Magnetic-field-induced breakdown of equivalence of multidimensional motion

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In this paper, we have studied Brownian motion in multidimension phase space in presence of a magnetic field. The nonequilibrium behavior of thermodynamically inspired quantities along the individual component of motion has been studied in detail. Based on the Fokker-Planck description of the stochastic process and entropy balance equation, we have calculated information entropy production and entropy flux at nonequilibrium state. The dependence of these quantities on time, magnetic field, and thermal bath is studied. In this context, we have observed that there exists extremum behavior in the dynamics and the applied magnetic field breaks the equivalence in motion of the components in the nonequilibrium state.

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I. INTRODUCTION

In recent years the stochastic dynamics [1–5] community is becoming increasingly interested to study the role of noise in dissipative dynamical systems, because of its potential applications on various noise-induced phenomena, such as stochastic resonance [6], coherence resonance [7,8], resonant activation [9], directed motion of a Brownian particle [10], noise-induced pattern formation [11], self-induced aggregation kinetics [12] etc. Study of the Brownian motion of a charged particle under the influence of a magnetic field is also an important issue. It has been considered in various situations [13–21]. In the present paper, we have studied the relaxation behavior of a charged Brownian particle in terms of information entropy [22,23]. The well known information entropy is Shanon's information measure [22,23]

$$S = -\int \rho(\mathbf{q},t)\ln \rho(\mathbf{q},t)d\mathbf{q}, \tag{1}$$

which typically is not a conserved quantity. $\rho(\mathbf{q},t)$ is the phase-space distribution function. It is similar to thermodynamic entropy, because of the "H-theorem" of the Fokker-Planck equation, and again S defined in Eq. (1) is simply a Lyapounov function characterizing the stability of the equilibrium solution [22,23]. Thus the H-theorem implies that the whole of statistical mechanics can be elegantly reformulated by extremization of S, subject to the constraints imposed by the *a priori* information one may possess concerning the system of interest. Using the above definition in the Fokker-Plank equation one can easily have the information entropy balance equation [24,25]. From this equation it is possible to identify thermodynamically inspired quantities like entropy flux and production. Making use of the time-dependent solution of the Fokker-Planck equation in these quantities we are lead to understand the relaxation mechanism in detail. In the present paper, we have investigated the effect of applied magnetic field in this context both presence and absence of nonequilibrium constraint. Although a major advantage of this method lies in its simplicity it suffers from a limitation since using the present method only relaxation near equilibrium can be studied. As mentioned above the information entropy has no quantitative relation with the thermodynamic entropy, the information entropy production cannot be used in the same spirit of thermodynamic entropy production in the fields of refrigeration, air conditioning, heat pump systems liquid chiller, etc. [26]. However, it is a useful tool to study the details of relaxation process in a stochastic system. For example, heat conductivity in a medium has recently been studied when its constituents are stochastic [27]. Thus information entropy production may be useful to understand the time required to reach the steady state as well as the steady state thermal conductivity of the medium.

Based on information entropy a method was described in Ref. [24] for the study of the relaxation processes in mesoscopic system. The mesoscopic system has also been studied in recent papers [28–34] on the basis of Gibbs entropy. An important application of information entropy in the context of Brownian motion is to solve the Fokker-Planck equation using the maximum entropy principle [35,36]. The von Newmann equation in quantum mechanics was also solved using this principle [37–40]. Based on information entropy a method for the global optimization of stochastic function has been developed very recently [41]. In general, entropy measures the information content of a probability distribution, and thus gives a criterion for decision: we have to choose the one which yield the most information concerning location and value of the global maximum sought from several possibilities. As a point of digression we may also note that in Ref. [42], it was shown that the Legendre-transformation structure of thermodynamics can be replaced without any change if one replaces the entropy S by Fisher's information measure (FIM) which obeys the important thermodynamic property of concavity. This method seems to be able to treat equilibrium and nonequilibrium situations in a manner entirely similar to the conventional one. Moreover, interesting relationships exist that connect FIM and the relative Shanon information measure invented by Kullback [43,44]. These have been shown to have some bearing on the time evolution of arbitrary systems governed by quite general continuity equation [45–47]. Thus information entropy and related quantities are vital to study the stochastic process.

To put the present discussion in an appropriate perspective, we first note that the time evolution of S takes care of

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the rate of phase-space expansion and contraction when the dynamical variables are governed by the stochastic processes. This implies that the specific nature of the force field has a strong role to play with S. In view of the immediate connection between information entropy and probability distribution function of the phase-space variables, it is worthwhile to explore about the effect of interplay of dynamical system, thermal environment, and an applied magnetic field on the information entropy and related quantities. Forces due to friction and magnetic field (which does not induce dissipation of energy) are velocity dependent and their interplay may attribute special significance. To check our expectation, we consider Brownian motion of a particle with out any force from potential energy (a free particle) and a particle in a three-dimensional harmonic potential in presence of magnetic field. We have observed that there exists extremum behavior in the dynamics and the applied magnetic field breaks the equivalence in motion of the components in the nonequilibrium state.

Our results for free particle are relevant for understanding the diffusion behavior of plasma. It also may be applicable to explain transport properties of charge particles in presence of magnetic field since it is related to diffusive behavior of free particles. The insights that we perceive for another case usually have a wide impact, as the harmonic oscillator constitutes much more than a mere example. It has been considered in many contexts for the model study in the literature. However, in the real non-linear system usually has a linear part. Thus understanding of dynamics of linear part may be important. As for example, the attempt frequency, entering in the prefactor of the Kramers' rate, in some cases can be approximated by the linearization around the fixed point of the corresponding potential. But if the potential minimum, which contains the powers larger than 2 on the coordinate then the linearization approximation can lead to serous errors. Recently the Kramers' problem had been studied exactly [48–51]. To be mention here that our study does not related directly to escape problem but it may lead to understand the steady state barrier-crossing rate as we mentioned above. We will discuss it with our results. Another case to be consider here. Artificial atoms and molecules or quantum dots in which electron-nuclear interaction has been replaced by a confine potential are interesting objects on many counts [52–56]. The confinement model was first introduced in Refs. [57,58]. In this model particle is confined in a two-dimensional (2D) harmonic potential in presence of magnetic field. Thus our present study is related to dynamics of quantum dot which is coupled to a heat bath at high temperature.

The outline of the paper is as follows: in Sec. II, we calculate entropy flux and entropy production of a Brownian particle with two applications. Relaxation behavior of a small electric field driven equilibrium state is investigated in Sec. III. The paper is concluded in the Sec. IV.

