Markovian iterative method for degree distributions of growing networks

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Currently, simulation is usually used to estimate network degree distribution $P(k)$ and to examine if a network model predicts a scale-free network when an analytical formula does not exist. An alternative Markovian chain-based numerical method was proposed by Shi *et al.* [Phys. Rev. E **71**, 036140 (2005)] to compute time-dependent degree distribution $P(k,t)$. Although the numerical results demonstrate a quick convergence of $P(k,t)$ to $P(k)$ for the Barabási-Albert model, the crucial issue on the rate of convergence has not been addressed formally. In this paper, we propose a simpler Markovian iterative method to compute $P(k,t)$ for a class of growing network models. We also provide an upper bound estimation of the error of using $P(k,t)$ to represent $P(k)$ for sufficiently large t, and we show that with the iterative method, the rate of convergence of $P(k,t)$ is root linear.

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I. INTRODUCTION

It has been observed that additions of nodes and edges in most real (and modeled) growing networks are random, but follow certain rules. For example, two basic network-generating mechanisms for the Barabási-Albert (BA) model are growth and preferential attachment [1]. At each time step, these rules are applied only according to the current state of the network, so that the state of the network at the next time step can be determined probabilistically, as in [1–5]. This shows that the evolution of growing networks is intrinsically Markovian, i.e., the future evolution of the network depends only on the current network state. Furthermore, the degree distribution as a key network topological measure corresponds to the steady-state probabilities of a set of Markovian chains. Thus, Markovian chains provide a convenient modeling and analysis framework for growing networks. Shi *et al.* [6] first discovered this relationship between a growing network and a set of Markovian chains. They also developed an efficient rectangle-iterative algorithm to compute time-dependent network degree distributions and show numerically that the degree distributions can stabilize when the computation time is sufficiently long. However, no rigorous discussion is provided in [6] to quantify how fast the time-dependent degree distribution of a certain network converges to the real steady-state degree distribution.

In this paper, we examine some questions arising within the Markovian chain framework for a general class of growing network models. We mainly provide an explicit expression of the time-dependent degree distribution and an upper bound estimation on the time (or the network size) required for it to converge to the steady-state degree distribution. This solves the open problem left in [6] and provides an efficient and reliable method to compute the real steady-state degree distribution for a general class of growing networks.

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II. DEGREE-GROWING MARKOVIAN CHAINS

Consider a general class of growing networks in which multiple edges and loops are not permitted. Suppose that the initial network consists of $m_0 \geq m \geq 1$ nodes which are numbered as $-m_0, \ldots, -1$, where m is the minimum degree of all the nodes in the network except those of the initial network. Let n_k be the number of initial nodes with degree k and k_0 be the sum of the degrees of all the initial nodes. Let $k_i(t)$ be the degree at time t of the node added at time step i. For $i = -m_0, \ldots, -1$, $k_i(t)$ represents the degree of an initial node at time t. Obviously, $k_i(t)$ is nondecreasing and can increase at most by 1 at each time step. It is not difficult to see that $k_i(t)$ for any i is a nonhomogeneous Markovian chain [6] with transition probability

$$P\{k_i(t+1)=l|k_i(t)=k\} = \begin{cases} 1-f_i(k,t), & l=k \\ f_i(k,t), & l=k+1 \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Here, $f_i(k,t)$ is the conditional probability that the degree of node i becomes $k+1$ at time step $t+1$, given that the degree was k at time t, i.e., the degree of node i will increase by 1 at time step $t+1$. Let $\alpha_i(h)$ be the probability that the number of edges node i obtained is h when it is first introduced into the network, where $0 \leq h \leq i-1+m_0$. Clearly, $f_i(k,t)$, $\alpha_i(h)$, and Eq. (1) are determined by the network-generating mechanism of a growing network model and, together with the initial network, they completely define the Markovian chain for node i. The family of Markovian chains $\{k_i(t)\}$ for all nodes represents a growing network model completely. We call $\{k_i(t)\}$ *the degree-growing Markovian chain* (DGMC) for the fact that it captures the degree evolution of every individual node of a growing network model.

