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Fractional Brownian motion run with a nonlinear clock

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We construct a family of stochastic processes with nonstationary, correlated increments which allow *a priori* independent selections of both fractal dimension and mean-square displacement. The family is essentially fractional Brownian motion (fBm) run with a nonlinear clock (fBm-nlc). The fractal dimension of fBm-nlc is shown to be the same as that of the underlying fBm process. We also compute the p-variation and discuss the problems in using this to differentiate between diffusive processes. The fBm-nlc process illustrates that the range of anomalous diffusive processes has not been adequately explored.

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I. INTRODUCTION

The standard description of an anomalous process is one where the mean-square displacement goes asymptotically as time to some power, β, other than 1. If $0\le\beta<1$, then the process is called subdiffusive and if $\beta>1$, then it is superdiffusive. Lévy processes [1,2], that do not have a finite second moment fall into the superdiffusive category. Fractional Brownian motion (fBm) was introduced by Mandelbrot [3] and allows for β between 0 and 2 exclusive. The exponent for classical Brownian motion is $\beta=1$.

Subdiffusive and superdiffusive processes often occur in heterogeneous or preasymptotic systems. Anomalous diffusion occurs in confined nanofilms [4–6], transport in porous media [7,8], fractal structures with holes over all length scales [9], charge carrier transport in anomalous semiconductors [10,11], drifters in near-surface ocean currents [12], in the atmosphere [13,14], geologic formations [15–17], vortex arrays in rotating flows [18], layered velocity fields [19] and mRNA molecules in live *E. coli* cells [20] to name a handful of examples. An overview of continuous-time random walks (CTRWs) focused primarily on subdiffusion is given in [21].

In one dimension, a process $B_H(t)$ is said to be fBm with Hurst exponent H, $(0<H<1)$ if [22] (a) with probability 1, $B_H(t)$ is continuous and $B_H(0)=0$; (b) for any $t>s\ge 0$ the increment $B_H(t)-B_H(s)$ is normally distributed with mean zero and variance $(t-s)^{2H}$.

Note that the second condition implies the increments are stationary. The correlation structure is given by

$$E\{B_H(s)[B_H(t)-B_H(s)]\}=\frac{1}{2}[t^{2H}-s^{2H}-(t-s)^{2H}]. \quad (1)$$

It can be shown [22] that

$$\dim(graphB_H|_{[0,1]})=2-H, \quad (2)$$

where dim denotes the Hausdorff (fractal) dimension and $graphB_H|_{[a,b]}=\{(x,t):x=B_H(t),t\in[a,b]\}$.

The p-variation is a generalization of the total variation. It has been suggested [23] that the underlying diffusive process can be determined by examining the p-variation of random walks. The p-variation from 0 to T of a function H is defined to be

$$V_p(H)(T)=\lim_{|\Delta|\to 0}\sum_{i=1}^{N}|H(t_i)-H(t_{i-1})|^p, \quad (3)$$

where Δ is a partition of $[0,T]$ given by $0=t_0<t_1<\cdots<t_{N-1}<t_N=T$ and $|\Delta|=\max_{1\le i\le N}|t_i-t_{i-1}|$.

We create a process that has arbitrary but *a priori* specified fractal dimension and mean-square displacement, the latter of which need not be related to the former, and compute the p-variation for the process. We show that observations of the p-variation may eliminate some diffusive processes from consideration, but cannot conclusively determine the underlying process. A brief discussion of how fBm-nlc may be used to model diffusion in a confined nanofilm is presented.

II. FRACTIONAL BROWNIAN MOTION RUN WITH A NONLINEAR CLOCK

Let $B_H(t)$ be a fractional Brownian motion with Hurst exponent H and let $F(t)$ be an absolutely continuous function with non-negative derivative, so that $F(t)=\int_0^t f(s)ds$ for some function $f\ge 0$. Given an fBm and the function $F(t)$, the corresponding fBm run with a nonlinear clock (fBm-nlc) is given by $X(t)=B_H[F(t)]$. From (a) and (b) above, it can be shown that $X(t)$ has the following properties:

(i) With probability 1, $X(t)$ is continuous and $X(0)=0$.