II. INFORMATION ENTROPY PRODUCTION OF A BROWNIAN PARTICLE IN PRESENCE OF A MAGNETIC FIELD

The Langevin equation of motion of a Brownian particle in three dimension in presence of a magnetic field $\mathbf{B}$, can be written for the velocity vector $\mathbf{u}$ as

$$\dot{\mathbf{u}} = \mathbf{F} - \gamma\mathbf{u} + \frac{Q}{mc}\mathbf{u}\times\mathbf{B} + \mathbf{f}(t), \tag{2}$$

where

$$\mathbf{F} = -\frac{1}{m}\boldsymbol{\nabla} V(\mathbf{r}) - \gamma\mathbf{u} + \frac{Q}{mc}\mathbf{u}\times\mathbf{B}. \tag{3}$$

Here $\boldsymbol{\nabla} V$ is the gradient operation of potential V, γ is the friction constant, Q is the charge of the particle and m its mass, c is the speed of the light, $\mathbf{f}(t)$ is the Gaussian fluctuating force per unit mass. The component of the random force satisfies the properties of white Gaussian noise with zero mean value, $\langle f_i(t)\rangle=0$ and the standard fluctuation-dissipation relation with a correlation function,

$$\langle f_i(t)f_j(t')\rangle = \epsilon D_{ij}\delta_{ij}\delta(t-t'). \tag{4}$$

Here D_{ij} being a positive definite matrix related to the thermal bath and our natural demand is that all the diagonal elements are same. We represent them as D_d. The parameter ϵ is used to control the noise strength. Here we use $f_1 \equiv f_x$, $f_2 \equiv f_y$, and $f_3 \equiv f_z$.

The Fokker-Planck equation corresponding to the above Langevin equation of motion (2) can be written as

$$\frac{\partial\rho}{\partial t} = -\sum_{i=1}^{6}\frac{\partial F_i\rho}{\partial q_i} + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\frac{\partial^2\rho}{\partial q_i\,\partial q_j}. \tag{5}$$

For $i=1$ to 3, q_i corresponds position coordinates x,y and z respectively. The components of velocity (u_x,u_y and u_z) are represented by q_i choosing $i=4$ to 6. However, using the above Fokker-Planck equation we now proceed to identify information entropy flux and production. The time evolution of S with Eq. (5) can be written as

$$\frac{dS}{dt} = -\int d\mathbf{q}\left[-\sum_{i=1}^{6}\frac{\partial F_i\rho}{\partial q_i} + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\frac{\partial^2\rho}{\partial q_i\,\partial q_j}\right]\ln\rho - \frac{d}{dt}\int\rho d\mathbf{q}. \tag{6}$$

The second term in above equation equals to zero as the total probability is always one. However, making use of partial integration in the above equation we have

$$\frac{dS}{dt} = \int d\mathbf{q}\rho\boldsymbol{\nabla}\cdot\mathbf{F} - \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\left[\ln\rho\frac{\partial\rho}{\partial q_j}\right]_{limit} + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial q_i}\right)\left(\frac{\partial\rho}{\partial q_j}\right). \tag{7}$$

To proceed further we use the usual boundary condition. We consider the system with a finite phase-space volume as usually happens in reality. Hence there should have a well defined boundary on which and beyond the distribution function must be zero. We assume the derivatives of the distribution function at the boundary to vanish. This leads us to the following form of entropy balance equation

$$\frac{dS}{dt} = \int d\mathbf{q}\rho \boldsymbol{\nabla} \cdot \mathbf{F} + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial q_i}\right)\left(\frac{\partial\rho}{\partial q_j}\right). \quad (8)$$

The first term in Eq. (8) has no definite sign while the second term is positive definitely, because of positive definiteness of D_{ij}. Then one can identify the first and the second terms as entropy flux (S_F) and entropy production (S_P), respectively.

$$S_F = \int d\mathbf{q}\rho \boldsymbol{\nabla} \cdot \mathbf{F} \quad (9)$$

$$S_P = \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial q_i}\right)\left(\frac{\partial\rho}{\partial q_j}\right). \quad (10)$$

Thus the entropy flux defined here calculates average of divergence of deterministic force involved in the system i.e., it considers time evolution of the average of phase-space expansion or contraction rate by virtue of the deterministic force. On the other hand entropy production measures the rate of phase-space expansion due to the random force. We then examine the connection between the information entropy production and the phase-space collapse of system at equilibrium state. In this state we have (for details we refer Ref. [24])

$$S_P = -S_F = -\int d\mathbf{q}\rho \boldsymbol{\nabla} \cdot \mathbf{F} = -\overline{\boldsymbol{\nabla}\cdot F^{\infty}} = -\sum_i \sigma_i' + \bigcirc(\epsilon) > 0, \quad (11)$$

in the limit $\epsilon \ll 1$.

Here σ_i' is the Lyapunov exponent of the ith component of the phase space. Thus information entropy production as defined by Eq. (10) is equal to the negative of the Lyapunov exponent or equivalently to the rate of phase-space volume contraction plus a correction term vanishing as the noise strength goes to zero [24] at stationary state. It is a link between thermodynamically inspired quantities and the quantities involved in the underlying dynamics in phase space. At the same time this explains how finite phase-space volume is possible at long time in presence of dissipative force. Furthermore, following [24], the connection between the entropy production of irreversible thermodynamics and the underlying dynamics in phase space for the Langevin description may be established.

Using the identity

$$\frac{\partial^2\rho}{\partial q_i\,\partial q_j} = \frac{\partial}{\partial q_i}\left[\rho\frac{\partial \ln\rho_e}{\partial q_j}\right] + \frac{\partial}{\partial q_i}\left[\rho_e\frac{\partial}{\partial q_j}\frac{\rho}{\rho_e}\right] \quad (12)$$

in Eq. (6) we have

$$\frac{dS}{dt} = -\int d\mathbf{q}\ln\rho\left[-\sum_{i=1}^{6}\frac{\partial F_i\rho}{\partial q_i} + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\frac{\partial}{\partial q_i}\left(\rho\frac{\partial \ln\rho_e}{\partial q_j}\right)\right] - \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\ln\rho_e\frac{\partial}{\partial q_i}\left(\rho_e\frac{\partial}{\partial q_j}\frac{\rho}{\rho_e}\right) + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\left(\frac{\partial}{\partial q_i}\ln\frac{\rho}{\rho_e}\right)\left(\frac{\partial}{\partial q_j}\ln\frac{\rho}{\rho_e}\right), \quad (13)$$

where ρ_e is the stationary solution of the Fokker-Planck Eq. (5). Here it is to be noted that the first, second and third integrals in Eq. (13) are of zeroth, first, and second order, respectively, with respect to the deviation from equilibrium. Doing partial integration of the above equation, one obtains

$$\frac{dS}{dt} = \overline{\boldsymbol{\nabla}\cdot\mathbf{F}^t} + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\rho\left[-\frac{\partial\ln\rho_e}{\partial q_i}\frac{\partial\ln\rho_e}{\partial q_j} + 2\frac{\partial\ln\rho}{\partial q_i}\frac{\partial\ln\rho_e}{\partial q_j}\right] + \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\left(\frac{\partial}{\partial q_i}\ln\frac{\rho}{\rho_e}\right)\times\left(\frac{\partial}{\partial q_j}\ln\frac{\rho}{\rho_e}\right). \quad (14)$$

