Hub synchronization in scale-free networks

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Heterogeneity in the degree distribution is known to suppress global synchronization in complex networks of symmetrically coupled oscillators. Scale-free networks display a great deal of heterogeneity, containing a few nodes, termed hubs, that are highly connected, while most nodes receive only a few connections. Here, we show that a group of synchronized nodes may appear in scale-free networks: hubs undergo a transition to synchronization while the other nodes remain unsynchronized. This general phenomenon can occur even in the absence of global synchronization. Our results suggest that scale-free networks may have evolved to complement various levels of synchronization.

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The last decade has witnessed a tremendous growth of interest in various kinds of collective dynamics in networks with complex structures, ranging from physical, biological to social and engineering systems [1–10]. Real-world complex systems have been modeled as networks of interacting nodes. Synchronized activities have a major impact on the network with important fitness consequences to all nodes and network functioning. The network structure exerts dramatic influence on its synchronization properties [3–5].

Recent studies reveal that disparate real-world networked systems share important structural features such as the scale-free property [11,12]. Scale-free networks are characterized by a high level of heterogeneity in the node's degree—the number of connections of a node. Such networks contain a few high-degree nodes, termed hubs, while most nodes receive only a few connections. The hubs serve specific purposes within their networks, such as regulating the information flow and providing resilience during attacks. They severely affect the dynamical processes taking place over scale-free networks, particularly the emergence of global synchronized motion [3–5].

Heterogeneity in the degree distribution may lead to a hierarchical transition toward global synchronization, with hubs synchronizing first, followed by the low-degree nodes [6]. In large scale-free networks, however, the heterogeneity inhibits global synchronization [5]. This turns out to be a desirable property, since in most real-world networks where synchronization is relevant, global synchronization can be related to pathological activities, such as epileptic seizures [7] and Parkinson disease [8] in neural networks. The study of collective behavior apart from global synchronization is thus of substantial interest.

In this paper, we show a general cluster synchronization in scale-free networks—only the hubs undergo a transition to synchronization even in the absence of global synchronization. Interestingly, the very heterogeneity that may prevent global synchronization is the primary ingredient of hub synchronization. We provide conditions for the onset of hub synchronization and determine the persistence under small perturbations. One direct consequence of our theoretical analysis is that hub synchronization is both dynamically and structurally stable, thus, allowing the network to function in a flexible and robust way.

Our approach is to introduce nonlinear dynamics on each node and then perform stability analysis to determine when the hubs synchronize. From the point the view of stability, reasonable arguments show that the network dynamics acts as a small noiselike coupling. Hence, the linear stability of the synchronized hubs is maintained. Later on, in the large size limit, we provide a rigorous treatment on the linear stability problem. Our analysis is based on the new results of the theory differential equations and spectral graph theory.

We consider a network compose of n nodes, and label the nodes according to their degrees $k_1 \le k_2 \le \cdots \le k_n$, where k_1 and k_n denote the minimal and maximal node degree, respectively. Hence, the ith node has degree k_i. A scale-free network is characterized by the degree distribution $P(k)$, the probability that a randomly chosen node within the network has degree k, that follows a power-law $P(k)=ck^{-\gamma}$, for $k_1 \le k_i \le k_n$, where c is the normalization factor. The degree distribution is normalizable for $\gamma > 1$, and for large k_n we have $c \approx (\gamma-1)k_1^{\gamma-1}$. The mean degree $\langle k \rangle$ attains a finite limit for large k_n provided $\gamma > 2$. We consider only connected networks with well defined mean degree, that is, $\gamma > 2$.

The dynamics of a general network of n identically coupled elements is described by

$$\dot{x}_i = F(x_i) + \frac{\alpha}{k_n}\sum_{j=1}^{n} A_{ij}[E(x_j) - E(x_i)], \tag{1}$$

here $x_i \in \mathbb{R}^m$ is the m-dimensional vector describing the state of the ith node (node with degree k_i), $F:\mathbb{R}^m \to \mathbb{R}^m$ governs the dynamics of the individual oscillator and is assumed to be smooth, $E:\mathbb{R}^m \to \mathbb{R}^m$ is the coupling function (without loss of generality assumed to be a constant matrix), α is the normalized overall coupling strength [13], and A is the adjacency matrix. A encodes the topological information of the network, defined as $A_{ij}=1$ if nodes i and j are connected and $A_{ij}=0$ otherwise. Note that A is symmetric, and by definition $k_i = \sum_j A_{ij}$.

