Stochastic resonance in coupled underdamped bistable systems

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We study onset and control of stochastic resonance (SR) phenomenon in two driven bistable systems, mutually coupled and subjected to independent noises, taking into account the influence of both the inertia and the coupling. In the absence of coupling, we found two critical damping parameters: one for the onset of SR and another for which SR is optimum. We then show that in weakly coupled systems, emergence of SR is governed by chaos. A strong coupling between the two oscillators induces coherence in the system; however, the systems do not synchronize no matter what the coupling is. Moreover, a specific coupling parameter is found for which the SR of each subsystem is optimum. Finally, a scheme for controlling SR in such coupled systems is proposed by introducing a phase difference between the two coherent driving forces.

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I. INTRODUCTION

Among a large variety of phenomena which has been attracting researchers in coupled nonlinear systems over several decades, synchronization [1], chaos and bifurcations structures [2], and recently stochastic resonance (SR) are the most prominent. SR, however, has been widely explored in single nonlinear systems such as in bistable lasers [4,5], chemical reactions [3], semiconductors devices, and mechanoreceptor cells in the tail of the crayfish [6]. This now well-established effect requires three main ingredients: (i) a weak coherent signal, (ii) a noise source, and (iii) an energetic activation barrier. In the absence of noise, the signal should be weak enough such that the effect of signal-induced switching is not observed. Likewise, the noise-induced switching should not be appreciable in the absence of the signal. It is the interplay of both the signal and the noise that results in a sharp enhancement of the power spectrum within a narrow range about the forcing frequency. This observation was explained by matching the forcing frequency with the switch rate (Kramer's rate) of the unperturbed system [7]. To distinguish this from the dynamical resonance, one speaks of SR. Due to its simplicity and robustness, SR has been implemented by mother nature on almost every scale, thus enabling interdisciplinary interest from physicists, geologists, engineers, biologists, and medical doctors, who nowadays exploit it as an instrument for their specific purposes[8].

The first experimental observation of SR was performed while investigating the noise dependence of the spectral line of an ac-driven Schmitt-Trigger [9]. Although SR has been largely explored in various dynamical systems [3,8], little has been done for coupled stochastic systems [10–13]. The case of coupled stochastic bistable systems taking into account its full inertial dynamics has hitherto not yet been considered. Another motivation for this study comes by several recent observations of SR in nanomechanical silicon resonators where the inertial term needs to be taken into account to understand its dynamics fully [14–16]. One can also couple two such systems of nanomechanical resonators [17], which make it relevant to study signal amplification and synchronization dynamics of SR. The inertial term adds interesting features as the system becomes chaotic, in some parameter regimes, whose interference with the externally injected noise might affect its ability to detect weak signals using SR mechanism.

In this paper, we demonstrate the constructive role of noise assisted by a weak signal in a coupled bistable system in which chaos plays a role. SR has been studied in such a system [11] but essentially in the overdamped regime. Here we revisit the same system but study its full dynamics by focusing mainly on the weak damping regime, i.e., the regime where the inertia plays a major role, thereby rendering the system richer in that chaos is likely to show up for some parameter values. The conditions for the onset and control of SR in such a coupled system are explored by varying the mutual coupling and the damping parameters.

The paper is organized as follows. Section II is devoted to the description of the model for two coupled forced bistable oscillators. Section III discusses our results using signal-to-noise ratio as the indicator of SR in both systems. The onset and control of SR in the system are studied and analyzed for a variety of coupling and dissipation parameters. We also compute the synchronization quantifier and explore the regime of coupling and dissipation. A scheme for the control of SR is also discussed toward the end of the paper. Finally, Sec. IV concludes the paper.

II. MODEL SYSTEM

Our system consists of two coupled underdamped bistable oscillators which are forced by two periodic signals and statistically independent noise sources. This system is governed by the following dimensionless coupled stochastic differential equations:

$$\ddot{x} = -\gamma\dot{x} - \frac{dV_1(x)}{dx} + k(y-x) + \xi_1(t) + F_1(t), \tag{1}$$

$$\ddot{y} = -\gamma\dot{y} - \frac{dV_2(y)}{dy} - k(y-x) + \xi_2(t) + F_2(t), \tag{2}$$

where k is the coupling strength and γ is the damping parameter. The potentials of the two subsystems

