

Pulse propagation in a chain of o-rings with and without precompression

Italo'Ivo Lima Dias Pinto and Alexandre Rosas*

Departamento de Física, CCEN, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-900 João Pessoa, PB, Brazil

Aldo H. Romero

Cinvestav-Querétaro, Libramiento Norponiente 200, 76230, Fracc. Real de Juriquilla, Querétaro, Querétaro, Mexico

Katja Lindenberg

Department of Chemistry and Biochemistry and BioCircuits Institute, University of California–San Diego, La Jolla, California 92093-0340, USA

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We implement a binary collision approximation to study pulse propagation in a chain of o-rings. In particular, we arrive at analytic results from which the pulse velocity is obtained by simple quadrature. The predicted pulse velocity is compared to the velocity obtained from the far more resource-intensive numerical integration of the equations of motion. We study chains without precompression, chains precompressed by a constant force at the chain ends (constant precompression), and chains precompressed by gravity (variable precompression). The application of the binary collision approximation to precompressed chains provides an important generalization of a successful theory that had up to this point only been implemented to chains without precompression, that is, to chains in a sonic vacuum.

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I. INTRODUCTION

The dynamics of pulse propagation in granular chains continues to be of keen interest to physicists and engineers both because of the theoretical challenges that it poses, and because of the prospect of related technological applications. Among other applications, it has been argued that understanding of pulse propagation in granular media may lead to the development of new shock absorbers and of instruments for detection of buried objects [1]. Most recently, a new nonlinear acoustic lens consisting of ordered arrays of granular chains has been shown to be capable of generating high-energy acoustic pulses and may point the way to improved capability in a variety of devices [2]. While these applications involve propagation in higher dimensional granular realms, a thorough understanding of the dynamics in one dimension is often pursued as a way to lay the groundwork for the problem, or even because the systems are actually composed of arrays of granular chains. Theoretically, a granular chain appears simple, and yet its discrete nonlinear character immediately imposes severe difficulties on the development of analytic methods to explain and predict the behavior of energy pulses in even these simplest granular environments. As a result, the available information is obtained mostly from numerical [3–8] and occasionally from experimental [9,10] results. The numerical approaches are notoriously resource intensive, and the experiments are constrained by the particular experimental setup.

The numerical and experimental results have been supplemented with some analytic results that have proven extremely helpful and accurate. The accuracy is particularly interesting in view of the fact that the two most successful

theories approach the problem from two essentially orthogonal points of view, both of which have provided excellent results. Historically, the first was based on a long-wavelength approximation [11,12] that assumes that a chain can be treated as a continuum augmented with first-order corrections due to the discreteness of the system. This approach was applied to granular chains in which the granules just touch, so that there are no intergranular gaps but also no initial precompression. The long-wavelength solution to this problem brought to light the existence of solitary waves in granular chains. This purely nonlinear scenario was dubbed a “sonic vacuum” because it supports no sound waves. The continuum approach has been applied in the absence [11–13] as well as in the presence of dissipation [14]. The success of the continuum approach is particularly noteworthy because it presumes a pulse width that is large compared to the size of the granules, and yet the observed pulse as well as the solitary wave solution that emerges from this approach extends over only a few granules. Further merits of this approximation include the fact that it allows an estimation of omitted terms, it subsumes the weakly nonlinear and even linear cases, and it in fact successfully predicts not only solitary waves but also periodic waves and shock waves. By its very nature, the continuum methodology is most easily applicable to monodisperse or nearly monodisperse chains. While it has been applied to a variety of chains with different granular configurations and initial conditions [13], the mathematics becomes quite cumbersome.

A second alternative approach developed more recently is particularly useful when the continuum approximation becomes more difficult to implement. It is based on a binary collision approximation that presumes that intergranular collisions involve only two granules at a time. This extremely-short-wavelength approach has in fact been shown to yield results that are even closer to numerical simulation results for the “canonical” case of a chain of spherical granules with

*arosas@fisica.ufpb.br

initially no gaps and no precompression [15]. Not only does this method yield highly accurate results for that case, but it has been successfully applied to a number of other configurations that are not easily accessible to the continuum approach, namely, to tapered [16], decorated [17], and randomly decorated chains [18] and to one-dimensional granular gases [19]. However, it has not yet been applied to any kind of precompressed chain.