We wish to show that a group of oscillators having nearly the same number of connections as the main hub may display a synchronized motion. Consider $\xi_i = x_n - x_i$, thus, synchronization is possible between the nodes i and n if $\xi_i \to 0$. Stability of this synchronized state is determined by analyzing the variational equations governing the perturbations, which read

$$\dot{\xi}_i = K_i(t;\alpha)\xi_i + \alpha\eta_i, \tag{2}$$

where the matrix $K_i(t;\alpha)=\{DF[x_n(t)]-\alpha\mu_i E\}$ depends continuously on t, DF stands for the Jacobian matrix of F, $\mu_i = k_i/k_n$ is the normalized degree, and

$$\eta_i = \frac{1}{k_n}\sum_j (A_{ij}-A_{nj})E(\xi_j)$$

is the coupling term.

Neglecting the coupling term η_i the equations governing the evolution of the perturbations ξ_i and are decoupled from the other perturbations and read

$$\dot{\xi}_i = [DF(x_n(t)) + \alpha\mu_i E]\xi_i. \tag{3}$$

We now assume that Eq. (3) is Lyapunov regular and that its fundamental matrix is integrally separated [14]. The stability of the zero solution of Eq. (3) is determined by its largest Lyapunov exponent $\Lambda(\alpha\mu_i)$, which can be regarded as the *master stability function* of the system [3,4]. The perturbation ξ_i is damped out if $\Lambda(\alpha\mu_i)<0$.

For many widely studied oscillatory systems the master stability function $\Lambda(\alpha\mu_i)$ is negative in an interval $\alpha_1 < \alpha\mu_i < \alpha_2$ for general coupling function E [3,4]. The perturbation ξ_i is damped out if $\alpha_1 < \alpha\mu_i < \alpha_2$. Moreover, normalization imposes $\mu_n=1$ and $\mu_1 \propto k_n^{-1}$, hence, as k_n increases, μ_1 converges to zero. Not only μ_1, but most of the normalized degrees μ_i will converge to zero. Therefore, it will be impossible, for large k_n, to have $\alpha_1 < \alpha\mu_i < \alpha_2$ for all $i = 1,2,\ldots,n$. Hence, in the thermodynamic limit no stable global synchronization is possible in scale-free networks.

Now take α in the stability region. Then, the state $x_n = x_{n-1}$ is linearly stable. This is true as long as we can neglect the coupling term η_i. Under the effect of η_i local mean field arguments show that $x_n \approx x_{n-1}$ is stable. The argument goes as follows. If $\Lambda(\alpha\mu_{n-1})<0$, we guarantee the linear stability of ξ_{n-1}. Moreover, if the remaining oscillators are not synchronized, the coupling term η_{n-1} can be viewed as a small coupling noise, as long as the signals x_i are uncorrelated, with α fixed and k_n large [16]. Results from ordinary differential equations state that the linear stability is maintained under small perturbations [15,17]. Therefore, if at $t=0$ we have $x_n(0)-x_{n-1}(0)\approx 0$, then for all $t \geq 0$ it yields $x_n(t) - x_{n-1}(t) \approx 0$.

These arguments cannot be applied to low-degree nodes. The reason is that to set the low-degree nodes into the stability region we must have $\alpha\mu_1 \approx \alpha_1$, requiring α to be as large as k_n. Hence, the coupling term $\alpha\eta_i$ cannot be made small for low-degree nodes.

The mean field arguments also hold for *correlated scale-free networks*. The node correlation does not play a major role to the onset of hub synchronization. For instance, the Barabási-Albert (BA) scale-free model is known to present finite size node correlation, hubs are likely connected [12]. If we rewire the connections between the hubs, connecting the hubs with the low-degree nodes, the mean field argument is still valid, that is, hub synchronization still takes place.