According to their network-generating mechanisms, we can easily define the DGMC for many growing-network models discussed in the existing literature. For example:

Growth and degree-preferential model. This is the first growing network model [1] and is commonly referred to as the BA model. At each time step, a new node with m edges is added to the network and is linked to m different existing nodes. The probability that an edge is linked to an existing node with degree k_i is $\Pi(k_i)=k_i/\sum_j k_j$. For more complete definition of the model, we add the following requirement: the initial network consists of $m_0 \geq m \geq 1$ nodes with k_0 being the sum of the degrees of all the initial nodes. For the model, $\alpha_i(m)=1$, because the number of new edges brought in by each new node is fixed at m.

For the BA model, the probability that a new edge is linked to an existing node with degree k is $\Pi(k)$. Approximately, the probability that one of m new edges is linked to an existing node with degree k is $m\Pi(k)=k/(2t+k_0/m)$. But it can be shown that the error tends to zero as $t\to\infty$. In fact, consider a model with the same initial network and $\alpha_i(m)=1$ but allowing multiple edges. By the degree-preferential rule, the probability that an existing node i with degree k will receive l edges from a new node is $C_m^l[\Pi(k)]^l[1-\Pi(k)]^{m-l}$. Clearly, the probability that node i with degree k in the BA model will receive a new edge at step $t+1$ is not smaller than the probability that the corresponding node in the above model receives one edge, i.e., $C_m^1\Pi(k)[1-\Pi(k)]^{m-1}=m\Pi(k)-o(1/t)$, and is not greater than the probability that the corresponding node in the above model receives at least one edge, i.e., $\sum_{l=1}^m C_m^l[\Pi(k)]^l[1-\Pi(k)]^{m-l}=1-C_m^0[1-\Pi(k_i)]^m=m\Pi(k)-o(1/t)$. Thus, for the BA model, the conditional probability is given by $f_i(k,t)=k/(2t+k_0/m)-o(1/t)$ for all i.

Copying growing model. For modeling citation networks, Krapivsky and Redner [7] proposed copying instead of citing popular papers as the network growing mechanism. In this model, suppose that the initial network consists of a single node. At each time step, a target node is chosen randomly from the existing nodes to be copied to make the new node, and the new node links to the target node as well as to all the ancestor nodes of the target node. Let $k_i(t)$ represent the in-degree of node i after time step t. Because a new node has no in-degree, we have $\alpha_i(h)=\delta_{h0}$. The probability that an existing node is randomly copied is $1/(t+1)$. Consider a node with in-degree k. If it is chosen to be copied or is one of the k nodes with an edge directed to the node being copied, the in-degree of this node will increase by 1 at time step $t+1$. Hence, the conditional probability $f_i(k,t)=(k+1)/(t+1)$ for all i.

Saturated growing model. Although the number of edges added each time increases in some observed real networks, the increase cannot continue forever and will likely slow down at some point. We use $[M(1-e^{-ri})]+1$ to replace the constant m in the BA model, where $M\geq 2$ is an upper boundary, r gives the accelerating rate, and $[x]$ represents the integral part of x; hence, $\alpha_i(h)=\delta_{h\{[M(1-e^{-ri})]+1\}}$, $i,h=1,2,\ldots$. We call this model a saturated model. In this model, the total number of new edges added after time step t is $\int_0^t\{[M(1-e^{-rx})]+1\}dx$; thus, $m\Pi(k)=\{[M(1-e^{-rt})]+1\}k/(2\int_0^t\{[M(1-e^{-rx})]+1\}dx+k_0)$. Using a similar argument as above for the BA model, we can show that $f_i(k,t)=\{[M(1-e^{-rt})]+1\}k/(2\int_0^t\{[M(1-e^{-rx})]+1\}dx+k_0)-o(1/t)$ for all i for the saturated growing model.

III. TIME-DEPENDENT DEGREE DISTRIBUTION

Now, we focus on growing network models with $f_i(k,t)\equiv f(k,t)$, e.g., models with the degree-preferential mechanism. We first write the following master equation for the degree distribution of node i defined by Eq. (1):

$$P(k,i,t+1)=f(k-1,t)P(k-1,i,t)+[1-f(k,t)]P(k,i,t), \tag{2}$$

where $P(k,i,t)=P\{k_i(t)=k\}$. The initial conditions are $P(k,i,i)=\alpha_i(k)$, for $i\geq 1$, and δ_{kl}, for $i<1$, where l is the degree of initial node i.

