PHYSICAL REVIEW E **82**, 040105(R) (2010)

Renormalization group computation of the critical exponents of hierarchical spin glasses

Michele Castellana[1,2] and Giorgio Parisi[1]
[1]*Dipartimento di Fisica, Università di Roma 'La Sapienza', 00185 Rome, Italy*
[2]*LPTMS–CNRS–Universite Paris-Sud, UMR 8626, Bât. 100, 91405 Orsay, France*

(Received 29 June 2010; revised manuscript received 16 September 2010; published 15 October 2010)

The large scale behavior of the simplest non-mean-field spin-glass system is analyzed, and the critical exponent related to the divergence of the correlation length is computed at two loops within the ϵ-expansion technique with two independent methods. The techniques presented show how the underlying ideas of the renormalization group apply also in this disordered model, in such a way that an ϵ-expansion can be consistently set up. By pushing such calculation to high orders in ϵ, a consistent non-mean-field theory for such disordered system could be established, giving a substantial contribution the development of a predictive theory for real spin glasses.

DOI: 10.1103/PhysRevE.82.040105 PACS number(s): 64.60.ae, 75.10.Nr, 05.10.Cc

The understanding of glassy systems and their critical properties is a subject of main interest in statistical physics. The mean-field theory of spin-glasses [1] and structural glasses [2] provides a physically and mathematically rich theory. Nevertheless, real spin-glass systems have short-range interactions, and thus cannot be successfully described by mean-field models [1]. This is the reason why the development of a predictive and consistent theory of glassy phenomena going beyond mean field is still one of the most hotly debated, difficult and challenging problems in this domain [3–5], so that a theory describing real glassy systems is still missing. This is because nonperturbative effects are poorly understood and not under control, and the basic properties of large scale behavior of these systems still far from being clarified.

In ferromagnetic systems, the physical properties of the paramagnetic-ferromagnetic transition emerge in a clear way already in the original approach of Wilson [7], where one can write a simple renormalization group (RG) transformation. It was later realized that Wilson's equations are exact in models with ferromagnetic power-law interactions on hierarchical lattices as the Dyson model [8,9]. This model contains all the physical RG properties, and is simple enough to yield a solution of the RG equations within the ϵ-expansion [10].

The extension of this approach to random systems is available only in a few cases. An RG analysis for random models on the Dyson hierarchical lattice has been pursued in the past [11,12], and a systematic analysis of the physical and unphysical infrared (IR) fixed points has been developed within the ϵ-expansion technique. Unfortunately, in such models spins belonging to the same hierarchical block interact each other with the same [11] random coupling J, in such a way that frustration turns out to be relatively weak and they are not a good representative of realistic strongly frustrated system. Moreover, there has recently been a new wave of interest for strongly frustrated random models on hierarchical lattices [13–15]: for example, it has been shown [14] that a generalization of the Dyson model to its disordered version [the hierarchical random energy model (HREM)] has a random energy model-like phase transition.

In this letter we present a field theory analysis of the critical behavior of a generalization of Dyson's model to the disordered case, known as the hierarchical Edwards-Anderson Model (HEA) [13]. The HEA is of particular interest, since it is a non-mean-field strongly frustrated model with long-range interaction. It follows that its RG analysis pursued in this work makes a contribution to the development of a theory describing real glassy systems with short-range interaction. Indeed, the symmetry properties of the HEA make an RG analysis simple enough to be done with two independent methods, showing that its IR-limit is physically well-defined, independently on the computation technique that one uses. The same symmetry properties make the RG equations simple enough to make a high-order ϵ-expansion tractable by means of a symbolic manipulation program, resulting in a first predictive theory for the critical exponents for a strongly frustrated non-mean-field system mimicking a real spin-glass. It is possible that such a perturbative expansion turns out to be nonconvergent: if this happens, it may help us to pin down the nonperturbative effects. Motivated by this purpose, we show with a two-loop calculation that such ϵ-expansion can be set up consistently, and that the ordinary RG underlying ideas actually apply also in this case, so that the IR limit of the theory is well-defined independently on the regularization technique.