In the above new decomposition of the time evolution of information entropy the first term has no definite sign and contains, in principle, contributions of all orders in the deviation from equilibrium. But the third term is both positive and of second order in the deviation from equilibrium, thereby fulfilling the principal condition required for entropy production of irreversible process. Thus it is analogous to the entropy production of irreversible thermodynamics and we represent it as

$$S_{P'} = \frac{\epsilon}{2}\sum_{ij=1}^{6} D_{ij}\int d\mathbf{q}\left(\frac{\partial}{\partial q_i}\ln\frac{\rho}{\rho_e}\right)\left(\frac{\partial}{\partial q_j}\ln\frac{\rho}{\rho_e}\right). \quad (15)$$

We call it information entropy production which is due to irreversibility in the relaxation process. In the stationary state these two terms are related as follows:

$$S_{P'} = -\overline{\boldsymbol{\nabla}\cdot\mathbf{F}^{\infty}} - (\text{terms of 0th and 1st order in deviation from equilibrium}). \quad (16)$$

Using Eq. (11) in above equation we have

$$S_{P'} = -\sum_i \sigma_i' - (\text{terms of 0th and 1st order in deviation from equilibrium}). \quad (17)$$

This is the required connection between entropy production of irreversible process and phase-space dynamics. Now we explicitly enquire how entropy production evolves in time and what the role of magnetic field is. This evolution should signify the relaxation behavior of a noise driven dynamical system in presence of a magnetic field. However, to identify the signature of the applied magnetic field (if any) in this context we will consider two examples in next two subsections.

A. Free particle

First, we consider a very simple system, the Brownian motion of a charged free particle in presence of a constant magnetic field. If we assume the magnetic field, for simplicity, pointing along the z axis of the Cartesian reference frame, that is, $\mathbf{B}=(0,0,B)$ with B a constant, then Eq. (2) can be described by means of two independent processes. One is described on the x-y plane perpendicular to the magnetic field; the other is along the magnetic field. In these cases the Eq. (2) can be written in terms of its components as follows:

$$\dot{u_x} = -\gamma u_x + \Omega u_y + f_x(t), \tag{18}$$

$$\dot{u_y} = -\gamma u_y - \Omega u_x + f_y(t), \tag{19}$$

and

$$\dot{u_z} = -\gamma u_z + f_z(t), \tag{20}$$

where $\Omega = \frac{QB}{mc}$ is the Larmor frequency. Since we are interested in the effect of the applied magnetic field on the nonequilibrium properties of the charged Brownian particle, we will focus on the stochastic motion on the x-y plane. Now using Eqs. (18) and (19), it is easy to write the Fokker-Planck equation for arbitrary values of dissipation parameter, magnetic field strength and ϵD_{ij} in the following form:

$$\frac{\partial \rho}{\partial t} = \left[\gamma \frac{\partial u_x}{\partial u_x} - \Omega \frac{\partial u_y}{\partial u_x} + \gamma \frac{\partial u_y}{\partial u_y} + \Omega \frac{\partial u_x}{\partial u_y} + \frac{\epsilon D_d}{2}\left(\frac{\partial^2}{\partial u_x^2} + \frac{\partial^2}{\partial u_y^2} \right) \right] \rho. \tag{21}$$

To have the time-dependent solution of the above equation we then search for the Green's function or conditional probability solution for the system at u_x, u_y at time t given that it had the value at u_x', u_y' at $t=0$. This initial condition may be represented by the δ function,

$$\delta(u_x - u_x')\,\delta(u_y - u_y') = \lim_{\epsilon_1 \to \infty} \sqrt{\frac{\epsilon_1}{\pi}} \exp\{-\epsilon_1[(u_x - u_x')^2 + (u_y - u_y')^2]\}. \tag{22}$$

$\sqrt{\frac{\epsilon_1}{\pi}}$ is the normalization constant. We now look for a solution of the Eq. (21) of the form

$$\rho(u_x, u_y, t | u_x', u_y', 0) = \exp[G(t)], \tag{23}$$

where $G(t) = -\frac{1}{\sigma(t)}\{[u_x - \alpha_{ux}(t)]^2 + [u_y - \alpha_{uy}(t)]^2\} + \ln \nu(t)$.

We will see that by suitable choice of $\alpha_x(t), \alpha_y(t), \sigma(t), \nu(t)$ one can solve Eq. (21) subject to the initial condition

$$\delta(u_x - u_x')\,\delta(u_y - u_y') = \rho(u_x, u_y, 0 | u_x', u_y', 0) = \lim_{\epsilon_1 \to \infty} \sqrt{\frac{\epsilon_1}{\pi}} \exp\{-\epsilon_1[(u_x - u_x')^2 + (u_y - u_y')^2]\}. \tag{24}$$

Comparing Eq. (23) with Eq. (24) and $G(0)$ we have $\sigma(0) = \lim_{\epsilon_1 \to \infty} \frac{1}{\epsilon_1}$, $\alpha_{ux}(0) = u_x'$, $\alpha_{uy}(0) = u_y'$ and $\nu(0) = \lim_{\epsilon_1 \to 0} \sqrt{\frac{\epsilon_1}{\pi}}$.

If we put Eq. (23) in Eq. (21) and equate the coefficients of equal powers of u_x and u_y we obtain after some algebra

$$\dot{\sigma}(t) = -2\gamma\sigma(t) + 2\epsilon D_d, \tag{25}$$

$$\dot{\alpha_{ux}}(t) = -\gamma \alpha_{ux}(t) + \Omega \alpha_{uy}(t), \tag{26}$$

$$\dot{\alpha_{uy}}(t) = -\gamma \alpha_{uy}(t) - \Omega \alpha_{ux}(t), \tag{27}$$

and

$$\frac{1}{\nu(t)} \dot{\nu}(t) = 2\gamma + \frac{4\gamma}{\sigma(t)}(\alpha_{ux}^2 + \alpha_{uy}^2) - \frac{2\epsilon D_d}{\sigma(t)}. \tag{28}$$

It is pertinent to note that the width of the distribution function is solely governed by the characteristic of the thermal bath. Later on we will show that it depends only on the temperature at long time as expected from the kinetic theory. The time-dependent contribution in the distribution function from the characteristics of dynamical system is incorporated through α_{ux} and α_{uy}. In the present case these quantities take care of the effect of magnetic field. However, one can check very simply calculating the average of the velocity components whether the form of distribution function as given in Eq. (23) is correct or not. From Eq. (23) we have

$$\langle u_x \rangle(t) = \alpha_{ux}(t) \tag{29}$$

and

$$\langle u_y \rangle(t) = \alpha_{uy}(t) \tag{30}$$

Thus Eqs. (26) and (27) are the expressions for the rate of change of average of u_x and u_y with time. These are exactly the same as one expects from the Langevin equations of motion (18) and (19). This also ensures that the form of the distribution function in Eq. (23) is correct. Shortly we will show that it reduces to the Boltzmann distribution function at equilibrium as demanded by the kinetic theory.