Let the time-dependent degree distribution be defined by $P(k,t)=[1/(t+m_0)]\sum_i P(k,i,t)$. Summing over i on both sides of Eq. (2) and using the initial conditions, we have

$$\begin{aligned}(t+1+m_0)P(k,t+1)-(t+m_0)[1-f(k,t)]P(k,t)\\ =(t+m_0)f(k-1,t)P(k-1,t)+\alpha_{t+1}(k).\end{aligned} \tag{3}$$

Lemma 1. (Solution of the difference equation [8]) When $t\geq 1$, the difference equation $a_{t+1}-h_t a_t=d_t$ has a closed-form solution

$$a_{t+1}=\prod_{i=1}^t h_i\left[a_1+\sum_{l=1}^t d_l\prod_{j=1}^l h_j^{-1}\right]. \tag{4}$$

Now, we can give a recursive formula of the time-dependent degree distribution.

Theorem 1. For the DGMC of a growing network model with $f_i(k,t)\equiv f(k,t)$, let $h_i=\min\{h|\alpha_i(h)>0\}$ and $m=\min\{h_i\}$. Then the time-dependent degree distribution of the network may be computed recursively as follows: For $k=m-1$,

$$P(m-1,t)=\frac{n_{m-1}}{t+m_0}\prod_{i=0}^{t-1}[1-f(m-1,i)],$$

for $k\geq m$, $P(k,0)=n_k/m_0$,

$$P(k,1)=\frac{m_0}{1+m_0}\{f(k-1,0)P(k-1,0)+[1-f(k,0)]P(k,0)\}+\frac{\alpha_1(k)}{1+m_0},$$

and for $t\geq 2$,

$$P(k,t)=\frac{1+m_0}{t+m_0}\prod_{i=1}^{t-1}[1-f(k,i)]\left[P(k,1)+\sum_{l=1}^{t-1}\frac{(l+m_0)f(k-1,l)P(k-1,l)+\alpha_{l+1}(k)}{(1+m_0)\prod_{j=1}^{l}[1-f(k,j)]}\right]. \tag{5}$$

Proof. By definition, only nodes in the initial network may have degree $m-1$. To maintain degree $m-1$ of some initial nodes unchanged up to t, no edge can be linked to these nodes in all $t-1$ steps. Hence, we obtain $P(m-1,t)$. For $k\geq m$, $P(k,0)=n_k/m_0$ is obvious. Letting $t=0$, we obtain $P(k,1)$ from Eq. (3). Applying Lemma 1 to Eq. (3), we obtain Eq. (5) for $P(k,t)$. This completes the proof.

Next we give a sufficient condition of the existence of the steady-state degree distribution. We also give the conditions under which a modeled network exhibits a scale-free topological structure. We need the following lemma:

Lemma 2. (Stolz-Cesáro theorem [9]) Let $\{y_n\}$ be a monotone increasing sequence with $y_n \to \infty$; we have $\lim_{n\to\infty}(x_n/y_n)=l$ if $\lim_{n\to\infty}[(x_{n+1}-x_n)/(y_{n+1}-y_n)]=l$, where $-\infty \le l \le +\infty$.

Corollary 1. For a DGMC model with $f_i(k,t)\equiv f(k,t)$, if $\lim_{i\to\infty}\alpha_i(h)=\alpha(h)$ is a proper distribution and $\lim_{t\to\infty} tf(k,t)\equiv F(k)\ge 0$, the steady-state network degree distribution exists, i.e., the DGMC is stable. Furthermore, let $j\ge m$ be the minimum h such that $\alpha(h)>0$, we have the following recursive expressions:

