \frac{sQ}{z^2} + \frac{s(s+1)Q^2}{z^4} - \cdots\right). \tag{D7}$$

Thus, in the leading order, the current-correlation decays as z^{-1}. For the small z behavior, a general treatment valid for any s can be used by exploiting the series expansion of the incomplete gamma function $\gamma^*(a,x)$ (cf. [43]),

$$\Gamma(a)\gamma^*(a,x) = \sum_{n=0}^{\infty}\frac{(-x)^n}{(a+n)n!},$$

and its connection to the upper incomplete gamma function $\Gamma(a,x)$ and the lower one $\gamma(a,x)$,

$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x),$$

$$x^{-a}\gamma(a,x) = \Gamma(a)\gamma^*(a,x).$$

We obtain from these arguments

$$\frac{\langle J_k;J_k\rangle_z}{\langle J_k^* J_k\rangle} = \frac{z}{Q}e^{z^2/Q}\left[\left(\frac{z^2}{Q}\right)^{s-1}\Gamma(1-s) + \frac{1}{s-1} + \frac{z^2/Q}{2-s} - \frac{z^4/Q^2}{2(3-s)} + \cdots\right]. \tag{D8}$$

Thus, for $s>1$, the leading order is linear and given by $\langle J_k;J_k\rangle_z/\langle J_k^* J_k\rangle \approx (s-1)^{-1}Q^{-1}z$. More expansions also in terms of Bessel functions can be found in Ref. [52]. Note that there are special cases for the incomplete gamma function for $s=1, s=1/2, s=0$ (see Ref. [43]). These special cases have been found earlier by different means (cf. Ref. [31]).

Hong and Lee also found that the time correlation was Kummer's function $M(s,1/2,-Qt^2/4)$ or Φ by use of the inverse Laplace transform with Eq. (D2). Kummer's function $M(s,1/2,-Qt^2/4)$ with our case $Q=2\Omega^2 A$ and $s=3/2$ has the same short-time expansion of Eq. (13) as

$$\frac{\langle J_k^*(t)J_k(0)\rangle}{\langle J_k^* J_k\rangle} = 1 - \frac{3A}{2}(\Omega t)^2 + \frac{5A^2}{8}(\Omega t)^4 \cdots . \tag{D9}$$

This is related to the fact that Eq. (13) approximately satisfies the differential equation for $M(3/2,1/2,-A(\Omega t)^2/2)$ within the regime of $0<\Omega t \ll 1$ in the large A limit.

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