(ii) For any $t>s\ge 0$, the increment $X(t)-X(s)$ is normally distributed with mean zero and variance $[F(t)-F(s)]^{2H}$.

Note that the second condition implies that the increments are nonstationary unless $F(t)$ is linear in t.

If $F(t)=t$, the original fBm is obtained. The case of $H=1/2$ corresponds to compressed/stretched Brownian motion [24]. If $F(t)=t$ and $H=1/2$ the result is a Brownian motion. The correlation structure of fBm-nlc follows from that of fBm: $E\{X(s)[X(t)-X(s)]\}=E\{B_H[F(s)]\{B_H[F(t)]$

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$-B_H[F(s)]\}\}$. Upon application of Eq. (1), we see that

$$E\{X(s)[X(t)-X(s)]\} = \frac{1}{2}\{[F(t)]^{2H} - [F(s)]^{2H} - [F(t)-F(s)]^{2H}\}. \quad (4)$$

III. FRACTAL DIMENSION

The one-dimensional fBm-nlc process can be easily extended by taking $B_H(t)$ to be a multidimensional fBm. In this case, the path of $X(t)$ as t varies from a to b is that of $B_H(t)$ as t varies from $F(a)$ to $F(b)$. That is, assuming that $B_H(t)$ is an fBm in n dimensions,

$$\begin{aligned} X([a,b]) &= \{x \in \Re^n : x = X(t), a \le t \le b\} \\ &= \{x \in \Re^n : x = B_H(F(t)), a \le t \le b\} \\ &= \{x \in \Re^n : x = B_H(t), F(a) \le t \le F(b)\} \\ &= B_H([F(a),F(b)]). \end{aligned} \quad (5)$$

Therefore, the two paths have the same fractal dimension.

In the one-dimensional case, it is easy to show the fractal dimension of the graphs of fBm and fBm-nlc are the same. For brevity, we also make the assumption that $0 < L_{[a,b]} < f(t) < M_{[a,b]} < \infty$ for $0 < a \le t \le b < \infty$. The bounds L and M may depend on the interval $[a,b]$, and they may become very large or small as $a \to 0$. This permits the important case of power law mean-square displacement as well as many other possible clocks. It should be noted however that following result holds under more general conditions.

Recall that a function, $h(\mathbf{x})$, is called bi-Lipschitz if there is a constant C such that $\frac{1}{C}\|\mathbf{x}-\mathbf{y}\| \le h(\mathbf{x}-\mathbf{y}) \le C\|\mathbf{x}-\mathbf{y}\|$. Let $\alpha(x,t)=[x,F(t)]$, and note that $\alpha(x,t)$ is bi-Lipschitz on $(-\infty,\infty)\times[\varepsilon,1]$ for any $\varepsilon>0$ by the boundedness assumption on f. Observe that $\alpha[graph(X|_{[\varepsilon,1]})] = graph(B_H|_{[F(\varepsilon),F(1)]})$. It follows from [22], Corollary 2.4(b) that $\dim[graph(X|_{[\varepsilon,1]})] = \dim[graph(B_H|_{[F(\varepsilon),F(1)]})] = 2-H$. Let $A_0=\{(0,0)\}$ and $A_n = graph(X|_{[1/n,T]})$ for $n>0$. The Hausdorff dimension has countable stability, which means that the Hausdorff dimension of a countably infinite union of sets is equal to the supremum of the dimensions of the sets in the union, $\dim[\cup_{n=0}^{\infty} A_n] = \sup_{n\ge 0}[\dim A_n]$. This implies that

$$\begin{aligned} \dim[graph(X|_{[0,1]})] &= \dim[\cup_{n=0}^{\infty} A_n] \\ &= \sup_{0\le n<\infty} \dim A_n \\ &= \sup\{0, 2-H\}. \end{aligned}$$

We conclude that

$$\dim[graph(X|_{[0,1]})] = 2 - H. \quad (6)$$

The choice of the function $F(t)$ therefore does not affect the fractal dimension of the fBm-nlc. This means that the mean-square displacement and the fractal dimension can be, but do not have to be, related for fBm-nlc.