Now making use of the Eq. (23) in Eqs. (9), (10), and (15) we finally obtain explicit time dependence of the entropy flux (S_F) and the entropy production (S_P) having all order contribution with respect to deviation from equilibrium and the entropy production (S_P') due to irreversibility in the process as

$$S_F = S_{Fx} + S_{Fy}, \quad S_{Fx} = S_{Fy} = -\gamma, \tag{31}$$

$$S_P = S_{Px} + S_{Py},$$

$$S_{Px} = S_{Py} = \frac{\epsilon D_d}{\sigma(t)} = \frac{\epsilon D_d}{\frac{\epsilon D_d}{\gamma} + \left[\sigma(0) - \frac{\epsilon D_d}{\gamma} \right] \exp(-2\gamma t)}, \tag{32}$$

and

$$S_P' = S_{P'x} + S_{P'y}, \tag{33}$$

where

$$S_{P'x} = \frac{1}{\epsilon D_d \sigma}[\gamma^2 \sigma \alpha_{ux}^2 + \gamma^2 \sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2], \tag{34}$$

$$S_{P'y} = \frac{1}{\epsilon D_d \sigma}[\gamma^2 \sigma \alpha_{uy}^2 + \gamma^2 \sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2]. \tag{35}$$

In Eq. (31) we have separated total entropy flux into its corresponding x and y components and represented as S_{Fx} and S_{Fy}. Similarly other quantities are expressed in terms of their components. In Eq. (32) we have used the following time-dependent solution of the width of distribution function

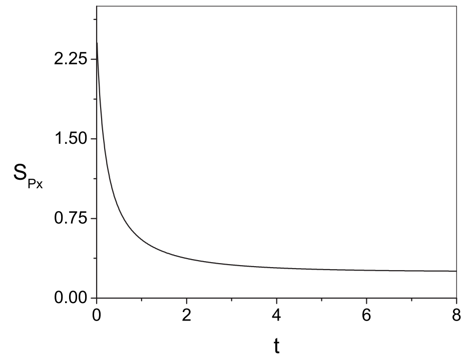

FIG. 1. Plot of (S_{Px}) vs time using Eq. (32) for the parameter set $\gamma=0.25$, $\sigma(0)=0.1$, and $k_BT=0.5$.

$$\sigma(t)=\frac{\epsilon D_d}{\gamma}[1-\exp(-2\gamma t)]+\sigma(0)\exp(-2\gamma t). \quad (36)$$

The entropy flux represented by Eq. (31) corresponds to the phase-space contraction rate which is governed by the dissipative force. Therefore the entropy flux (ENF) is independent of nondissipative force derived from the applied magnetic field. Since both the components of motion are coupled to the same thermal bath, ENF is same for each component. Now we consider the Eq. (32). It shows that the information entropy production is proportional to the Fisher information with the proportionality constant, D_d. However, it is apparent in Eq. (32) that the evolution of S_P or its' component is solely governed by the characteristic of the bath and therefore the components of S_P are same. Time evolution of the x component is presented in Fig. 1. Since the width of the distribution function σ increases with time, the information entropy production (S_{Px}) and the Fisher information decreases toward the stationary value (γ). When the width of the distribution function is small, the random force plays a strong role for expansion of phase space against the deterministic force and therefore the entropy production (the phase-space expansion rate) is highest during the onset of the motion of the particle.

Although S_P and S_F are independent on the applied magnetic field but the component of entropy production corresponding to irreversibility in the process depends on the average dynamics of the relevant coordinate of that component as it is evident from Eqs. (34) and (35). $\alpha_{ux}(t)$ and $\alpha_{uy}(t)$ appear in Eqs. (34) and (35) are average of components of velocity along x,y directions, respectively. Thus magnetic field (MF) has a role in the evolution of S_P' and its' components and MF breaks the symmetry of the average motion of the components of total motion. As a result of that the evolution of $S_{P'x}$ and $S_{P'y}$ are not identical and we have demonstrated it in Fig. 2. It shows a damping of an oscillation induced by nondissipative force due to applied magnetic field and y component (dashed curve) relaxes first compared to other (dotted curve). In absence of magnetic field the entropy production of both the components become equal for same initial values of α_{ux} and α_{uy} as expected and they are presented by solid curve. The monotonic decay of the solid curve implies that the magnetic field can induce an oscillation and then the phase-space expansion rate does not decay monotonically.

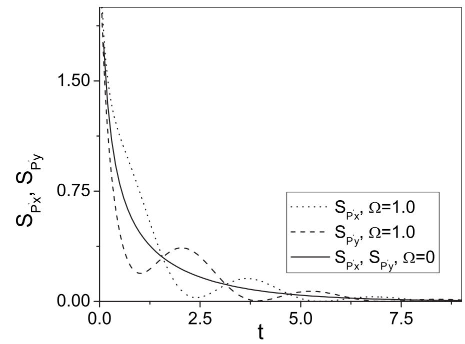

FIG. 2. Plot of $(S_{P'x},S_{P'y})$ vs time using Eq. (34) and (35) for the parameter set $\gamma=0.25$, $\sigma(0)=0.1$, $\alpha_{ux}(0)=\alpha_{ux}(0)=1.0$, and $k_BT=0.5$.

We now check whether our calculation satisfies or not the known results. In the long time limit Eqs. (25)–(27) show $\sigma(\infty)=\epsilon D_d/\gamma$, $\alpha_{ux}=\alpha_{uy}=0$. Thus the equilibrium distribution function corresponding to Eq. (23) is

$$\rho(u_x,u_y)=\nu\exp\left[-\frac{u_x^2+u_y^2}{2k_BT}\right]. \quad (37)$$

Here we have used $\epsilon D_d/\gamma=2k_BT$. The above equation is the equilibrium solution of Eq. (21). Equations (31) and (32) satisfy the equilibrium condition as follows:

$$\frac{dS}{dt}=S_F+S_P=0, \quad (38)$$

since at long time

$$S_{Px}=S_{Py}=\gamma. \quad (39)$$

Finally, Eq. (33) also reduces to the expected equilibrium condition(entropy production due to irreversible process is zero at equilibrium), $S_{P'}=0$ since $S_{P'x}=S_{P'y}=0$ at $t\to\infty$. Thus our calculation is consistent with the limiting results.