In this contribution, we study pulse propagation in a granular chain of toroidal rings (o-rings) placed between rigid cylinders that act as nonlinear springs. For such granules, the purely repulsive force characteristic of dry granular materials is modified by the topological properties of the o-rings, leading to a hard potential proportional to the seventh power of the compression in addition to the usual softer Hertz potential proportional to the compression to power 5/2. We are directed to this particular system because it has been studied both numerically and experimentally [5]. Here, we make use of the binary collision approximation to study the pulse velocity in such chains of o-rings, extending the approximation not only to a new geometry but also beyond the case of a sonic vacuum to precompressed chains. The extension of the binary collision approximation to precompressed chains provides an important generalization of a successful theory that had up to this point only been implemented for chains without precompression.

The paper is organized as follows. In Sec. II, we present the equations of motion for the granular chain of o-rings and we describe the binary collision approximation. Next, we compare the results of this approach with those of the numerical integration of the equations of motion in Sec. III. Finally, in Sec. IV, we briefly summarize our results.

II. MODEL

Consider a chain of N equal o-rings, hereafter also called the “granules.” The double power-law character of the elastic interaction of the o-rings leads to the following equation of motion for granule k in the interior of the chain,

$$\frac{dy_k^2}{d\tau^2} = A[(y_{k-1} - y_k)^{3/2} - (y_k - y_{k+1})^{3/2}] + B[(y_{k-1} - y_k)^6 - (y_k - y_{k+1})^6] + F_k/m, \quad (1)$$

where y_k is the displacement of the k th granule at time τ from its equilibrium position. Granule k is subject to the external force F_k , and m is the mass of the cylinders separating the o-rings (which is much larger than the mass of the o-rings). The constants $A = 1.25\pi DE/md^{1/2}$ and $B = 50\pi DE/md^5$ are constants that characterize the elastic properties of the material [5], where d and D are, respectively, the cross-section and mean diameter of the o-ring, and E is the Young’s modulus of the o-rings. For comparison with experiments, we present the values for these constants for the teflon o-rings used in [5]: $E = 1.46$ GPa, $D = 7.12$ mm, $d = 1.76$ mm, and $m = 3.276$ g.

The granules at the ends of the chain must be considered separately. While the leftmost granule does not have any granule pushing it to the right, a constant force F_1 may be

applied in this direction in the case of precompression. Therefore, its equation of motion reads

$$\frac{dy_1^2}{d\tau^2} = F_1/m - A(y_1 - y_2)^{3/2} - B(y_1 - y_2)^6. \quad (2)$$

Similarly, the rightmost granule moves according to

$$\frac{dy_N^2}{d\tau^2} = -F_N/m + A(y_{N-1} - y_N)^{3/2} + B(y_{N-1} - y_N)^6. \quad (3)$$

Without precompression, $F_k = 0$ for all granules, while for a constant precompression, $F_1 = -F_N = F$, and $F_k = 0$ for all other granules. In the case of a vertical chain subject to gravity with labels running downward, $F_k = kmg$ for $k = 1, 2, \dots, N-1$ and $F_N = -Nmg$, g being the acceleration due to gravity. Since we are interested in pulse propagation, initially all the granules are at rest except for the leftmost or top granule, which has an initial velocity v_0 . In the case of precompression, either by an external constant driving force or by gravity, the initial positions of the granules are the equilibrium positions. For a chain in a sonic vacuum (that is, without precompression), initially the granules just touch each other.