IV. *p*-VARIATION

In [23], a test was introduced to determine whether subdiffusive dynamics originate from an fBm or a CTRW scheme. The CTRW scheme is equivalent to a very special subordinated Brownian motion. Random walks from this scheme can be represented as $B_{1/2}(S_\alpha(t))$. The function $S_\alpha(t)$ is a stochastic α-self-similar (i.e., $S_\alpha(ct)$ has the same distribution as $c^\alpha S_\alpha(t)$) step function. Further details can be found in [23], and the references therein. The test is based upon the concept of p-variation mentioned earlier. Looking only at fBm and CTRW processes creates a false dilemma because the actual range of subdiffusive processes is much larger than these two options. In fact, the p-variation of either of these processes can be approximated by an fBm-nlc with completely different dynamics. In [23] the p-variation for fBm, $B_H(t)$, and the CTRW, $Z_\alpha(t)$, are shown to be

$$V_p(B_H)(T) = \begin{cases} \infty & \text{if } p < \frac{1}{H} \\ C_H T & \text{if } p = \frac{1}{H} \\ 0 & \text{if } p > \frac{1}{H} \end{cases} \quad (7)$$

and

$$V_p(Z_\alpha)(T) = \begin{cases} \infty & \text{if } p < 2 \\ S_\alpha(T) & \text{if } p = 2 \\ 0 & \text{if } p > 2, \end{cases} \quad (8)$$

where C_H depends on H, but is independent of T, and $S_\alpha(T)$ is a stochastic step function which plays a role similar to $F(t)$ in fBm-nlc.

The p-variation for fBm-nlc, $X(t)$, is given by

$$V_p(X)(T) = \begin{cases} \infty & \text{if } p < \frac{1}{H} \\ C_H F(T) & \text{if } p = \frac{1}{H} \\ 0 & \text{if } p > \frac{1}{H}. \end{cases} \quad (9)$$

This can be seen by recalling $F(t)$ is absolutely continuous, and hence uniformly continuous. It is a direct consequence of the uniform continuity of $F(t)$ that if Δ_n is a sequence of partition refinements given by $0=t_0^n<t_1^n<\dots<t_{N_n-1}^n<t_{N_n}^n=T$ such that $|\Delta_n|\to 0$, then $|F(\Delta_n)|\to 0$ where $F(\Delta_n)$ is a sequence of partition refinements given by $0=F(t_0^n)<F(t_1^n)<\dots<F(t_{N_n-1}^n)<F(t_{N_n}^n)=F(T)$. With this in mind, we apply the definition of p-variation.

We adopt the notation $V_p^\Delta(H)(T)=\sum_{n=1}^N |H(t_i)-H(t_{i-1})|^p$ where the partition Δ is given by $0=t_0<t_1<\dots<t_{N-1}<t_N=T$. Let Δ_n be a sequence of partitions of $[0,T]$ with the same notation as above and which converges to zero in norm. It follows that

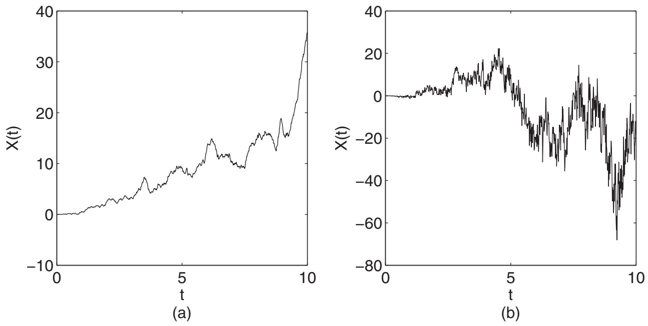

FIG. 1. (a) The graph of $X(t)$ with $H=0.75$ and $F(t)=t^2$ and (b) the graph $X(t)$ with $H=0.25$ and $F(t)=t^6$.