The Hamiltonian of the HEA is defined [13] as $H_J[S]=-\sum_{i,k}J_{i,k}S_iS_k$ where the spin S_is take values ± 1 and $J_{i,k}$ are Gaussian random variables with zero mean and variance $\sigma^2_{i,k}$. Everything depends on the form of $\sigma^2_{i,k}$ that will be chosen in such way to make the model simple enough, and a good candidate mimicking a real glassy system. At large distance we have that $\sigma^2_{i,k}=O(|i-k|^{-2\sigma})$, where σ is a parameter tuning the decay of the interaction strength with distance: we recover the mean-field regime for $\sigma=1/2$, while no transition is present for $\sigma>1$ [13]. We will thus be interested in the case $1/2<\sigma<1$, where the interaction strength mimics the non-mean field forces of a real spin glass. The form of $\sigma^2_{i,k}$ is given by the following expression: if only the last m digits in the binary representation of the points i and k are different, $\sigma^2_{i,k}=2^{-2\sigma m}$. This form of the Hamiltonian corresponds in dividing the system in hierarchical embedded blocks of size 2^m, such that the interaction between two spins depends on the distance of the blocks to which they belong. The quantity $\sigma^2_{i,k}$ is not translational invariant, but it is invariant under a huge symmetry group and this will be crucial in the study of the model.

We reproduce the IR behavior of the HEA by two different methods. The first method is analogous to the coarse-graining Wilson's method for the Ising model: the IR limit is obtained by imposing invariance with respect to the composition operation taking two systems of 2^k spins and yielding a system of 2^{k+1} spins, for which one can obtain closed formulas because of hierarchical structure of the Hamiltonian. The second method is more conventional: we perform the IR-limit of the theory by constructing an IR-safe renormalized theory and performing its IR limit by the Callan-Symanzik equation.

Wilson's method. As mentioned before, the hierarchical symmetry structure of the model makes the implementation of a recursionlike RG equation simple enough to be solved within an approximation scheme, yielding [13] a recurrence relation for the probability distribution of the overlap [1,2] $Q_{ab}, a=1,\cdots,n$

$$Z_k[Q] = e^{\beta^2/4\,\mathrm{Tr}[Q^2]}\int [dP] Z_{k-1}\left[\frac{Q+P}{C^{1/2}}\right] \times Z_{k-1}\left[\frac{Q-P}{C^{1/2}}\right], \tag{1}$$

where $C\equiv 2^{2(1-\sigma)}$, $\beta\equiv 1/T$ is the inverse-temperature and $\int[dP]$ stands for the functional integral over P_{ab}. The recursion relation (1) can be solved by supposing $Z_k[Q]$ to be a mean-field solution, i.e., a Gaussian function of Q. As it will be explicitly shown in the following, the resulting fixed point $Z_*[Q]$ of Eq. (1) turns out [13] to be stable just for $\epsilon\equiv\sigma-\frac{2}{3}<0$. For $\epsilon>0$ the stable fixed point is no more Gaussian, and we search for a solution to Eq. (1) as a small perturbation to the mean-field solution

$$Z_k[Q] = \exp\{-[r_k\,\mathrm{Tr}[Q^2] + w_k/3\,\mathrm{Tr}[Q^3]]\}. \tag{2}$$

General RG arguments [7] suggest that the corrections to the mean-field solution must be proportional to ϵ.

A complete reconstruction of the function $Z_k[Q]$ for $\epsilon>0$ stems from the following systematic expansion procedure. In first approximation, we write $Z_k[Q]$ as in Eq. (2), and take into account only the cubic term. By inserting Eq. (2) into Eq. (1), and expanding in terms of w_{k-1} to up to $O(w_{k-1}^3)$, we find that $Z_k[Q]$ has the same functional form as in Eq. (2), where the coefficients r_k, w_k are given by some functions of r_{k-1}, w_{k-1} that can be directly computed. It follows that the recursion Eq. (1) yields a relation between r_k, w_k and r_{k-1}, w_{k-1}. In particular, the recursion relation giving w_k as a function of r_{k-1}, w_{k-1}