We proceed by defining the scaled variables [20]

$$y_k = \left(\frac{v_0^2}{A}\right)^{2/5} x_k, \quad \tau = \frac{1}{v_0} \left(\frac{v_0^2}{A}\right)^{2/5} t, \quad (4)$$

in terms of which the equations of motion are written as

$$\begin{aligned} \ddot{x}_1 &= f_1 - (x_1 - x_2)^{3/2} - b(x_1 - x_2)^6, \\ \ddot{x}_k &= f_k + [(x_{k-1} - x_k)^{3/2} - (x_k - x_{k+1})^{3/2}] \\ &\quad + b[(x_{k-1} - x_k)^6 - (x_k - x_{k+1})^6], \\ \ddot{x}_N &= -f_N + (x_{N-1} - x_N)^{3/2} + b(x_{N-1} - x_N)^6, \end{aligned} \quad (5)$$

where a dot denotes a derivative with respect to t and where we have defined

$$b = \frac{B}{A} \left(\frac{v_0^2}{A}\right)^{9/5}, \quad f_k = \frac{F_k}{mv_0^2} \left(\frac{v_0^2}{A}\right)^{2/5}. \quad (6)$$

Consequently, the parameter b measures the relative strength of the two power-law terms vs the initial velocity, and f_k plays the role of the external force.

Binary collision approximation

The binary collision approximation is based on the assumption that the pulse propagates through the chain by a sequence of binary collisions. Since this is not exactly the case, we must also specify the moment of passage of the pulse from one granule to the next. We say that the pulse moves from granule k to granule $k+1$ when the velocity of the latter surpasses the velocity of the former. With this approximation, instead of having to take into account all the equations of motion at once, we can focus on the interaction of just two granules at a time. Since the granules in the original chain are initially in equilibrium, so are the two

granules in the binary collision approximation. Consequently, the equations of motion of these two particles during the collision may be written as

$$\ddot{x}_k = f - (x_k - x_{k+1})^{3/2} - b(x_k - x_{k+1})^6, \quad (7)$$

$$\ddot{x}_{k+1} = -f + (x_k - x_{k+1})^{3/2} + b(x_k - x_{k+1})^6, \quad (8)$$

where f is the force causing the compression of the two granules at the beginning of the collision. In the case of a sonic vacuum, this force is zero. When a constant external compression force is applied at the ends of the chain, the force is positive and f equals this external force for each pair of granules. In the presence of a gravitational force it equals $k\mathcal{G}$, the scaled gravitational constant \mathcal{G} being related to the unscaled gravitational constant g by

$$\mathcal{G} = \frac{g}{v_0^2} \left(\frac{v_0^2}{A} \right)^{2/5}. \quad (9)$$

We define $z = x_k - x_{k+1}$ and, for simplicity of notation, for now we drop the k subscripts on z and f . Subtracting the two equations, we have

$$\ddot{z} = 2f - 2z^{3/2} - 2bz^6, \quad (10)$$

which describes the motion of a fictitious particle of unit mass whose displacement is z and which is moving in the potential

$$V(z) = -2fz + \frac{4}{5}z^{5/2} + \frac{2}{7}bz^7. \quad (11)$$

Next we make use of conservation of energy to write

$$\frac{1}{2}\dot{z}^2 + V(z) = \frac{1}{2}\dot{z}_0^2 + V(z_0), \quad (12)$$

where $\dot{z}_0=1$ is the velocity of the incoming granule at the beginning of the collision, and z_0 is obtained from the equilibrium condition

$$f = z_0^{3/2} + bz_0^6. \quad (13)$$

As discussed above, the pulse is said to reside on granule k until the velocities of the two granules become equal. At that moment $\dot{z}=0$, the pulse is said to move onto the next granule, and the compression is maximum. Hence, the maximum compression z_m is obtained from the energy conservation condition Eq. (12) as the solution of

$$\frac{8}{5}(z_m^{5/2} - z_0^{5/2}) + \frac{4}{7}b(z_m^7 - z_0^7) - 4f(z_m - z_0) = 1. \quad (14)$$

Once we know the initial and maximum compression, the residence time, i.e., the time spent by the pulse on a given granule, may be obtained as

$$T_k = \int_{z_0,k}^{z_{m,k}} \frac{dz}{\dot{z}} = \int_{z_0,k}^{z_{m,k}} \frac{dz}{\sqrt{1 + V(z_0,k) - V(z)}}, \quad (15)$$

where we have made use of Eq. (12) and reinstated the subscript k .