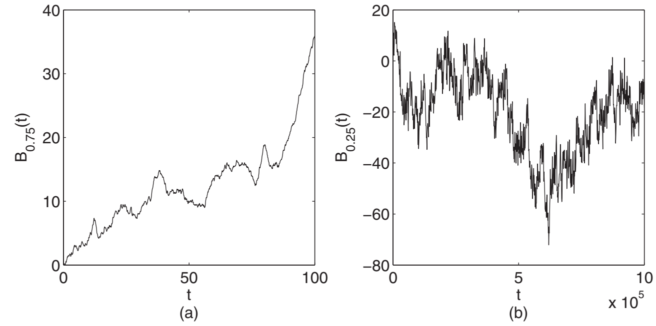

FIG. 2. (a) The graph of $B_{0.75}(t)$ on which Fig. 1(a) is based and (b) the graph of $B_{0.25}(t)$ on which Fig. 1(b) is based.

$$\begin{aligned} V_p(X)(T) &= \lim_{n\to\infty} V_p^{\Delta_n}(B_H \circ F)(T) \\ &= \lim_{n\to\infty} \sum_{i=1}^{N_n} |B_H[F(t_i^n)] - B_H[F(t_{i-1}^n)]|^p \\ &= \lim_{n\to\infty} V_p^{F(\Delta_n)}(B_H)[F(T)] \\ &= V_p(B_H)[F(T)]. \end{aligned} \tag{10}$$

Equation (9) follows immediately from this and Eq. (7).

Comparing Eqs. (6) and (8), we see that over a relatively small time interval the function $F(T)$ could appear to be linear. In such a case, the p-variation of an fBm-nlc could be mistaken for that of an fBm with the same Hurst exponent. However, over a longer time the nonlinearity will show itself. This concept meshes well with the notion that the behavior of a diffusive process depends on the scale at which it is observed.

Nonstationary fractional Brownian motion can also imitate the p-variation of the CTRW scheme simply by choosing $F(T)$ to be an approximation to $S_\alpha(T)$ and $H=1/2$. The diffusion of such a process would have long periods of slow diffusion followed by short periods of very fast diffusion.

These observations on the p-variation of fBm-nlc show that the p-variation test described in [23] cannot be used to determine the underlying dynamics, but it may be used effectively to eliminate certain processes from consideration.

V. EXAMPLES

A. Anomalous diffusion with power law mean-square displacement

Fix H and set $F(t)=t^{\beta/2H}$ with $\beta>0$, then by property (ii), $X(t)-X(0)\sim N(0,t^\beta)$. Therefore $\langle X^2(t)\rangle = t^\beta$, and the fractal dimension and power law mean-square displacement may be chosen arbitrarily. In the case where $\beta>1$, we have superdiffusion and when $\beta<1$ we have subdiffusion. When $\beta=3$ we obtain a version of Richardson superdiffusion [13]. Figure 1 presents two such possibilities; the Hausdorff dimension of the graph in Fig. 1(a) is 1.25 corresponding to $H=0.75$, and the Hausdorff dimension of the graph in Fig. 1(b) is 1.75 corresponding to $H=0.25$. Figure 2 shows the fBms from which Fig. 1 was generated. Despite the difference in complexity, these two processes have the same mean-square displacement. Generally, we may choose the Hausdorff dimension of the graph to be anything between 1 and 2 exclusive, regardless of the selection for the power law exponent of mean-square displacement.

It is interesting to note that if you choose $F(t)$ such that the mean-square displacement is proportional to t, and yet choose $H\neq 1/2$, then the process is considered classically diffusive, but would have a fractal dimension other than that of Brownian motion.

B. Logarithmic subdiffusion

In this case we fix H and set $F(t)=[\ln(t+t_0)-\ln(t_0)]^{1/2H}$ with $t_0>0$, so that $\langle X^2(t)\rangle = \ln(t+t_0)-\ln(t_0)$. The result is a very slow subdiffusion whose graph can have any Hausdorff dimension between 1 and 2 exclusive.

C. Arbitrary mean-square displacement

To achieve an arbitrary mean-square displacement, we fix H (which amounts to a choice of the fractal dimension) and set $F(t)=[\sigma^2(t)]^{1/2H}$ where $\sigma^2(t)$ is the desired mean-square displacement. The only requirements that need to be placed on $\sigma^2(t)$ are the same as those placed on $F(t)$ above. The chain rule then implies that $F(t)$ has the desired properties. This allows for a wide array of mean-square displacements, and as above, the graph can have any Hausdorff dimension between 1 and 2 exclusive.