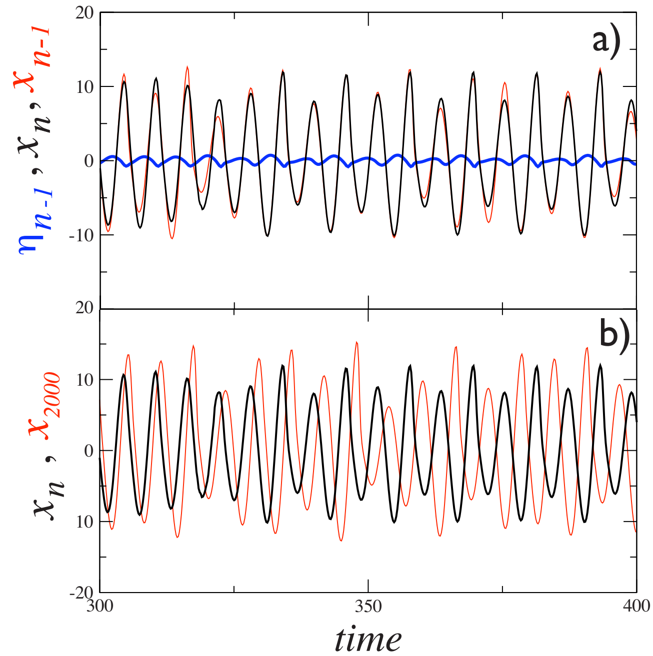

FIG. 1. (Color online) Hub synchronization in a BA scale-free network of 3000 coupled Rössler oscillators with coupling parameter $\alpha=0.3$. (a) Time series of the largest hub x_n (full line) and the second largest x_{n-1} (light gray line). The coupling term η_{n-1} (bold line) spoiling the stability of the hub synchronization is small as predicted by the local mean field arguments. (b) Time series of the largest hub x_n (full line) and of a low-degree node x_{2000} (light gray line) The corresponding node degrees are $k_n=k_{n-1}=165$ and $k_{2000}=3$.

We illustrate this phenomenon with numerical experiments. We generate a Barabási-Albert (BA) scale-free network with 3×10^3 nodes and $m=3$ [12]. The network has largest degrees $k_n=k_{n-1}=165$. Each node x_i is modeled as a Rössler oscillator, for $x_i=(x_{1i},x_{2i},x_{3i})^T$ we have $F(x_i)=(x_{2i} - x_{3i}, x_{1i}+0.2x_{2i}, 0.2+x_{3i}(x_{1i}-7))^T$. We consider E to be a projector in the first component, i.e., $E(x,y,z)^T=(x,0,0)^T$. The master stability function $\Lambda(\alpha)$ has a stability region for $\alpha\in(\alpha_1,\alpha_2)$ with $\alpha_1\approx 0.13$ and $\alpha_2\approx 4.55$. Global synchronization in this network is impossible [18].

For $\alpha=0.30$ we have observed the hub synchronization $x_n \approx x_{n-1}$. In Fig. 1(a) the time series x_n is depicted in full line while x_{n-1} is depicted in light gray line and η_{n-1} in bold line. Figure 1(a) shows that the local mean field approximation on η_{n-1} indeed holds, as shown in the times series $x_{n-1}\approx x_n$. In Fig. 1(b) $-x_n$ is depicted in bold line while x_{2000} in full line. Clearly $\xi_{n-1}\approx 0$ whereas ξ_{2000} presents large fluctuations.

All this reasoning can be set into a rigorous frame in the thermodynamic limit, for uncorrelated scale-free networks. To tackle the problem let us introduce $\zeta_i(t)=x_i(t)-s(t)$, where $s(t)$ is a given typical trajectory of $\dot{x}=F(x)$. Consider $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\ldots,\zeta_n)^T$ and $\boldsymbol{\mu}=diag(\mu_1,\mu_2,\ldots,\mu_n)$. Hence, $\boldsymbol{\zeta}\in\mathbb{R}^{mn}$. The variational equations of the perturbations $\boldsymbol{\zeta}$ can be written in a convenient block form

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\Omega}(t;\alpha)\boldsymbol{\zeta} + \alpha\boldsymbol{B}\boldsymbol{\zeta} \tag{4}$$

where $\boldsymbol{\Omega}(t;\alpha)=I_n\otimes DF[s(t)]-\alpha\boldsymbol{\mu}\otimes E$, with $\otimes$ standing for the Kronecker product, and $\boldsymbol{B}=k_n^{-1}A\otimes E$ is the coupling among the variational equations. We shall demonstrate that for large scale-free network with $\gamma > 2$, the term coupling term can be made arbitrarily small.

According to the aforementioned arguments $\mathbf{\Omega}(t;\alpha)$ splits into independent blocks as in Eq. (3). By choosing a fixed α such that nodes with degree larger than $k_{n-\ell}$ have their perturbations damped out, we guarantee that ℓ nodes display a synchronous behavior with the main hub x_n. In other words, $\mathbb{R}^{nm} = U \oplus S$, where U and S respectively the unstable and stable spaces, clearly $\dim(U) = (n-\ell)m$ and $\dim(S) = \ell m$. Notice that on the subspace S all Lyapunov exponents are negative.

It remains to show that the coupling term can be made as small as one wishes whenever k_n is large enough. Thus, results of qualitative theory of ordinary differential equations guarantee that the linear stability is not affected by small continuous perturbations [19].

By our hypothesis on the symmetry of the matrix A the spectral theorem guarantees that

$$A = NJN^{-1}.$$

where N is an orthogonal matrix and $J = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the matrix of the eigenvalues of A ordered according to their magnitudes $\lambda_1 \leq \lambda_2 \leq \cdots < \lambda_n$.