B. Particle in a three-dimensional harmonic potential

Now we consider the time evolution of the thermodynamically inspired quantities of stochastic motion which is bounded in a three dimensional harmonic potential well $V(x,y,z)=k_xx^2+k_yy^2+k_zz^2)/2$ in presence of a magnetic field. k_x,k_y and k_z are force constants along x, y, and z directions, respectively. Using this potential we start with the following relevant equations of motion [Eq. (2)]:

$$\dot{x}=u_x, \quad (40)$$

$$\dot{y}=u_y, \quad (41)$$

$$\dot{u_x}=-\gamma u_x-\omega_1^2x+\Omega u_y+f_x(t), \quad (42)$$

$$\dot{u_y}=-\gamma u_y-\omega_2^2y-\Omega u_x+f_y(t). \quad (43)$$

Here we have used $\omega_1=\sqrt{k_x/m}$ and $\omega_2=\sqrt{k_y/m}$. Equation of motion along the z component is not considered since we are

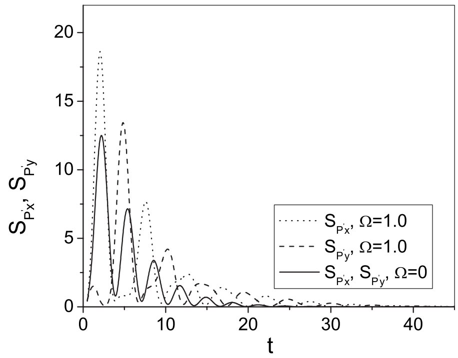

FIG. 3. Plot of $(S_{P'x}, S_{P'y})$ vs time using Eq. (44) and (45) for the parameter set $\gamma=0.25$, $\sigma(0)=0.1$, $\alpha_{ux}(0)=\alpha_{ux}(0)=1.0$, and $k_BT=0.5$.

interested in motion on x-y plane. However, it is to be noted that the above model may correspond to a quantum dot which is coupled to a high temperature thermal bath [57,58].

Repeating the above calculation for this set of equations we finally obtain the following form of entropy production due to irreversibility of motion,

$$S_{P'x} = \frac{1}{\epsilon D_d \sigma}[2\gamma^2\sigma\alpha_x^2 + \gamma^2\sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2], \quad (44)$$

$$S_{P'y} = \frac{1}{\epsilon D_d \sigma}[2\gamma^2\sigma\alpha_y^2 + \gamma^2\sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2], \quad (45)$$

where α_x and α_y are the solutions of the following coupled equations:

$$\dot{\alpha_{cx}}(t) = \left[-2\gamma + \frac{2\epsilon D_d}{\sigma(t)}\right]\alpha_{cx} + \alpha_x, \quad (46)$$

$$\dot{\alpha_{cy}}(t) = \left[-2\gamma + \frac{2\epsilon D_d}{\sigma(t)}\right]\alpha_{cy} + \alpha_y, \quad (47)$$

$$\dot{\alpha_x}(t) = -\gamma\alpha_x(t) - \omega_1^2\alpha_{cx} + \Omega\alpha_x(t), \quad (48)$$

$$\dot{\alpha_y}(t) = -\gamma\alpha_y(t) - \omega_2^2\alpha_{cy} - \Omega\alpha_y(t), \quad (49)$$

$\alpha_{cx}, \alpha_{cy}, \alpha_x$, and α_y are related to four dimensional distribution function as follows:

$$\rho(x,y,u_x,u_y,t|x',y',u_x',u_y',0) = \exp[G(t)], \quad (50)$$

where $G(t)=-\frac{1}{\sigma(t)}\{\omega_1^2[x-\alpha_{cx}(t)]^2+\omega_2^2[y-\alpha_{cy}(t)]^2+[u_x-\alpha_x(t)]^2+[u_y-\alpha_y(t)]^2\}+\ln\nu(t)$. Here $\sigma(t)$ is the same as described in Eq. (36). It is consistent with our earlier explanation. The forms of S_F and S_P remain also same as discussed above. The time dependence of Eqs. (44) and (45), is made more explicit in Fig. 3. Here for dotted, dashed and solid curves same convention is followed as in Fig. 2. Figures 2 and 3 show that the additional harmonic force increases the amplitude of the damped oscillation as well as the relaxation time. Increase of deterministic force leads to enhancement of amplitude. This is consistent with our explanation. Slow relaxation for the present case does not need any discussion. It is important to note that Fig. 3 also describes how the equivalence of the components of motion breaks down by the magnetic field. A closer look into the theoretical scheme also assures that the treatment is consistent with the results obtained for equilibrium condition.

III. ENTROPY PRODUCTION OF AN IRREVERSIBLE PROCESS FOR RELAXATION OF A SMALL EXTERNAL ELECTRIC FIELD DRIVEN EQUILIBRIUM STATE TO A NEW STATIONARY STATE

It is now interesting to examine the time dependence of entropy flux and entropy production during the relaxation of small mechanical force-driven thermostatted equilibrium state. To this end we consider the constant electric field $(\mathbf{F}_1)$ in Eq. (2) due to external force so that the total drift in Eq. (2) now becomes

$$\mathbf{F} = \mathbf{F}_0 + h\mathbf{F}_1, \quad (51)$$

where $\mathbf{F}_0=-\frac{1}{m}\nabla V(\mathbf{r})-\gamma\mathbf{u}+\frac{Q}{mc}\mathbf{u}\times\mathbf{B}$, h is smallness parameter. When $h=0$, $\rho=\rho_e$, ρ_e is the equilibrium solution of Eq. (5). The deviation of ρ from ρ_e in presence of nonzero small h can be explicitly taken into account once we make use of the identity for the diffusion term Eq. (12). Then for the above definition of deterministic force the Fokker-Planck Eq. (5) becomes

$$\frac{\partial\rho}{\partial t} = -\sum_{i=1}^{6}\frac{\partial}{\partial q_i}\left[\left(F_{0i} - \frac{\epsilon}{2}\sum_{j=1}^{6}D_{ij}\frac{\partial\ln\rho_e}{\partial q_j}\right)\rho\right] - h\sum_{i=1}^{6}\frac{\partial}{\partial q_i}(F_{1i}\rho) + \frac{\epsilon}{2}\sum_{ij=1}^{6}D_{ij}\frac{\partial}{\partial q_i}\left(\rho_e\frac{\partial}{\partial q_j}\frac{\rho}{\rho_e}\right) \quad (52)$$