VI. CONCLUSIONS

A process was constructed, fBm-nlc, with arbitrary but *a priori* known mean-square displacement and arbitrary fractal dimension. We achieved this by running fBm with a nonlinear clock. This allows one to choose the mean-square displacement while maintaining the fractal dimension of the original fBm. An application of this model to diffusion in laterally confined rare gas nanofilms is presented in [25]. The most important facts for the discussion here is that on relatively short time scales subdiffusion occurs, but over longer periods of time the diffusion becomes Brownian [25]. The variable power law mean-square displacement and the convergence to Brownian imply that fBm-nlc with $F(t)$ chosen to fit the mean-square displacement and $H=1/2$ would serve as a reasonable model.

The p-variation of fBm-nlc can mimic the p-variation of other processes. Therefore, a study of a process' p-variation cannot determine the underlying stochastic process. Matching a physical phenomenon with a mathematical model of diffusion cannot be achieved by examining a single property. The process presented here demonstrates that there may be other models that share the same property but differ in other ways. When matching physical phenomena with models it would be prudent to consider as many diffusive models as possible and choose a model which matches data in as many ways as possible. fBm-nlc can match both the mean-square displacement and complexity (fractal dimension) of a diffusive phenomenon.

It is apparent from the recent literature [23] that there is an implicit assumption that only a few anomalous stochastic models exist. The erroneous nature of this assumption is illustrated by the introduction of a novel process like fBm-nlc. The possibilities are endless and certainly not limited to fBm, continuous-time random walks or Levy motions.

[1] G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance* (Chapman and Hall, New York, 1994).

[2] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Studies in Advanced Mathematics (Cambridge University Press, Cambridge, England, 1999), Vol. 68.

[3] B. B. Mandelbrot and J. W. Van Ness, SIAM Rev. **10**, 422 (1968).

[4] M. Schoen *et al.*, Mol. Phys. **81**, 475 (1994).

[5] J. H. Cushman, Nature (London) **347**, 227 (1990).

[6] D. J. Diestler *et al.*, J. Chem. Phys. **100**, 9140 (1994).

[7] J. H. Cushman, *The Physics of Fluids in Hierarchical Porous Media: Angstroms to Miles* (Kluwer Academic, Boston, 1997).

[8] B. Berkowitz *et al.*, Rev. Geophys. **44**, RG2003 (2006).

[9] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).

[10] H. Scher and M. Lax, Phys. Rev. B **7**, 4491 (1973).

[11] Q. Gu, E. A. Schiff, S. Grebner, F. Wang, and R. Schwarz, Phys. Rev. Lett. **76**, 3196 (1996).

[12] J. H. LaCasce and C. Ohlmann, J. Mar. Res. **61**, 285 (2003).

[13] L. F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).

[14] J. H. Cushman *et al.*, Geophys. Res. Lett. **32**, L19816 (2005).

[15] J. H. Cushman *et al.*, J. Stat. Phys. **75**, 859 (1994).

[16] F. W. Deng and J. H. Cushman, Water Resour. Res. **31**, 1659 (1995).

[17] M. M. Meerschaert, D. A. Benson, and B. Baumer, Phys. Rev. E **59**, 5026 (1999).

[18] E. R. Weeks and H. L. Swinney, Phys. Rev. E **57**, 4915 (1998).

[19] G. Zumofen *et al.*, J. Stat. Phys. **65**, 991 (1991).

[20] I. Golding and E. C. Cox, Phys. Rev. Lett. **96**, 098102 (2006).

[21] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).

[22] K. Falconer, *Fractal Geometry* (Wiley, Chichester, 2003).

[23] M. Magdziarz, A. Weron, K. Burnecki, and J. Klafter, Phys. Rev. Lett. **103**, 180602 (2009).

[24] J. H. Cushman, D. O'Malley, and M. Park, Phys. Rev. E **79**, 032101 (2009).

[25] J. H. Cushman and M. Park, Phys. Rev. E (to be published).