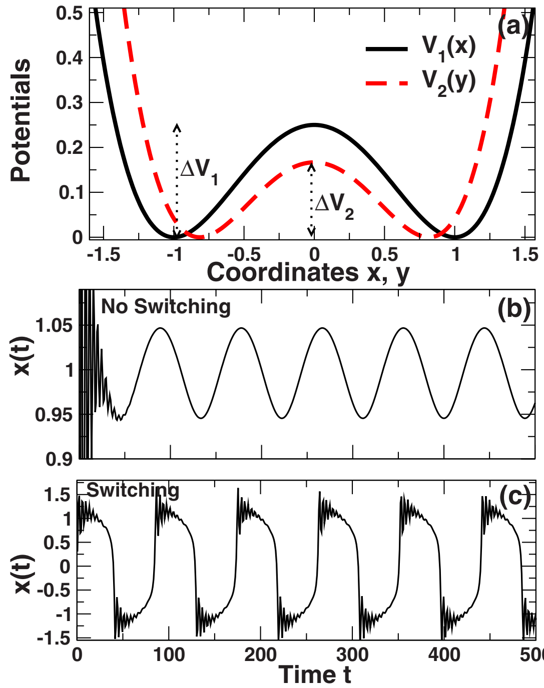

FIG. 1. (Color online) (a) Potentials $V_1(x)$ (solid) and $V_2(y)$ (dashed), with $\Delta V_1=0.25$ and $\Delta V_2=0.17$, respectively. (b) Prototypical scenarios of no switching, $A_0=0.15$, and (c) switching, $A_0=0.50$, of y for $k=0.0$, $D=0.0$, $\gamma=0.25$.

$V_i(x)=-a_ix^2/2+b_ix^4/4$ for $i=1,2$ are sketched in Fig. 1(a), with $a_1=b_1=1$ and $a_2=1$, $b_2=1.5$. This choice leads to different activation barrier energies $\Delta V_1=0.25$ and $\Delta V_2=0.17$. The stochastic terms $\xi_1(t)$ and $\xi_2(t)$ are zero-mean independent Gaussian white noises defined as follows:

$$\langle \xi_i(t) \rangle = 0, \tag{3}$$

$$\langle \xi_i(t)\xi_i(t') \rangle = 2D_i\delta(t-t'), \tag{4}$$

$$\langle \xi_i(t)\xi_j(t') \rangle = 0, \quad i \neq j, \tag{5}$$

where $i,j=1,2$. The parameters D_1 and D_2 are the intensities of the two noises $\xi_1(t)$ and $\xi_2(t)$, respectively. In the following, we set the noise intensities equal: $D_1=D_2=D$. The periodic driving signals are

$$F_i(t) = A_i \cos(\Omega_i t + \phi_i), \tag{6}$$

characterized by the amplitude A_i, the angular frequency Ω_i, and the phase Φ_i with $i=1,2$. In the following, to observe SR we set $A_1=A_2=A$ and choose a subthreshold driving amplitude $A<\Delta V_{1,2}$. The intrawell relaxation frequencies of the two different subsystems are identical and equal to $\omega_r=\sqrt{2a_{1,2}}$ since $a_1=a_2=1$. To allow for the adiabatic driving, we set the modulation frequency smaller than the relaxation one, say $\Omega=\omega_r/20$. Considering the subsystem x, for instance, with $D=0$ and $\gamma=0.25$, vivid scenarios of no switching with $A=0.15$ in Fig. 1(b) and switching with $A=0.5$ in Fig. 1(c) are shown. Unlike the well-studied overdamped bistable oscillator, the characteristic intrawell relaxation oscillations are clearly visible whenever the system switches from one state to the other[18–20].