$$w_k = \frac{2w_{k-1}}{C^{3/2}} + \frac{n-2}{16C^{3/2}}\left(\frac{w_{k-1}}{r_{k-1}}\right)^3 + O(w_{k-1}^5),$$

shows that for $\epsilon<0$ the fixed point is Gaussian, while for $\epsilon>0$ a non-Gaussian fixed point arises. It is important point out that the value of $\epsilon=\sigma-2/3$ is different by the $\epsilon'=\sigma-1/2$, arising in the generalization Dyson model to its disordered version that has been already pursued in the literature [11,12]. This is because in the latter the frustration is much weaker than in the HEA, in such a way that the IR-behavior of the theory turns out to be generally different.

Higher order corrections to the Gaussian solution can be handled systematically: inserting Eq. (2) into Eq. (1), and expanding to $O(w_{k-1}^4)$, we generate in $Z_k[Q]$ four monomials $\{I_4^l[Q]\}_{l=1,\ldots,4}$ of fourth degree in Q. In order to close the recursion relation (2), it is then natural to set

$$Z_k[Q] = \exp\left\{-r_k\left[\mathrm{Tr}[Q^2] + w_k/3\,\mathrm{Tr}[Q^3] + \frac{1}{4}\sum_{l=1}^{4}\lambda_k^l I_4^l[Q]\right]\right\}, \tag{3}$$

where $\lambda_k^l = O(w_k^4)$. By plugging Eq. (3) into Eq. (1), we obtain a recursion equation relating $r_k, w_k, \{\lambda_k^l\}_{l=1,\cdots,4}$ to $r_{k-1}, w_{k-1}, \{\lambda_{k-1}^l\}_{l=1,\cdots,4}$. This procedure can be pushed to arbitrary high order p in w_k, yielding an p-degree polynomial for

$$Z_k[Q] = \exp\left(-\sum_{j=2}^{p}\sum_{l=1}^{n_j} c_{j,k}^l I_j^l[Q]\right), \tag{4}$$

where the number n_j of monomials proliferates for increasing j. Following the method explained above, a recursion equation relating $\{c_{j,k}^l\}_{j,l}$ to $\{c_{j,k-1}^l\}_{j,l}$ can be obtained, and the critical fixed point $\{c_{j,*}^l\}_{j,l}$ computed by solving perturbatively in ϵ the fixed-point equations. Following the standard RG, we suppose that the system has a characteristic correlation length ξ, diverging at the critical point, where the system is invariant under change in the scale length. By linearizing the recursion relation close to such fixed point, the critical exponent ν governing the power-law divergence of ξ for $T\to T_c$ can be obtained in terms of the largest eigenvalue λ of the matrix M linearizing such transformation next to the fixed point [7]: $\nu^{-1}=\log_2\lambda$.

We performed this systematic expansion to the order $p=5$, generating $n_4=4$ invariants of fourth degree, and $n_5=4$ invariants of fifth degree in Q. Such computation yields ν to the order ϵ^2. For $n\to 0$, we find

$$\nu = 3 + 36\epsilon + [432 - 27(50 + 55\cdot 2^{1/3} + 53\cdot 2^{2/3})\log 2]\epsilon^2 + O(\epsilon^3). \tag{5}$$

The one-loop result for ν is the same as that of the power-law interaction spin-glass studied in [5] (where $\epsilon\equiv 3(\sigma-2/3)$). Notwithstanding this, the coefficients of the expansion in these two models will be in general different at two or more loops. As a matter of fact, the binary tree structure of the interaction of the HEA emerges in the nontrivial $\log 2, 2^{1/3}$ factors in the coefficient of ϵ^2 in Eq. (5), that can't be there in the power-law case.

Before discussing the result in Eq. (5), we point out that Wilson's method explicitly implements the binary-tree structure of the model when approaching the IR limit. Nevertheless, if the IR limit is well-defined, physical observables like ν must not depend on the technique we use to compute them in such a limit. It is then important to reproduce Eq. (5) with a different approach.