$$P(j) \triangleq \lim_{t\to\infty} P(j,t) = \frac{\alpha(j)}{1+F(j)} > 0,$$

$$P(k) = \frac{F(k-1)P(k-1)+\alpha(k)}{1+F(k)}, \quad k>j. \tag{6}$$

Proof. For $k<j$, because $\lim_{i\to\infty}\alpha_i(h)=\alpha(h)$, there are only finitely many nodes with degree k in the network, and hence $\lim_{t\to\infty}\Sigma_i P(k,i,t)<\infty$. Thus,

$$P(k) = \lim_{t\to\infty}\frac{1}{t+m_0}\sum_i P(k,i,t) = 0.$$

In Eq. (5), let $y_t=\{[(1+m_0)/(t+m_0)]\Pi_{i=1}^{t-1}[1-f(k,i)]\}^{-1}$ and $x_t=P(k,t)y_t$. We have

$$\frac{x_{t+1}-x_t}{y_{t+1}-y_t} = \frac{(t+m_0)f(k-1,t)P(k-1,t)+\alpha_{t+1}(k)}{1+(t+m_0)f(k,t)}.$$

For $k=j$, using Lemma 2 and by conditions in Corollary 1 and $\lim_{t\to\infty}P(j-1,t)=0$, we get

$$P(j) = \lim_{t\to\infty}P(j,t) = \lim_{t\to\infty}\frac{x_{t+1}-x_t}{y_{t+1}-y_t} = \frac{\alpha(j)}{1+F(j)}.$$

Similarly, for $k>j$, by induction we have

$$P(k) = \frac{F(k-1)P(k-1)+\alpha(k)}{1+F(k)}.$$

This completes the proof.

Corollary 2. For a stable DGMC model with $f_i(k,t)\equiv f(k,t)$, if there is a constant J such that $\alpha(h)=0$ when $h>J$ and $F(k)=\beta k+B$, where β, referred to as the dynamic exponent, and B are two constants, we have:

(1) The network is scale-free when $0<\beta\le 1$ with degree distribution (for sufficiently large $k>J$)

$$P(k) \sim [k+(B/\beta)]^{-(1+1/\beta)}. \tag{7}$$

(2) The network is random when $\beta=0$ and $B>0$ with

$$P(k) \sim \frac{1}{B}e^{-k/B}. \tag{8}$$

Proof. Without loss of generality, let $m=1$. Substituting $F(k)=\beta k+B$ into Eq. (6) and noting that the gamma function $\Gamma(k)\sim\Gamma(k+\gamma)k^{-\gamma}$ for sufficiently large $k>J$ and $\lim_{j\to\infty}[1+(x/j)]^j=e^x$, we have, for $\beta\ne 0$,

$$P(k) = \frac{\beta(k-1)+B}{1+\beta k+B}P(k-1) = \frac{\Gamma[k+(B/\beta)]}{\Gamma[k+(B/\beta)+1+(1/\beta)]}P(1) \sim [k+(B/\beta)]^{-(1+1/\beta)},$$

and for $\beta=0$

$$P(k) = \frac{B}{1+B}P(k-1) = \frac{1}{B}\left(\frac{B}{1+B}\right)^k = \frac{1}{B}\left(\left[1+\frac{k/B}{k}\right]^k\right)^{-1} \sim \frac{1}{B}e^{-k/B}.$$

This completes the proof.

Remark. Using the first-passage probability from the *Markovian chain theory*, Hou *et al.* also obtained similar results [10]. A simpler and more direct proof using the limit theorem of difference equations is given in [11].

IV. ERROR AND THE RATE OF CONVERGENCE

We have shown that based on the DGMC framework, one can easily write down $\alpha_i(h)$ and $f(k,t)$ from network-generating mechanisms and then determine if a model generates a scale-free network by checking them against a set of simple conditions. The degree exponent can also be easily determined explicitly. If we also need the detailed numerical values of the degree distribution $P(k)$ for a network model, we can use the analytical expressions to compute $P(k,t)$ accurately and efficiently for any t. But a more important question remains, that is, we need to know how quickly the time-dependent degree distribution $P(k,t)$ converges to $P(k)$ in order to draw conclusions confidently from observations of finite networks. We tackle this problem next.