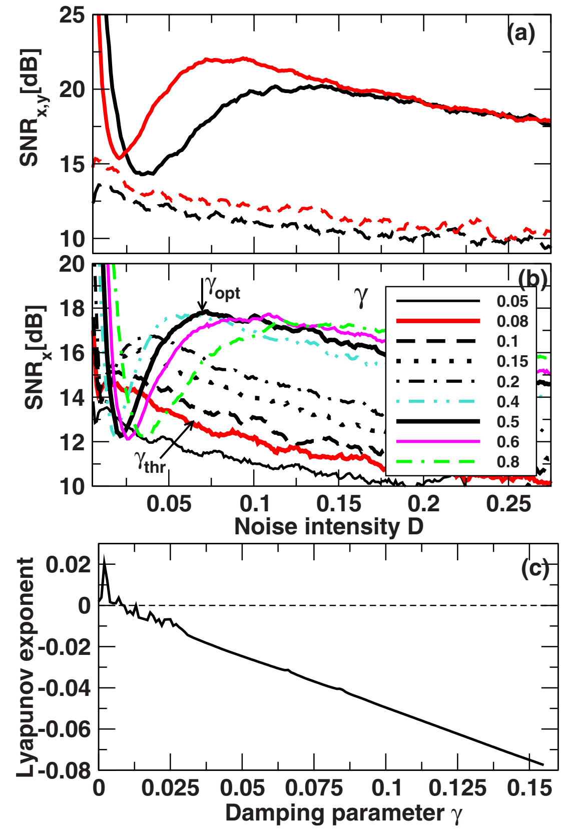

FIG. 2. (Color online) (a) $\mathrm{SNR}_{x,y}$ of x and y for $k=0.0$ and $A_0=0.15$. For a weak damping $\gamma=0.05$, SNR_x (dashed red/gray) and SNR_y (dashed black), while for a strong damping $\gamma=0.75$, SNR_x (solid red/gray) and SNR_y (solid black) (b) SNR_x as a function of D for various values of γ from 0.05 to 0.8. Onset of SR happens at $\gamma_{thr}=0.08$ and it is optimized for $\gamma_{opt}=0.5$. (c) The Lyapunov exponent of x as a function of γ for uncoupled system $k=0$, $D=0$, and $A_0=0.15$. Note the positive Lyapunov exponents for small γ.

III. RESULTS AND DISCUSSIONS

A. Stochastic resonance for uncoupled systems

Let us first consider that both subsystems are independent ($k=0$), driven by different noises and identical driving forces. In order to study SR, we consider a subthreshold signal amplitude $A_0=0.15$ that does not allow switching in the absence of noise. Note also that in the absence of the driving and for the overdamped regime, the stochastic switching time scale which is characterized by Kramer's rates, $\Gamma_{K_{1,2}} \propto \exp(-\Delta V_{1,2}/D)$, is too long due to low noise amplitude, i.e., the noise alone cannot induce synchronized switching. The time scale of switching being $1/\Gamma_{K_{1,2}}$, the time series for $D \in (0,0.5)$ (not shown) does not exhibit any switching. When both the noise and the driving force are applied, the signal-to-noise ratio (SNR) is indeed a good candidate commonly used for evaluating the constructive role of noise [3].

Figure 2(a) shows $\mathrm{SNR}_{x,y}$ in the weak damping regime (dashed black and red/gray) $\gamma=0.05$ and in the strong one

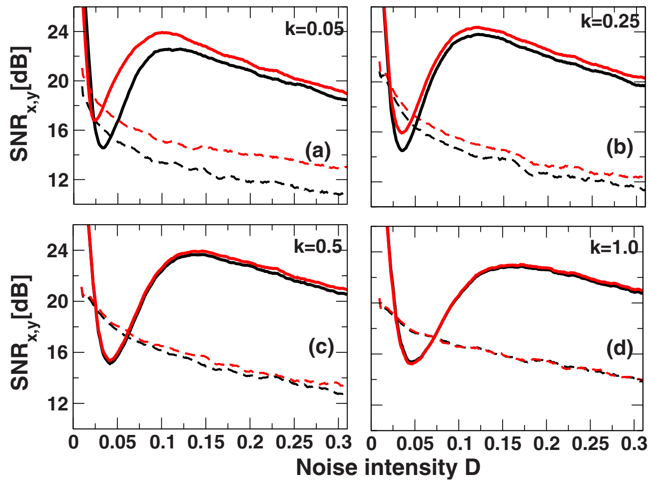

FIG. 3. (Color online) $SNR_{x,y}$ of x and y for $A_0=0.15$ and for different values of the coupling k as indicated on panels. In each panel, plots are for a weak damping $\gamma=0.05$, SNR_x (dashed red/gray) and SNR_y (dashed black), and for a strong damping $\gamma=0.75$, SNR_x (solid red/gray) and SNR_y (solid black).