Field-theoretical method. Here the ϵ-expansion is performed by constructing a functional integral field theory and by removing its IR divergences within the minimal subtraction scheme. The field theory is constructed by expressing the average of the replicated partition function as a functional integral over the local overlap field

$$Q_{iab} \equiv S_i^a S_i^b, \quad \mathbb{E}_J[Z^n] = \int [dQ] e^{-S[Q]}. \tag{6}$$

A short computation yields the IR-dominant terms in the effective Hamiltonian for a system of 2^k spins,

$$S[Q] = \frac{1}{2} \sum_{i,j}^{0,2^k-1} (\sigma_{i,j}^2 + \tau^{2\sigma-1}\delta_{ij}) \mathrm{Tr}[Q_i Q_j] + \frac{g}{3!} \sum_{i=0}^{2^k-1} \mathrm{Tr}[Q_i^3], \tag{7}$$

where Tr denotes the trace over the replica indexes and $\tau^{2\sigma-1} \propto T - T_c$. The field theory defined by Eq. (7) reproduces the Q^3 interaction term of the well-know effective actions describing the spin-glass transition in short-range [16] and long-range [5,17] spin glasses. Notwithstanding this similarity, the novelty of the HEA is that a high-order ϵ-expansion can be quiet easily automatized by means of a symbolic manipulation program solving the simple RG Eq. (1) to high orders in ϵ. This is not true for such short and long-range [5,16,17] models, where the only approach to compute the exponents is the field-theoretical one. Indeed, nobody ever managed to automatize at high orders a computation of the critical exponents within the field-theoretical minimal subtraction scheme, either for the simplest case of the Ising model.

To start our field-theoretical analysis, we observe that Eq. (7) presents an unusual quadratic term that is not invariant under spatial translations and it is difficult to perform explicit calculations. This difficulty can be overcome by a relabeling of the sites of the lattice $i = 0, \cdots, 2^k - 1$, following the same procedure of [18,19]. After relabeling one obtains that $\sigma_{i,j}^2 \propto |i-j|_2^{-2\sigma}$, where $|i|_2$ is the diadic norm of i, i.e., if 2^m divides i and $i/2^m$ is odd, $|i|_2 = 2^{-m}$. Even if this representation is quite unusual (if you are not an expert in p-adic numbers), in this way the variance of the couplings $J_{i,j}$ is translational invariant, since it depends only on $i-j$ (each realization of the system is not translational invariant). In the replica formalism we need to know only the variance of the couplings, not the actual couplings and therefore the effective Hamiltonian in replica space is translational invariant and we can use the standard Fourier transform [19,20] in order to compute loop integrals. The field theory defined by Eq. (7) can be now analyzed within the loop expansion framework. We expand the 1PI correlation functions

$$\Gamma^{(m,l)}_{a_1 b_1 i_1 \cdots a_m b_m i_m; j_1 \cdots j_l} \equiv 2^{-l} \langle Q_{i_1 a_1 b_1} \cdots Q_{i_m a_m b_m} \mathrm{Tr}[Q_{j_1}^2] \cdots \mathrm{Tr}[Q_{j_l}^2] \rangle_{1\mathrm{PI}},$$

in terms of the renormalized coupling constant g_r and take the small renormalized mass limit $\tau_r \to 0$. According to general results [6] concerning long-range models, the field Q is not renormalized, and all we need are the $\mathrm{Tr}[Q^2]$-renormalization constant Z_2, and the g-renormalization constant Z_g. An explicit evaluation of the loop integrals related to the action [Eq. (7)] shows that the IR divergences arising for $\epsilon > 0$, $\tau_r \to 0$ can be reabsorbed into Z_g, Z_2 by means of the minimal subtraction scheme [6]. An IR-safe renormalized theory can be constructed, and its IR fixed point g_r^* is computed as the zero of the β-function $\beta(g(\lambda)) = \lambda g'(\lambda)$, yielding the effective coupling constant $g(\lambda)$ of the theory at the energy scale λ. ν is given in terms of g_r^*, Z_2

$$\eta_2[g_r] \equiv \tau_r \left. \frac{\partial \log Z_2}{\partial \tau_r} \right|_{g,\tau}, \quad \frac{1}{\nu} = \eta_2[g_r^*] + 2\sigma - 1. \tag{8}$$