Lemma 3. (Product estimation [12]) For large enough t, the product

$$\prod_{i=1}^{t}\left[1-\frac{Ak}{i}\right] = O(1)_k t^{-Ak}, \tag{9}$$

where $O(1)_k$ is a bounded constant depending only on k.

Now we may give the upper bound estimation of the error of using $P(k,t)$ to represent $P(k)$ when $P(k,t)$ indeed converges to $P(k)$.

Theorem 2. For a stable scale-free network model, when t is large enough, we have the following upper bound:

$$|P(k,t)-P(k)| \le |P(m,t)-P(k)| \le \varepsilon_t = \begin{cases} ct^{-[\beta m+B]}, & \text{for } m>1 \\ ct^{-[\beta(m+1)+B]}, & \text{for } m\le 1, \end{cases} \tag{10}$$

where ε_t is the error at t, $m=\min\{h_i\}$, $h_i=\min\{h|\alpha_i(h)\rangle 0\}$, exponent β and constant B are defined in Corollary 2, and c is an unknown constant.

Remark. Obviously, sequence ε_t converges to zero, and $\lim_{t\to\infty}\varepsilon_{t+1}/\varepsilon_t=1$. By definition [13], Theorem 2 shows that the rate of convergence of $P(k,t)$ by the Markovian iterative method is R linear (R stands for "root"). Because each iteration is very fast, Theorem 2 guarantees that our method is reliable and efficient for practical applications.

Let $S(k,t)=\Sigma_i P(k,i,t)=(t+m_0)P(k,t)$ and set $\Delta_k(t)=S(k,t)-(t+m_0)P(k)$. To give an upper bound to the error of using $P(k,t)$ to approximate $P(k)$, we need the following three lemmas:

Lemma 4. For a stable scale-free network model, we have, for sufficiently large t,

$$\Delta_k(t) \cong \prod_{i=1}^{t-1}[1-f(k,i)]\left\{\Delta_k(1)+\sum_{l=1}^{t-1}[f(k-1,l)\Delta_{k-1}(l)] \times\prod_{j=1}^{l}[1-f(k,j)]^{-1}\right\}, \quad (11)$$

where $\Delta_k(1)$ is a constant and

$$\Delta_{m-1}(t)=\begin{cases} O_{m-1}(1)t^{-[\beta(m-1)+B]}, & m>1\\ 0, & m\le 1. \end{cases} \quad (12)$$

Proof. Noting that $\lim_{t\to\infty}(t+m_0)f(k,t)=F(k)$ and $\lim_{i\to\infty}\alpha_i(k)=\alpha(k)$, it is easy to get, from Eqs. (3) and (6)

$$\begin{aligned}\Delta_k(t+1)-\Delta_k(t) &= S(k,t+1)-S(k,t)-P(k)\\ &= f(k-1,t)S(k-1,t)-f(k,t)S(k,t)+\alpha_{t+1}(k)\\ &\quad -P(k)\\ &= f(k-1,t)\Delta_{k-1}(t)-f(k,t)\Delta_k(t)+(t+m_0)\\ &\quad \times[f(k-1,t)P(k-1)-f(k,t)P(k)]\\ &\quad +\alpha_{t+1}(k)-P(k)\\ &\cong f(k-1,t)\Delta_{k-1}(t)-f(k,t)\Delta_k(t).\end{aligned}$$

Solving the above difference equation, we get Eq. (11). When $m>1$, from the condition in Theorem 2, $f(k,t)\sim(\beta k+B)/t$; using Lemma 3 and noting that $P(m-1)=0$ and $\Delta_{m-1}(t)=n_{m-1}\Pi_{i=0}^{t-1}[1-f(m-1,i)]$, when $m\le 1$, $f(m-1,t)=0$; hence, Eq. (12) is obvious.