$\gamma=0.75$ (solid black and red/gray) as a function of noise intensity. In each panel, the signal amplitude is $A_0=0.15$. It turns out that the cooperative effect of noise and driving force does not show up for a weaker dissipation regime where chaos is present. Exploring the SNR as a function of γ in Fig. 2(b), which depicts SNR_x for various values of γ, two critical values have been revealed, namely, γ_{thr} for which SR appears and γ_{opt} for which SR is optimum. Here we found $\gamma_{res}=0.08$ and $\gamma_{opt}=0.5$. Finally, the Lyapunov exponent, a good indicator of chaos in dynamical systems, has been plotted in the absence of noise as a function of γ in Fig. 2(d). This clearly confirms that chaos is present in the weak damping regime and may prohibit the occurrence of SR. A similar conclusion was drawn in Ref. [21] but in a noisy underdamped double-well potential.

B. Stochastic resonance and synchronization for coupled systems

1. Stochastic resonance

SR is essentially based on the exploration of the power spectra of subsystems $\bar{x}(\omega)$ and $\bar{y}(\omega)$ computed using the time series of the coupled systems. Because of the coupling, another quantity of interest is the coherence function defined as $\Gamma^2=|S_{xy}(\omega)|/[S_{xx}(\omega)S_{yy}(\omega)]$, where $S_{xy}(\omega)$ is the cross spectrum of processes $x(t)$, $y(t)$ and $S_{xx}(\omega)$, $S_{yy}(\omega)$ are the power spectra of $x(t)$, $y(t)$, respectively. This quantity reaches unity in case both processes become coherent. Figures 3(a)–3(d) show SNR of the two subsystem for weak damping regime ($\gamma=0.05$) and strong damping regime ($\gamma=0.75$) as a function of noise intensity for four different values of coupling parameters k. Other parameters are kept same as in Fig. 2(a). It turns out that for weak couplings [Figs. 3(a) and 3(b)] both systems are quasi-independent. Remarkably, as k increases, SNR of both subsystems becomes identical [Figs. 3(c) and 3(d)]. The coupling does not affect the onset of disappearance of SR in the system, unlike the dependence of SR seen on the damping parameter. Similarly the coherence Γ^2 (not shown) exhibits the same trend.

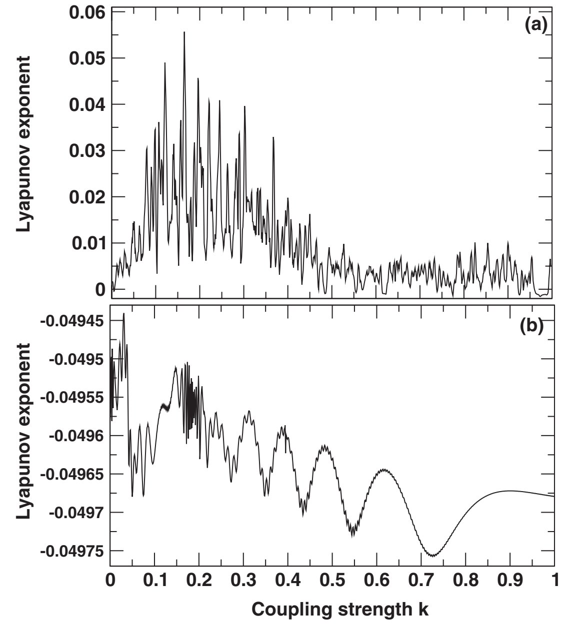

FIG. 4. The role of coupling parameter k on chaos in the absence of the noise ($D=0$). The Lyapunov exponents as a function of k for (a) a weak damping $\gamma=0.05$ and for (b) a strong damping $\gamma=0.75$. It turns out that the coupling does not influence at all the chaos's background of the system.

To shed more light into the understanding of the above result, we found it worthwhile to switch off completely the noise ($D=0$) and analyze the influence of the coupling on chaos that predominantly exists in the system. In this framework, we have computed the Lyapunov exponent as a function of the coupling strength k for both weaker and stronger regimes of the damping. A prototypical example is plotted in Fig. 4 which clearly demonstrates that chaos is persistent in the system in the weaker damping regime $\gamma=0.05$, irrespective of the coupling k [see Fig. 4(a)]. The Lyapunov is indeed positive for any k. Besides and as one can expect the Lyapunov exponent of the stronger damping regime $\gamma=0.75$ remains negative no matter what the coupling k [see Fig. 4(b)]. It thus turns out that, in our bistable oscillators, one cannot make use of the coupling to induce the transition from chaotic to deterministic dynamics.