As predicted by dimensional considerations, the fixed point $g_r^* = 0$ is stable only for $\epsilon < 0$, while for $\epsilon > 0$ a non-Gaussian fixed point g_r^* of order ϵ arises. By plugging the two-loop result for g_r^* and Z_2 into Eq. (8) and taking $n \to 0$, we reproduce exactly the result [Eq. (5)] derived within Wilson's method.

Conclusions. In this paper we consider a strongly frustrated non-mean-field spin-glass system, the HEA model, and performed an RG analysis yielding results and future developments for a predictive theory of the critical exponents for real spin-glass systems. We set up two perturbative approaches to compute the IR behavior of the HEA. The first explicitly exploits the hierarchical structure of the model, and implements a Wilson-like coarse-graining technique to reach the IR limit. The second relies on the construction of an effective field theory reproducing the IR limit by means of the Callan-Symanzik equation. In both methods, we implemented the basic RG underlying ideas. Among these, the existence of a characteristic length ξ diverging at the critical point, where the theory is invariant with respect to changes in its energy scale. The two approaches yield the same prediction for the critical exponent ν related to the divergence of ξ, showing that the IR limit of the theory is well-defined and independent on the actual method one uses to reproduce it.

Thanks to the hierarchical symmetry of the model, a high-order ϵ-expansion for the HEA could be automatized by means of a symbolic manipulation program. If such series could be made convergent [6] by means of some resummation technique, such high-order calculation would yield an analytical control on the critical exponents, resulting in a precise prediction for a non-mean-field spin-glass.

We are glad to thank S. Franz, M. Mézard and N. Sourlas for interesting discussions and suggestions.

[1] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[2] T. Castellani and A. Cavagna, J. Stat. Mech.: Theory Exp. (2005) P05012.
[3] C. De Dominicis and I. Giardina, *Random Fields and Spin Glasses: A Field Theory Approach* (Springer, New York, 2006).
[4] J. H. Chen and T. C. Lubensky, Phys. Rev. B **16**, 2106 (1977).
[5] G. Kotliar, P. W. Anderson, and D. L. Stein, Phys. Rev. B **27**, 602 (1983).
[6] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Vol. 113 of The International Series of Monographs on Physics (Clarendon, Oxford, 2002).
[7] K. G. Wilson and J. B. Kogut, Phys. Rep. **12**, 75 (1974).
[8] F. J. Dyson, Commun. Math. Phys. **12**, 91 (1969).
[9] P. Collet and J. Eckmann, *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics* (Springer-Verlag, Berlin, 1978).
[10] M. Cassandro and G. Jona-Lasinio, Adv. Phys. **27**, 913 (1978).
[11] A. Theumann, Phys. Rev. B **21**, 2984 (1980).
[12] A. Theumann, Phys. Rev. B **22**, 5441 (1980).
[13] S. Franz, T. Jörg, and G. Parisi, J. Stat. Mech.: Theory Exp. (2009) P02002.
[14] M. Castellana, A. Decelle, S. Franz, M. Mézard, and G. Parisi, Phys. Rev. Lett. **104**, 127206 (2010).
[15] Y. V. Fyodorov, A. Ossipov, and A. Rodriguez, J. Stat. Mech.: Theory Exp. (2009) L12001.
[16] A. B. Harris, T. C. Lubensky, and J. H. Chen, Phys. Rev. Lett. **36**, 415 (1976).
[17] M. C. Chang and J. Sak, Phys. Rev. B **29**, 2652 (1984).
[18] G. Parisi and N. Sourlas, Eur. Phys. J. B **14**, 535 (2000).
[19] Y. Meurice, J. Math. Phys. **36**, 1812 (1995).
[20] M. Taibleson, *Fourier Analysis on Local Fields* (Princeton University Press, Princeton, NJ, 1976).