In summary, in order to calculate the pulse velocity as it passes through granule k , we need to find the solutions z_0 and

z_m of Eqs. (13) and (14), respectively, and then we must numerically integrate Eq. (15) (since analytic integration appears impossible) to find the residence time. The pulse velocity is the inverse of the residence time,

$$c_k = 1/T_k. \quad (16)$$

III. NUMERICAL RESULTS

In this section, we compare the pulse velocity predicted by the binary collision approximation, Eq. (16), with the results of the numerical integration of the equations of motion, Eqs. (5). We consider three cases: chains without precompression, chains with precompression caused by a constant force at the edges of the chain, and chains with precompression caused by gravity.

A. Chains without precompression

This is the simplest case. In the absence of precompression, z_0 and f vanish. The k -independent potential then becomes

$$V(z) = \frac{4}{5}z^{5/2} + \frac{2}{7}bz^7, \quad (17)$$

and the equation for the maximum compression is simplified to

$$\frac{8}{5}z_m^{5/2} + \frac{4}{7}bz_m^7 = 1. \quad (18)$$

These simplifications lead to the residence time

$$T = \int_0^{z_m} \frac{dz}{\sqrt{1 - \frac{8}{5}z^{5/2} - \frac{4}{7}bz^7}}. \quad (19)$$

In the limit of small and large b , one or the other of the two terms in the potential may be neglected. In these limits z_m can be calculated exactly from Eq. (18). Furthermore, the integral [Eq. (19)] can then be performed exactly [15], leading to the low- b pulse velocity

$$c = \frac{2^{1/5}5^{3/5}\Gamma(9/10)}{\sqrt{\pi}\Gamma(2/5)} \simeq 0.820, \quad (20)$$

and the large- b velocity

$$c = \frac{2^{2/7}7^{6/7}b^{1/7}\Gamma(9/14)}{\sqrt{\pi}\Gamma(1/7)} \simeq 0.779b^{1/7}. \quad (21)$$

In Fig. 1 we show the pulse velocity as a function of the parameter b . This parameter measures the relative weight of the two terms in the granular interaction. For small values of b , the interaction is almost Hertzian while for large values the z^6 force is dominant. This behavior is clearly illustrated by the solid and dashed lines, which are the plots of Eqs. (20) and (21), respectively. The excellent agreement of the prediction of the binary collision approximation is evident over the entire range of values of b . In the inset we show the

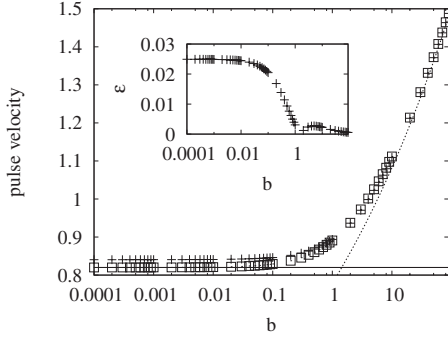


FIG. 1. Pulse velocity as a function of the parameter b in the absence of precompression. In the main figure, the plus signs refer to the numerical simulation results and the open squares to the binary collision approximation. The solid line is the plot of Eq. (20) and the dashed line is the plot of Eq. (21). The inset shows the relative error ε .

relative error $\varepsilon = |(c_b - c_n)/c_b|$ between the pulse velocity predicted by the binary collision approximation, which we call c_b , and the value obtained from the numerical integration of the equations of motion, which we call c_n . The plot shows that ε is always smaller than 3%. Furthermore, as b increases the error decreases because the harder potential becomes more important and the associated pulse is narrower. Consequently the binary collision approximation turns out to be increasingly more precise, that is, the idea that only two granules participate in each collision becomes increasingly more correct [15].