Having established that resonances exclusively occur in the stronger damping regime as demonstrated in Fig. 3, we then wish to see how the coupling influences this resonance phenomenon at the SNR level. For the same noise level, we have recorded for a given damping parameter $\gamma=0.75$ the maximum of SNR for both x and y. The resulting plot, shown in Fig. 5, clearly demonstrates that the maxima of SNR are optimum at moderate values of the coupling $k=k_{opt}$, where $k_{opt}\approx 0.25$ and $k_{opt}\approx 0.3$ for SNR_x and SNR_y, respectively. Note that one should not confuse these optima with the one obtained with the noise. This plot shows that for a given system there exists an optimum coupling between the two systems that would maximize the SNR for both the subsystems.

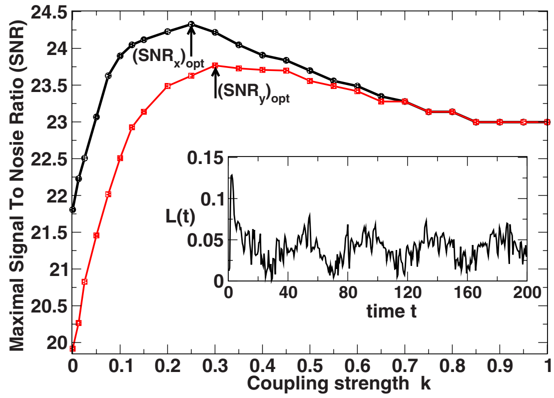

FIG. 5. (Color online) The role of coupling parameter k on the optimal SNR. Optimal values $(\mathrm{SNR}_x)_{opt}$ and $(\mathrm{SNR}_y)_{opt}$ of SNR_x (black) and SNR_y (red/gray), respectively, plotted as a function of k for a fixed noise intensity. Inset: a measure of synchronization $L(t)$ as a function of time for $k=1$, $A_0=0.15$, and $\gamma=0.75$.

2. Synchronization

Since we have seen that for strongly damped and for strongly coupled subsystems the SNR_x and SNR_y become identical, we would now like to see if this also implies that the two systems are perfectly synchronized in this regime. To proceed we define the quantity $L^2(t)=[x(t)-y(t)]^2+[\dot{x}(t)-\dot{y}(t)]^2$ which is a good measure of the synchronization [1]. A perfectly synchronization for the two subsystems would be achieved when $L(t)=0$ for all times. The inset in Fig. 5 shows an example of the time series of $L(t)$ for $A_0=0.15$, $\gamma=0.75$, and $k=1.0$. No synchronization state has been achieved even at the very strong limit that shows strong coherence. Similar outputs are found for any other value of k, demonstrating that synchronization is not reached as $L(t)$ does not vanish. What makes this difficult to achieve is presumably because the two subsystems are not only topologically not identical but also nondeterministic. The opposite happens in deterministic coupled systems in which a strong coupling enforces the synchronization [22].

C. Phase control of stochastic resonance

1. Case of identical driving frequencies

Due to the coupling between the subsystems x and y, we have a possibility to control the onset of stochastic resonance in the total system dynamics by introducing a phase difference between the two periodic driving $F_1(t)$ and $F_2(t)$. The relative phase $\Delta\phi=\phi_1-\phi_2$ between two oscillators is an important control parameter for coupled systems which could also arise, in certain situations, due to the time delay between the two signals. To study the effect of the relative phase $\Delta\phi$ on the SR, we first optimize the stochastic resonance curves due to individual driving signals. Note that in the absence of noise, simultaneous applications of both signals cannot induce any periodic switching. Now, by keeping the noise level at its optimum point, in Fig. 6(a), SNR_x is plotted as a function of $\Delta\phi$. As one can see, the optimal SNR starts immediately to decrease as $\Delta\phi$ increases and reaches its minimum at