We endow the vector space $\mathbb{R}^{mn}$ with the norm $\|\cdot\|_*$ such that for $u \in \mathbb{R}^{mn}$ we have $\|u\|_* = \|N \otimes I_m u\|_\infty$, where $\|u\|_\infty = \sup_i |u_i|$ for $i = 1, 2, \cdots, nm$. We also make use of the induced matrix norms. Now we claim that given $\delta > 0$ there exists K such that for all $k_n > K$ we have

$$\|\boldsymbol{B}\|_* < \delta.$$

Indeed, by using the induced matrix norm we can obtain bounds in terms of the largest eigenvalue of A. We postpone the technical details and go directly to the result which reads $\|A \otimes E\|_* \leq \lambda_n \|E\|_\infty$.

Under mild conditions [20] the largest eigenvalue of a scale-free network scales almost surely as $\lambda_n = k_n^\beta$, where β depends on γ. We have two distinct cases: (i) $\beta = 3 - \gamma$ for $2 < \gamma < 2.5$; and (ii) $\beta = 1/2$ for $\gamma > 2.5$. Putting all estimates together yields

$$\|\boldsymbol{B}\|_* \propto \frac{1}{k_n^{1-\beta}}. \qquad (5)$$

Hence, for k_n large enough our claim follows.

This analysis is grounded on the fact that $\lambda_n / k_n \to 0$. This is also the case for *correlated scale-free networks* [10], whenever the correlations preserve the scale-free character. These moderate correlations are immaterial for hub synchronization, as finite size correlation in the BA scale-free model.

In summary, we analyzed a general phenomenon in the synchronization of large scale-free networks, namely, the synchronization of hubs even when the entire network is out of synchrony. Our theoretical analysis provides insights into further generalizations for the master stability function. The stability analysis of the synchronous hubs can be tailored to the master stability function and the coupling term due to the underlying network dynamics. We have shown that for large scale-free networks the coupling term can be controlled, effectively acting as a small noise-like perturbation on the hubs.

Hub synchronization has counterintuitive effects. For example, the hubs do not need to be directly connected to synchronize. Remarkably, when the hubs synchronize, the low-degree nodes are out of synchrony; these nodes, however, are responsible for mediating the exchange of information between the hubs. This seems to challenge our understanding of the role of synchronization in the exchange of information within complex networks [9].

We believe that our findings provide strong evidence that incomplete, hub-driven, synchronization may be at least as important and persistent in real-world networks as other forms of synchronization and collective behaviors previously examined in the literature.

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[13] The choice of normalized coupling is immaterial. The choice does not play any role in the analysis. Note that we use the same normalization for all nodes, so we could have written $\sigma = \alpha / k_n$, as is usually done in the literature.

[14] In our context these are natural assumptions. Lyapunov regularity basically assures that the Lyapunov exponents exist. The integral separation is a generic property in the space of continuous bounded matrix valued functions. See [15] for a detailed discussion.

[15] *Collected Lectures on the Preservation of Stability under Discretization*, edited by D. J. Estep and S. Tavener (SIAM, Philadelphia, PA, 2002).

[16] Without attempt at rigor, the local mean field argument is the following. First remember that $\eta_{n-1} = k_n^{-1} \Sigma_j (A_{(n-1)j} - A_{nj}) E(\xi_j)$, since the oscillators are chaotic and unsynchronized (at least for small values of α) once can think of ξ_j as identically distributed random numbers. For $k_n \gg 1$, by the center limit theorem $\eta_i = O(k_n^{-1/2})$.

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[18] The stability of global synchronization is formulated in terms of the spectrum of Laplacian matrix. Let L be the Laplacian matrix of the graph. The spectrum of L is real and can be ordered as $0 = \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$. Global synchronization is possible if $\gamma_n / \gamma_2 < \alpha_2 / \alpha_1$, see [3,5] for details. For this network we have, $\gamma_n / \gamma_2 \approx 180$, while $\alpha_2 / \alpha_1 \approx 35$.

[19] The unique solution of the homogeneous part of Eq. (4) can be written in terms of the principal matrix $\boldsymbol{\zeta}(t) = T(t,s) \boldsymbol{\zeta}(s)$. Moreover, under our hypotheses the operator $T(t,s)$ admits a dichotomy being exponentially stable on the subspace S [17]. Since $\|\boldsymbol{B}\|_* < \delta$, it follows that the Lyapunov exponents of the perturbed equation remain negative [15].

[20] The conditions require k_n to grow faster than some powers of $\log n$. See [21] for further details. These conditions are all natural for large scale-free graphs, almost surely, it holds $k_n \propto n^{1/(\gamma-1)}$.

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