We end our discussion of uncompressed chains with a comparison of our results with the experiments of Herbold and Nesterenko [5]. In their work, the pulse was generated by the impact of a stainless steel sphere (not part of the chain) of 0.455 g of mass with velocities ranging from 1 to 800 m/s. In our model the pulse is generated by the first granule of the chain, whose mass is 3.276 g in their experiment. We adjust our initial momentum transferred to the chain to theirs by choosing the parameter b appropriately, in the range $[10^{-10}, 5]$. At the lower limit the dominant potential term is the Hertzian, and the pulse velocity is $c=0.82$ (see Fig. 1), which translates to a pulse velocity in physical units of around 250 m/s. The difference between our results and the experimental ones in this regime is about 172 m/s and is primarily due to the difference in the impulse generating method. However, it is reassuring that our results are within a factor of 2 of the experimental results. Experimental results were not shown for the large b limit so we are not able to compare with our theory. Our binary collision approximation predicts that the pulse velocity in this regime should be around 1100 m/s.

B. Constant precompression

When we apply a constant force at the ends of the chain, thus pressing the granules together, the precompression of any pair of granules is the same, $z_0 = \delta$, as is the force f on each granule. Therefore, the pulse again travels with a constant (k -independent) velocity along the chain. Figure 2 shows that this constant pulse speed increases as the precom-

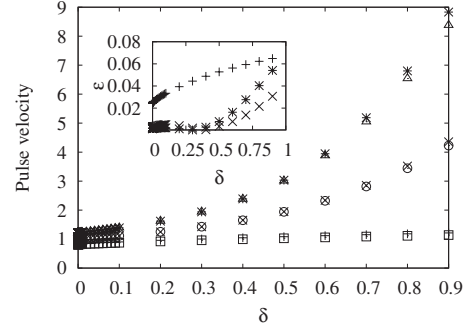


FIG. 2. Pulse velocity as a function of the compression. In the main figure, the plus signs (open squares), crosses (open circles), and stars (triangles) refer to the numerical simulations (binary collision approximation) for $b=10^{-5}$, 4, and 20, respectively. The inset shows the relative error ε for the same values of b .

pression (or, alternatively, the force) increases. The inset shows the relative error, which increases with increasing precompression. This error increase is a reflection of the fact that as precompression increases, more than two granules become actively engaged in any collision event. The force caused by granule $k+2$ on granule $k+1$ for a power-law potential proportional to z^n is $(n-1)(\delta_0/\delta_z)$, where δ_0 is the precompression between granules $k+1$ and $k+2$ and δ_z is the additional compression between them caused by the traveling pulse. Hence, for large δ_0 , this interaction is enhanced and the binary collision approximation is less effective. The inset also shows that for larger values of b , the relative error ε is initially smaller, as discussed in the previous section, but as $\delta = \delta_0 + \delta_z$ increases, the increase in the relative error is more prominent. Further, the increase in ε is even faster for $b=20$ than for $b=4$. Nevertheless, the relative error stays below 7% even for large precompressions, so our approximation is still useful.

Finally, we point out that the values we have chosen for the precompression (up to 0.9) correspond at most ($b=20$) to 0.9 mm for the case studied in [5] (about 13% of the diameter of the o-rings). Therefore, Fig. 2 probably encompasses most of the experimentally feasible cases. Above this value, the elastic limit of the granular interaction would not hold.

C. Gravitational precompression

For a vertical chain of granules, gravity causes an ever increasing downward compression of the chain. That is, for the pair of granules k and $k+1$, the scaled force is $f = k\mathcal{G}$, where the scaled gravitational constant \mathcal{G} was defined in Eq. (9). We again rename the initial and maximum compression as $z_{0,k}$ and $z_{m,k}$ to indicate explicitly that these quantities now vary along the chain and hence depend on grain number k . These compressions are now, respectively, the roots of the equations

$$k\mathcal{G} = z_{0,k}^{3/2} + bz_{0,k}^6, \tag{22}$$

$$4k\mathcal{G}(z_{m,k} - z_{0,k}) = \frac{8}{5}(z_{m,k}^{5/2} - z_{0,k}^{5/2}) + \frac{4}{7}b(z_{0,k}^7 - z_{m,k}^7) - 1. \tag{23}$$