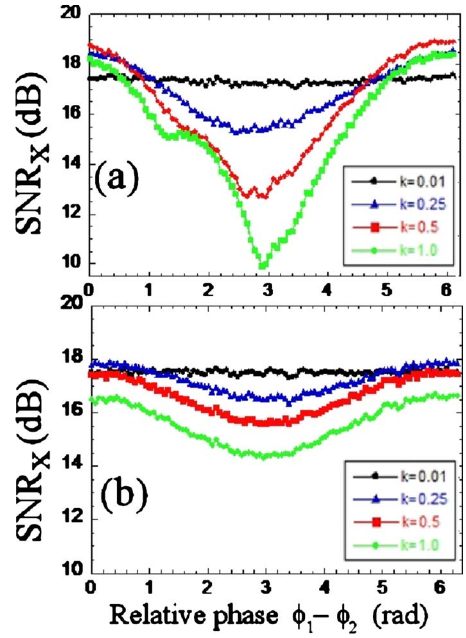

FIG. 6. (Color online) The control of optimum SNR using a phase difference between two driving signals. (a) SNR_x versus the relative phase $\Delta\phi=\phi_1-\phi_2$ between the two driving signals of identical frequencies $\Omega_1=\Omega_2$ and $A_1=A_2=0.15$, (b) SNR_x versus $\Delta\phi=\phi_1-\phi_2$ between two driving signals of different frequencies $\Omega_1=\Omega_2/2$ and $A_1=A_2=0.15$. The noise level for these results is kept fixed at its optimum value around $D_{opt}=0.15$.

$\Delta\phi\sim\pi$. Further increasing $\Delta\phi$ leads to a restoration of the optimal SNR at $\Delta\phi\sim 2\pi$. Remarkably, this collapse of the optimal SNR is more pronounced when the two driving forces are in antiphase $\Delta\phi\sim\pi$. It is thus the antagonist effects of these driving forces on corresponding oscillators which destroy the cooperative effects and are responsible of the collapse of the optimal SNR. This collapse is maximum for the stronger coupling between the two oscillators, suggesting that such a control scenario is the most efficient for strongly coupled systems. Here, one experiences a significant gap of about 8 dB as the coupling strength goes from its smallest value $k=0.01$ to the largest one $k=1$. The simple control scheme can further be extended by applying the control signal to the barrier height modulation, analogous to the control scheme of SR in the overdamped regime [23].

2. Case of different driving frequencies

To explore the sensitivity of the SR control to different driving frequencies, we consider a scenario when one of the system is driven by the $F_1(t)=A\cos(\Omega t+\phi_1)$ and the other one by its second harmonic $F_2(t)=A\cos(2\Omega t+\phi_2)$. By varying the relative phase $\Delta\phi$ and keeping the noise fixed at its optimum, the SNR_x is shown in Fig. 6(b). Again for very weak coupling almost no control is obtained but as the coupling increases, one can see that the SNR at the fundamental frequency Ω drops by more than 2 dB. In this case, whenever the two driving signals lead to opposing contribution, a drop in SNR is still observed. However, these opposing contributions do not lead to a good cancellation in the case of different frequencies which leads to a less noticeable control when compared to the case of identical driving frequencies.

IV. SUMMARY AND CONCLUSION

We have investigated the stochastic resonance dynamics of two coupled bistable systems which can have arbitrary damping and coupling. The SR which is central has been already considered in a similar system but in the overdamped regime [11] in which chaos is inhibited. Dealing first with the uncoupled system, we found two critical damping parameters: one indicating the threshold for the appearance of SR and another for its optimum. We show that the weak damping regime prohibits onset of SR and that the nonmanifestation of SR is due to the presence of chaos in the system. Then when the coupling is turned on, SR is in general not affected; however, the strong coupling regime induces SNR of subsystems to match, thereby showing a very high coherence. We also found, for each subsystem, a specific coupling parameter for which the SR is optimum. Exploring the systems along the same lines, the synchronization has not been reached for any value of the coupling.

Furthermore, the influence of the relative phase $\Delta\phi$ (or time delay) of the coherent signals is exploited to control the optimal SR effect. When the two subsystems are driven out of phase, the coupling cancels out the noise-induced oscillations, leading to a collapse of SR. Clearly, such a control is the most effective when the systems are in antiphase, driven by identical driving frequencies and are strongly coupled. Indeed, the mutual suppression of SNR by dephasing can serve as an indicator of their coupling constant. The results presented here are of generic importance to other coupled systems such as the coupled bistable lasers, the electronic circuits, and the nanomechanical systems where both the coupling and dissipation play an important role in the system dynamics [15,24]. The emergence and optimization of stochastic resonance for a network of a large number of coupled nonlinear subsystems, such as nanomechanical resonators [17], remains a problem of further research and importance.

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