Using the above equation one can write the rate of change of information entropy for the thermostatted system [24] as

$$\frac{dS}{dt} = h^2\int d\mathbf{q}\,\delta\rho\,\mathrm{div}\,\mathbf{F}_1 + h^2\int\mathbf{q}\left(\sum_{i=1}^{6}F_{1i}\frac{\partial\ln\rho_e}{\partial q_i}\right) + \frac{\epsilon}{2}\sum_{ij=1}^{6}D_{ij}\int\mathbf{dq}\left(\frac{\partial}{\partial q_i}\ln\frac{\rho}{\rho_e}\right)\left(\frac{\partial}{\partial q_j}\ln\frac{\rho}{\rho_e}\right). \quad (53)$$

where $h\delta\rho=\rho-\rho_e$. Comparing Eq. (53) with Eq. (14) one can easily identify that the third term as the entropy production $S_{P'}$ of irreversible process and the remaining terms correspond to the entropy flux like quantity

$$S_{F'} = h^2\int d\mathbf{q}\,\delta\rho\,\mathrm{div}\,\mathbf{F}_1 - h^2\int\mathbf{q}\left(\sum_{i=1}^{6}F_{1i}\frac{\partial\ln\rho_e}{\partial q_i}\right). \quad (54)$$

Here the first term presents the rate of phase-space volume contraction to the second order, whereas the second one can be read as the average of the work per unit time of the external forcing acting (tangentially) along the i component of motion. In the steady state, we have from Eq. (53)

$$S_{P'} = -S_{F'}. \quad (55)$$

Thus the above equation establishes a connection between thermodynamically inspired quantities of an irreversible process and phase-space dynamics. For an illustration of the present section as before we again consider the above two cases in the following subsections.

A. Free particle

For the from of **F** as given in Eq. (51) the Eqs. (18) and (19) become

$$\dot{u_x} = -\gamma u_x + \Omega u_y + hF_{1x} f_x(t), \tag{56}$$

$$\dot{u_y} = -\gamma u_y - \Omega u_x + hF_{1y} f_y(t), \tag{57}$$

where F_{1x} and F_{1y} are the components of the applied electric field $\mathbf{F}_1$ along x and y direction, respectively. Now using the procedure described in Sec. II A we obtain

$$S_{F'} = S_{F'x} + S_{F'y}, \tag{58}$$

$$S_{F'x} = -\frac{2\gamma\alpha_{ux}F_{1x}}{\epsilon D_d}, \tag{59}$$

$$S_{F'y} = -\frac{2\gamma\alpha_{uy}F_{1x}}{\epsilon D_d}, \tag{60}$$

$$S_{P'x} = \frac{1}{\epsilon D_d \sigma}[2\gamma^2\sigma\alpha_{ux}^2 + \gamma^2\sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2], \tag{61}$$

and

$$S_{P'y} = \frac{1}{\epsilon D_d \sigma}[2\gamma^2\sigma\alpha_{uy}^2 + \gamma^2\sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2]. \tag{62}$$

Here α_{ux} and α_{uy} are solution of the following differential equations:

$$\dot{\alpha_{ux}}(t) = -\gamma\alpha_{ux}(t) + \Omega\alpha_{uy}(t) + F_{1x}, \tag{63}$$

$$\dot{\alpha_{uy}}(t) = -\gamma\alpha_{uy}(t) - \Omega\alpha_{ux}(t) + F_{1y}. \tag{64}$$

We now explore explicit dependence of the above thermodynamically inspired quantities on time, magnetic field strength and others. In Fig. 4, we have plotted components of entropy production vs time. It shows that in presence of magnetic field the entropy production along x component exhibits both minimum and maximum as a function of time before reaching the nonzero steady state value. But for y component it simply decreases to zero. In absence of magnetic field the entropy production becomes equal for both the components of motion (for same initial values of α_x and α_y and $F_{1x} = F_{1y}$) and attains the same nonzero steady state value (as attained by the x component in the presence of magnetic field). Thus the relaxation behavior in presence of magnetic field is drastically different compared to its absence and the field can break the equivalence in entropy production strongly for the x and y components of motion. Entropy flux also behaves similarly. This is demonstrated in the inset of Fig. 4. Thus our present study highlights on distinctive relaxation mechanism (if any) of a nonequilibrium constraint driven equilibrium state.

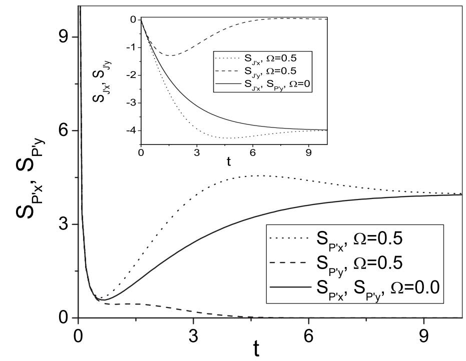

FIG. 4. Plot of $(S_{P'x}, S_{P'y})$ vs time using Eq. (61) and (62) for the parameter set $\gamma=0.5$, $\sigma(0)=0.025$, $\alpha_{ux}(0)=\alpha_{ux}(0)=0$, $F_{1x}=F_{1y}=1.0$, and $k_B T=0.5$. The variation of entropy flux along x,y directions with time is presented in inset for the same parameter set as given for the main figure.

The maximum for the x component in Fig. 4 may be a signature of oscillating behavior induced by magnetic field. Another optimum behavior in the same figure (which is almost common for both presence and absence of magnetic field) can be traced out by simplifying the Eqs. (61) and (62) in the following way. In the limit $\Omega \to 0$, $\alpha_{ux}(0) \to 0$, $\alpha_{uy}(0) \to 0$, and $\sigma(0) \to 0$, these equations become

$$S_{P'x} = \frac{[2F_{1x}^2(1 - 2e^{-\gamma t} + 2e^{-3\gamma t} - e^{-4\gamma t}) + \gamma\epsilon D_d e^{-4\gamma t}]}{\epsilon D_d(1 - e^{-2\gamma t})}. \tag{65}$$

and

$$S_{P'y} = \frac{[2F_{1y}^2(1 - 2e^{-\gamma t} + 2e^{-3\gamma t} - e^{-4\gamma t}) + \gamma\epsilon D_d e^{-4\gamma t}]}{\epsilon D_d(1 - e^{-2\gamma t})}. \tag{66}$$

First term in the numerator in both Eqs. (65) and (66), which vanishes as $t \to 0$, implies that the external force drives the equilibrium state by increasing entropy production while the second term corresponds to the relaxation to the old equilibrium state by decreasing of entropy production with time due to dissipative action as happens in absence of $\mathbf{F}_1$. Because of these two opposite effects, a system thrown away from an equilibrium state by a small external force to a new steady state through a minimum in entropy production with time. This is a surprising result since one expects that entropy production (phase-space expansion rate) would decrease monotonically to steady state for a simple system like free particle. Thus our calculation exposes a detail of relaxation mechanism. Now we show how the steady state values of entropy production become different for x and y components in presence of magnetic field. In the long time limit Eqs. (63) and (64) yield

$$\alpha_{ux}(\infty) = -\frac{\Omega}{\gamma}\frac{[\Omega F_{1x} - \gamma F_{1y}]}{\gamma^2 + \Omega^2} + \frac{F_{1x}}{\gamma}, \tag{67}$$

$$\alpha_{uy}(\infty) = -\frac{[\Omega F_{1x} - \gamma F_{1y}]}{\gamma^2 + \Omega^2}. \tag{68}$$