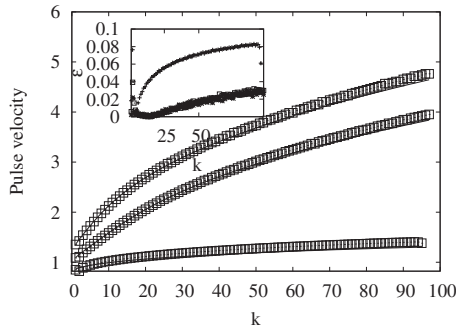


FIG. 3. Pulse velocity as a function of granule number moving downward along the chain. In the main figure, the value of b increases from bottom to top, $b=10^{-5}$, 4, and 20. The squares are the results of numerical simulations and the lines are the results of the binary collision approximation. The inset shows the relative error ϵ for $b=10^{-5}$ (stars), $b=4$ (squares), and $b=20$ (plus signs).

Furthermore, the full potential also becomes k -dependent,

$$V(k; z) = -2k\mathcal{G}z + \frac{4}{5}z^{5/2} + \frac{2}{7}bz^7. \quad (24)$$

Therefore, the residence time Eq. (15), and consequently the pulse velocity, both become k -dependent as well. In Fig. 3 we plot the pulse velocity as a function of the granule index. This dependence shows yet again the good agreement between the binary collision approximation and the numerical integration of the equations of motion. For this figure, we chose $\mathcal{G}=0.024$. This value is too large when compared with reasonable values for teflon o-rings. In fact, for teflon o-rings the gravitational effects are negligible. However, it is interesting to understand the effects of gravity since the theory is also applicable to other materials. The inset shows that the error increases along the chain. This is not surprising since the precompression also increases along the chain. The velocity of granules along the chain increases because of the increasing compression, and the binary collision approximation fails to fully capture this increase. Nevertheless, the effect of this failure is not very strong for the experimental situations that have been tested, which tend to involve far fewer than the 90 granules that we have followed in our work. Even for chains of up to 90 granules and even with the exaggerated gravitational effects assumed here, the relative error hardly exceeds 8% in the worst case scenario (small b).

For single power-law materials, it has been shown that for large k the pulse velocity scales as $k^{[1-1/(n-1)]/2}$ [3,21]. We have verified that even for our relatively short chains (short for the expectation of scaling behavior), the pulse velocity is indeed a power law of k . For $b=10^{-5}$, $c \sim k^{0.13}$, while for $b=20$, $c \sim k^{0.36}$. These values are not far from the values 0.17 and 0.42, respectively, which correspond to the limits of

a Hertzian or a power 7 potential. Hence, we can see that the binary collision approximation already approaches the asymptotic behavior for chains of 90 granules.

IV. CONCLUSION

The binary collision approximation has been very successful in predicting pulse propagation behavior in granular chain, providing analytic results where only numerical ones were previously available. In this paper we have accomplished two goals in the further application of the binary collision approximation to granular chains. One goal has been the extension of the method to chain in which toroidal rings (o-rings) are placed between rigid cylinders that act as nonlinear springs, resulting in potentials of interaction that contain two contributions rather than a single Hertzian one. This system is inspired by the availability of experimental results with which we can compare our analytic outcomes. The second, perhaps more important, goal is the extension of the binary collision approximation methodology to chains with precompression, that is, beyond the sonic vacuum cases considered in our earlier work. This extension is a challenging test for the binary collision approximation because precompression necessarily leads to situation in which more than two granules participate substantially in each collision event. When precompression is constant, the point of eventual failure of the binary collision approximation must occur when the precompression force is sufficiently strong. In the case of gravitational precompression, failure must occur when the chain is sufficiently long. However, we find that for parameters that expansively cover experimental regimes the binary collision approximation errs by relatively little. The errors are of at most a few percent when resulting pulse velocities are compared with those obtained by numerical integration of the equations of motion of the full granular chain. Given the differences in the experimental and theoretical initial setups, the velocities predicted by the binary collision approximation are gratifyingly close to the experimental values where the latter are available. We thus conclude that the binary collision approximation provides a powerful analytic method for the study of pulse propagation in granular chain even in the presence of precompression. We continue to examine the limits of applicability of this powerful methodology.

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