Lemma 5. For any $k\ge m$ in a stable scale-free network, when t is large enough, we have

$$\Delta_k(1)+\sum_{l=1}^{t}\frac{f(k-1,l)\Delta_{k-1}(l)}{\prod_{j=1}^{l}[1-f(k,j)]}=\begin{cases} O(t^\beta), & m>1\\ O(1), & m\le 1. \end{cases} \quad (13)$$

Proof. First, we prove the case of $k=m$. When $m>1$, by Eq. (12), using $f(k,t)\sim(\beta k+B)/t$ and Lemma 3, for the left-hand side in Eq. (13), we have

$$\begin{aligned}&\Delta_k(1)+\sum_{l=1}^{t}\frac{f(k-1,l)\Delta_{k-1}(l)}{\prod_{j=1}^{l}[1-f(k,j)]}\\ &=\Delta_k(1)+\frac{[\beta(m-1)+B]O(1)_{m-1}}{O(1)_m}\sum_{l=1}^{t-1}\frac{l^{-1}l^{-[\beta(m-1)+B]}}{l^{-[\beta m+B]}}\\ &=\Delta_k(1)+O(1)_m\sum_{l=1}^{t-1}l^{\beta-1}=O(t^\beta).\end{aligned}$$

TABLE I. The BA model.

Time t	10^3	10^4	10^5
Method in [6]	0.501001	0.500100	0.500010
Formula (5)	0.499073	0.499902	0.499990
Real errors of [6]	0.001001	0.000100	0.000010
Real errors of Eq. (5)	0.000927	0.000098	0.000010
Upper bounds	10^{-3}	10^{-4}	10^{-5}

When $m\le 1$, because the summation in Eq. (11) is zero, this shows that Eq. (13) holds for $k=m$. We leave the case of $k>m$ to the proof of the next lemma.

Lemma 6. For any $k\ge m$ in the stable scale-free network, when t is large enough, there is a positive constant M_k such that

$$\Delta_k(t)\le\begin{cases} M_k t^{-[\beta(k-1)+B]}, & m>1\\ M_k t^{-(\beta k+B)}, & m\le 1. \end{cases} \quad (14)$$

Proof. First, we prove the case of $k=m$. When $m>1$, by Eqs. (9), (11), and (13), we have

$$\Delta_m(t)=O(1)_m t^{-(\beta m+B)}t^\beta\le M_m t^{-[\beta(m-1)+B]}.$$

Now, suppose that $k>m$ holds. Similarly, we first have $\Delta_k(t)=O(1)_k t^{-(\beta k+B)}t^\beta$. Using it, we can prove Lemma 5 for $k+1$, and by Eqs. (9), (11), and (13), we have

$$\Delta_{k+1}(t)=O(1)_{k+1}t^{-[\beta(k+1)+B]}t^\beta\le M_{k+1}t^{-(\beta k+B)}.$$

When $m\le 1$, the proof is the same except we cancel the factor t^β. Thus, Lemma 6 also holds for $k+1$.

Proof of Theorem 2. Because m is the minimum of network degree except the initial nodes, by Lemma 6, and taking $c=\max_k M_k$, we have

$$\begin{aligned}|P(k,t)-P(k)|\le\Delta_k(t)t^{-1}&\le\begin{cases} M_k t^{-(\beta k+B)}, & m>1\\ M_k t^{-[\beta(k+1)+B]}, & m\le 1\end{cases}\\ &\le\begin{cases} ct^{-(\beta m+B)}, & m>1\\ ct^{-[\beta(m+1)+B]}, & m\le 1.\end{cases}\end{aligned}$$

This completes the proof of the theorem.

V. NUMERICAL EXAMPLES

We now apply our results to a few network models. For the BA model, since $\alpha_i(h)\equiv\alpha(m)=1$ and $F(k)=k/2$, the steady-state degree distributions of the model exist and predict a scale-free network with $\gamma=3$. For numerical results, we set $m_0=3$, $n_1=2$, and $n_2=1$ in the initial network. Letting $m=2$, we have $\beta=1/2$, $B=0$, $j=m=2$, and $P(2)=1/2$ exactly. Table I gives the numerical comparisons. Clearly, for $t=10^3$, we have $0.000\,927<c\times10^{-3\beta m}$, and hence taking $c=1$ the upper bound estimation of the error is t^{-1}.

For the saturated model, since $\lim_{i\to\infty}\alpha_i(h)=\delta_{Mh}$ and

TABLE II. The saturated model.