Thus, in presence of magnetic field, the average of the components of velocity at steady state becomes different even for the same value of the components of the applied constant electric field. This explains why steady state value of entropy production is not same along x,y directions. To read this explicitly we consider the long time limit of Eqs. (61) and (62) as

$$S_{P'x} = \frac{2\gamma^2\alpha_{ux}^2}{\epsilon D_d}, \tag{69}$$

and

$$S_{P'y} = \frac{2\gamma^2\alpha_{uy}^2}{\epsilon D_d}. \tag{70}$$

Thus the above equations implies that for $\Omega F_{1x} = \gamma F_{1y}$ the entropy production along y direction is zero at the steady state. In other words, for a certain condition the applied electric field may not be effective to drive the equilibrium state corresponding to motion in y direction. But under this condition x component has a nonzero value. For another limit, $\Omega=0$ and $F_{1x}=F_{1y}$, Eqs. (65) and (66) lead to the same entropy production ($S_{P'x}=S_{P'y}$). These are shown in Fig. 4 for the given parameter set. However, using similar analysis one can also explain the inset of Fig. 4 in which the entropy flux vs time has been plotted. It is to be noted from Eqs. (59), (60), and (67)–(70) that in presence of magnetic field the entropy flux and entropy production of the individual component do not balance each other at the steady state. But the total of them balance each other as follows:

$$S_{P'} = -S_{F'} = \frac{2\gamma^2(F_{1x}^2 + F_{1y}^2)}{\epsilon D_d(\gamma^2 + \Omega^2)}. \tag{71}$$

This shows that our calculation is consistent with the steady state condition ($\frac{dS}{dt}=0$). Now we check what happens in this context in absence of magnetic field. In this limit, there are following balance equations for the individual components.

$$S_{P'x} = -S_{F'x} = \frac{F_{1x}^2}{\gamma k_B T}, \tag{72}$$

$$S_{P'y} = -S_{F'y} = \frac{F_{1y}^2}{\gamma k_B T}. \tag{73}$$

These equations simultaneously satisfy two important checks. The above expressions are the standard results of irreversible thermodynamics and consistent with the steady state entropy balance. Now we explore how the entropy production varies with magnetic field at the steady state. Figure 5 presents this variation. It shows a maximum along x direction. But in y direction there is a minimum. These are signature of inequivalence in steady state average values of components of velocity as expressed in Eqs. (67) and (68). Similar signature is also observed in the plot of entropy production as a function of damping strength. It has been presented in the inset of Fig. 5. In presence of magnetic field the

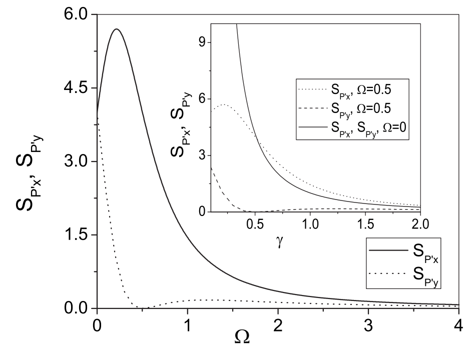

FIG. 5. Plot of $(S_{P'x}, S_{P'y})$ vs magnetic field strength Ω using Eq. (69) and (70) for the parameter set $\gamma=0.5$, $F_{1x}=F_{1y}=1.0$, and $k_B T=0.5$. Inset describes variation of entropy production $(S_{P'x}, S_{P'y})$ with γ for the same parameter set

inset is quite similar to the Fig. 5. We are now in a position to make the following interesting comment: Although magnetic field does not induce dissipation of energy but to determine the nonequilibrium state it has similar role as that of damping strength. This may be due to the fact that both the forces related to these quantities are velocity dependent. Based on the above comment one may expect that the magnetic field would have similar effect as that of damping on the steady state barrier-crossing rate. In our very recent study we have observed this effect [59].

In Fig. 6, we have presented the dependence of the entropy production on the applied nonequilibrium constraint (electric field). For the change of electric field (EF) in x direction there is a minimum in y direction in presence of magnetic field (MF). The applied EF is not effective at the minimum. But in x direction the entropy production increases monotonically both in presence and absence of magnetic field. There is a polynomial curve for nonzero MF and it becomes simple parabolic in nature when the MF vanishes. Quite similar plot is also observed for the change in electric field along y direction. Finally, in Fig. 7, we present the variation of entropy production as a function of temperature. It is apparent that at low temperature the applied electric field is more efficient to drive the x component than y component in presence of MF. This efficiency becomes equal as the magnetic field vanishes. At high temperature the distribution

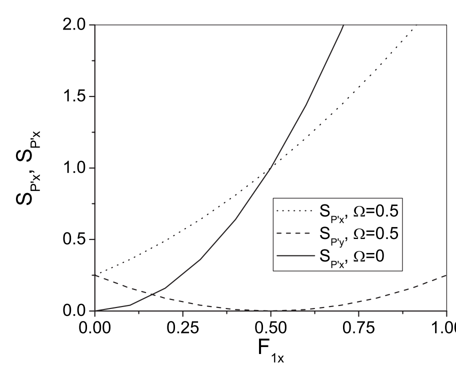

FIG. 6. Plot of $(S_{P'x}, S_{P'y})$ vs magnetic field strength F_{1x} using Eq. (69) and (70) for the parameter set $\gamma=0.5$, $F_{1y}=1.0$, and $k_B T=0.5$

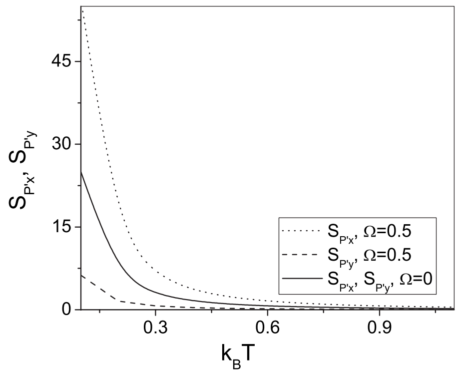

FIG. 7. Plot of $(S_{P'x}, S_{P'y})$ vs magnetic field strength F_{1x} using Eq. (69) and (70) for the parameter set $\gamma=0.5$ and $F_{1x}=1.0=F_{1y}=1.0$

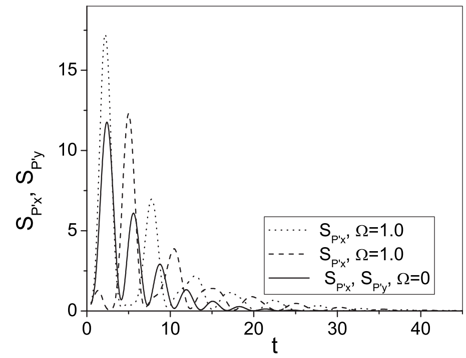

FIG. 8. Plot of $(S_{P'x}, S_{P'y})$ vs time using Eq. (80) and (81) for the parameter set $\gamma=0.25$, $\sigma(0)=0.025$, $\alpha_{ux}(0)=\alpha_{ux}(0)=0$, $F_{1x}=F_{1y}=1.0$, $\omega_1=\omega_2=1.0$, and $k_BT=0.5$.

function becomes wide and therefore the equilibrium state is very robust to the nonequilibrium constraint and the entropy production is zero at this limit. This is also consistent with our earlier discussion that the entropy production decreases as the width of the distribution function increases.

B. Particle in a three-dimensional harmonic potential

In this subsection we investigate the fate of the particle in a harmonic potential if it is driven by a constant electric field $\mathbf{F}_1$ from the equilibrium state. Making use of the harmonic potential in $\mathbf{F}$ as given in Eq. (51) the Eqs. (40)–(43) are modified as follows:

$$\dot{x} = u_x, \tag{74}$$

$$\dot{y} = u_y, \tag{75}$$

$$\dot{u_x} = -\gamma u_x - \omega_1^2 x + \Omega u_y + hF_{1x} + f_x(t), \tag{76}$$

$$\dot{u_y} = -\gamma u_y - \omega_2^2 y - \Omega u_x + hF_{1y} + f_y(t), \tag{77}$$

Applying the previous procedure we obtain the following expressions for entropy flux and production for the irreversible relaxation process

$$S_{F'x} = -\frac{2\gamma\alpha_{x'}F_{1x}}{\epsilon D_d}, \tag{78}$$

$$S_{F'y} = -\frac{2\gamma\alpha_{y'}F_{1x}}{\epsilon D_d}, \tag{79}$$

$$S_{P'x} = \frac{1}{\epsilon D_d \sigma}[2\gamma^2\sigma\alpha_{x'}^2 + \gamma^2\sigma^2 - \gamma\sigma\epsilon D_d + \epsilon^2 D_d^2], \tag{80}$$

$$S_{P'y} = \frac{1}{\epsilon D_d \sigma}[2\gamma^2\sigma\alpha_{y'}^2 + \gamma^2\sigma^2 - 2\gamma\sigma\epsilon D_d + \epsilon^2 D_d^2], \tag{81}$$

where α_x' and α_y' are the solution of the following coupled equations:

$$\dot{\alpha_{cx'}}(t) = \left[-2\gamma + \frac{2\epsilon D_d}{\sigma(t)}\right]\alpha_{cx'} + \alpha_x', \tag{82}$$

$$\dot{\alpha_{cy'}}(t) = \left[-2\gamma + \frac{2\epsilon D_d}{\sigma(t)}\right]\alpha_{cy'} + \alpha_y', \tag{83}$$

$$\dot{\alpha_{x'}}(t) = -\gamma\alpha_{x'}(t) - \omega_1^2\alpha_{cx'} + \Omega\alpha_{x'}(t) + F_{1x}, \tag{84}$$

$$\dot{\alpha_{y'}}(t) = -\gamma\alpha_{y'}(t) - \omega_2^2\alpha_{cy'} - \Omega\alpha_{y'}(t) + F_{1y}. \tag{85}$$

To make the physical content of the expressions in Eqs. (78)–(81) we have plotted entropy production *vs.* time in Fig. 8. It is similar to Fig. 3 in which the stationary state is the equilibrium state. Thus in presence of the bound harmonic potential the electric field $(\mathbf{F}_1)$ drives the equilibrium state to a new equilibrium state instead of a steady state as obtained in case of free particle. Before leaving this subsection we would like to mention another point. Both in Figs. 3 and 8, the frequency of the damped oscillation for entropy production decreases in presence of magnetic field for the particle in a harmonic potential. From this observation, one may anticipate that the attempt frequency, entering in the prefactor of barrier-crossing rate may decrease with increase of magnetic field. This has exactly been observed in our earlier study [59].

IV. CONCLUSION

In conclusion, we have studied the stochastic motion of a particle in presence of a magnetic field. Based on the Fokker-Planck description of the stochastic process and entropy balance equation thermodynamically inspired quantities are calculated. We summarize the major points.

(i) The entropy flux (S_F) and production (S_P) having all order contribution with respect to deviation from equilibrium solely depend on the characteristics of the thermal bath. The entropy flux is only a function of damping strength (γ) and time independent whereas the S_P depends on both γ and temperature of the thermal bath and it decreases monotonically with time to the equilibrium value. However, the entropy production $(S_{P'})$ which is related to irreversibility of the relaxation process is governed by the characteristic of the dynamical system and the thermal bath. The relaxation mechanism is quite different in presence of magnetic field. The MF strongly breaks the equivalence in motion of the components of a multidimensional system.

(ii) If a constant electric field drives an equilibrium state to a steady state then it follows that optimum behavior in the variation of the x component of $S_{P'}$ as function of time with a nonzero stationary value in presence of MF can be observed. But along the y direction there is a monotonically decaying of $S_{P'y}$ [y component of $S_{P'}$)] to the equilibrium value.

In presence of a bound potential the applied constant electric field drive the equilibrium state to a new equilibrium state instead of a steady state (as observed in case of free particle).

(iii) In the presence of nonequilibrium constraint we have observed extremum behavior in the variation of entropy production as a function of magnetic field strength and damping strength, respectively. Along the x direction there is a maximum. But in case of y direction the entropy a minimum appears with the equilibrium value. Although magnetic field does not induce dissipation of energy but to determine the nonequilibrium state it has a similar role as that of damping strength.

(iv) As the component of electric field along x direction (F_{1x}) increases the entropy production $S_{P'x}$ grows in the form of polynomial function of F_{1x} in presence of MF. But for $S_{P'y}$ there is a minimum at zero value. Thus at the minimum the applied electric field becomes ineffective. However, in absence of magnetic field $S_{P'x}$ increases as a parabolic function of F_{1x}. The same conclusion also can be done for the y component of nonequilibrium constraint.

(v) At low temperature the applied electric field is more efficient to drive the x component than y component in presence of MF. This efficiency becomes equal as the magnetic field vanishes. At high temperature as the equilibrium state is very robust to the nonequilibrium constraint the entropy production is zero.

The following aspects may be worthy issues of further research in the context of extension of present study. The nonmonotonically of the entropy production vs time implies that the phase-space contraction is not monotonous, as we have explained above. A direct computation of the Lyapunov exponents for the present systems will surely shed some light on the nonmonotonically of the phase-space contraction. Another interesting possibility would be to explore the effect of an oscillating external magnetic field on the results herein presented.

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