Time t	10^3	10^4	10^5
Method in [6]	0.377937	0.397154	0.399705
Formula (5)	0.370514	0.396905	0.399690
Real errors of [6]	0.022063	0.002846	0.000295
Real errors of Eq. (5)	0.029486	0.003095	0.000310
Upper bounds	0.031	0.0031	0.00031

TABLE III. The copying model.

Time t	10^3	10^4	10^5
Method in [6]	0.500000	0.500000	0.500000
Formula (5)	0.500000	0.500000	0.500000
Real errors of [6]	0	0	0
Real errors of Eq. (5)	0	0	0

$$F(k) = \lim_{t\to\infty}\left[\frac{t\{[M(1-e^{-rt})]+1\}k}{2\int_0^t\{[M(1-e^{-rx})]+1\}dx + k_0} + o(t)\right] = \frac{k}{2},$$

the steady-state degree distributions of the model exist and are the same as those of the BA model with the constant M. When $m_0=3$, $n_1=2$, and $n_2=1$ in the initial network, taking $M=3$ and $r=0.01$, we have $\beta=1/2$, $B=0$, $m=1$, $j=M=3$, and $P(3)=2/5$ exactly. Numerical comparisons are given in Table II. For $t=10^4$, we have $0.003\,095 < c\times 10^{-4\beta(m+1)}$, and hence taking $c=31$ the upper bound estimation of the error is $31t^{-1}$.

For the copying model, since $\alpha_i(h)\equiv\alpha(0)=1$ and $F(k)=k+1$, the steady-state degree distributions of the model exist and predict a scale-free network with $\gamma=2$. Obviously, we have $\beta=1$, $B=1$, $j=m=0$, and $P_{in}(0)=1/2$ exactly. Some numerical results are given in Table III.

The above numerical results show that although the rates of convergence are consistent with the predictions of Theorem 2, the real errors for different models vary substantially because of the time dependence of the degree distributions [11]. For the BA model, the errors are caused by the initial network only. The errors of the saturated model are caused by both the initial network and the varying m; hence, it has larger errors. The degree distribution of copying model is independent of time, so its error equals zero. In fact, by the mean-field [1] argument we have $\partial k_i/\partial t=(k_i+1)/(t+1)$ and $k_i(i)=0$. Then, $k_i(t)=[(t+1)/(i+1)]-1$ and $P\{k_i(t)<k\}=P\{i>(t+1)/(k+1)\}=1-[1/(k+1)]$, and hence the density function is $f_{in}(k,t)=\partial P\{k_i(t)<k\}/\partial k=1/(k+1)^2$. The degree distribution $P_{in}(k,t)=\int_k^{k+1}(x+1)^{-2}dx=1/(k+1)(k+2)$ is thus independent of time.

VI. DISCUSSION

Based on a general Markovian chain framework, we provide an exact expression of the time-dependent degree distribution for a general class of growing network models. This expression enables the approximation of the steady-state degree distribution iteratively. We show that the error of the iterative method converges to zero at least R linearly.

We also provide some general criteria for judging whether a set of network-generating mechanisms can ensure the existence of the steady-stage degree distribution and whether the steady-stage degree distribution is scale-free. For modeled networks, the criteria can be easily verified. The criteria can also be applied to observable real networks. The criteria translate to two intuitive conditions: (1) the number of edges each node has when it is first introduced into a network can stabilize and be characterized by some finite distribution, and (2) the probability that a node will receive an edge when new edges are introduced into the network is proportional to the degree of the node and inversely proportional to the network size. When the two conditions can be verified for a real network, it is necessarily a scale-free network.

Our results show that Markovian chain provides a powerful framework for theoretical analysis of complex networks. We may similarly discuss the degree-growing Markovian chain of the weighted network, the degree-evolving Markovian chain (birth-and-death process) of evolving networks [14], and the other types of Markovian chains for different complex networks [15].

The unknown constant c in the error bound involves several limiting processes that depend on specific network-generating mechanism and initial network conditions. Finding it for any specific model remains a challenging open problem.

Some types of networks that the current Markovian chain framework cannot handle are when the network-evolving rules depend on the age of nodes and/or geography location. How to build the network connectivity theory for these types of networks is still an open problem.

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