Model for hydromagnetic convection in a magnetized fluid

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We consider convection in a horizontally magnetized viscous fluid layer in the gravitational field heated from below with a vertical temperature gradient. Following Rayleigh-Bénard scenario and using a general magnetohydrodynamic approach, we obtain a simple set of four ordinary differential equations. In addition to the usual three-dimensional Lorenz model a new variable describes the profile of the induced magnetic field. We show that nonperiodic oscillations are influenced by anisotropic magnetic forces resulting not only in an additional viscosity but also substantially modifying nonlinear forcing of the system. On the other hand, this can stabilize convective motion of the flow. However, for certain values of the model parameters we have identified a deterministic intermittent behavior of the system resulting from bifurcation. In this way, we have identified here a basic mechanism of intermittent release of energy bursts, which is frequently observed in space and laboratory plasmas. Hence, we propose this model as a useful tool for the analysis of intermittent behavior of various environments, including convection in planets and stars. Therefore, we hope that our simple but still a more general nonlinear model could shed light on the nature of hydromagnetic convection.

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The problem of convection in magnetized fluids is important not only for laboratory and space plasma physics, but also for geophysics and astrophysics. Examples of possible applications include magnetoconfined plasmas in tokamaks, nanodevices and microchannels in nanotechnology, liquid interiors of the earth core, interiors of the sun and stars, solar sunspots and coronal holes, granulation, the flow in the magnetosphere and heliosphere, and even in interstellar and intergalactic media. However, notwithstanding of progress in numerical simulations of convection, the nature of nonlinear dynamics of a viscous fluid with the embedded magnetic fields is still not sufficiently understood. In fact in magnetohydrodynamics a behavior of a conducting fluid is rather complex, and instead of a general theory we must still rely on some simplified models.

To gain insight into inherent unpredictability of the weather, Lorenz obtained a set of three nonlinear ordinary differential equations describing a cellular convection of a viscous hydrodynamic fluid [1]. Similar low-dimensional models of plasma convection have been recently derived for toroidal magnetic field configuration [2]. The possibility of deterministic aperiodic behavior has been also suggested in space plasmas [3,4] and in a magnetized plasma in laboratory [5,6]. After all, a deterministic approach has also been successful for explaining scaling of turbulence by phenomenological models (e.g., [7–9]). Low-dimensional models of fluid dynamics are strongly simplified and one should bear in mind that purely low-dimensional behavior is rather rarely observed in real physical systems. However, despite their simplicity, these models used as approximations may provide an insight into dynamical mechanisms appearing in aperiodic convective flows, describing self-consistently anomalous transport processes in fluids.

In this Brief Report we derive a generalization of the Lorenz model by including the magnetic field. We also present important results of numerical studies of basic properties of the derived model, which can be of relevance for magnetized fluid experiments. We focus here on the results concerning influence of the magnetic field on convective motion, including new types of strange attractors, and intermittency phenomenon observed in the system.

In general, time and space changes, $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$, of the velocity $\mathbf{v}$ of the flow, the magnetic field $\mathbf{B}$, or equivalently Alfvén velocity $\mathbf{v}_{\mathrm{A}} = \mathbf{B}/(\mu_o\rho)^{1/2}$ with a constant magnetic permeability μ_o, and the temperature T (with mass density ρ and pressure p) are described by the following partial differential equations [10]:

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla\left(p + \frac{\mathbf{B}^2}{2\mu_o}\right) + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu_o\rho} + \nu\Delta\mathbf{v} + \mathbf{f}, \tag{1}$$

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\cdot\nabla)\mathbf{v} + \eta\Delta\mathbf{B}, \tag{2}$$

$$\frac{dT}{dt} = \kappa\Delta T, \tag{3}$$

where $\rho\mathbf{f}$ is the volume density of additional external forces, while ν, η, and κ denote kinematic viscosity, magnetic diffusive viscosity, and thermal conductivity of the fluid, correspondingly.

In this Brief Report we look at the Rayleigh-Bénard problem [11] of a horizontal (x-axis) viscous fluid layer of height h heated from below with an applied vertical (z-axis) tem-

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perature gradient δT under the vertical gravitational field with a constant acceleration g resulting in the buoyancy term **f**. The schematics of this standard scenario can be found, e.g., in Appendixes to Refs. [7,12]. As usual, using a constant coefficient β, we take into account the volume expansion for **f** term, $\rho=\rho_o(1-\beta\delta T)$, but except that the fluid is treated as incompressible, $\rho=\rho_o$ (the Oberbeck-Boussinesq approximation) [13,14].

But now in this Brief Report we take into consideration the effect of the magnetic field. Naturally, in a case of an incompressible fluid we can use a stream (potential) function Ψ defined by $\mathbf{v}=\nabla\times\Psi$, and similarly a vector potential $\mathbf{A}$ for the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$, satisfying naturally the conditions $\nabla\cdot\mathbf{v}=0$ and $\nabla\cdot\mathbf{B}=0$ in Eq. (2). Surely, taking the rotation of Eqs. (1)–(3) both the thermal and isotropic parts of the magnetic pressure are eliminated, but an anisotropic tension of the magnetic field should still be important.

One can expect that in a case of a thin horizontal layer, the influence of an external horizontal magnetic field should be important. If we apply an initial magnetic field B_o along the x direction by adding the Alfvén velocity $v_{\mathrm{Ao}}=B_o/(\mu_o\rho_o)^{1/2}$, while neglecting a possible vertical field, we can write the perturbed respective potentials in the forms $\boldsymbol{\Psi}=\{0,\psi(x,z,t),0\}$ and $\mathbf{A}/(\mu_o\rho_o)^{1/2}=\{0,\alpha(x,z,t)-v_{\mathrm{Ao}}z,0\}$. Now, following Rayleigh we can look for solutions of the induced potentials for both the bulk and the Alfvén velocities in the double asymmetric (parameter a) Fourier representation [11,15],

$$\psi(x,z,t)=\sqrt{2}\frac{1+a^2}{a}\kappa X(t)\sin\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right), \qquad (4)$$

$$\alpha(x,z,t)=\sqrt{2}\frac{1+a^2}{a}\kappa W(t)\cos\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right). \qquad (5)$$

In the well-known three-dimensional Lorenz model, besides a time-dependent variable X proportional to the intensity of the convective motion, the other two variables Y and Z describe the temperature profile in Eq. (3) (see Ref. [1]). In addition, in this Brief Report we have introduced a time-dependent variable W describing the profile of the magnetic field induced in the convected magnetized fluid according to Eqs. (1) and (2).

Altogether in this way we obtain from the general magnetohydrodynamic equations (1)–(3) a simple set of four ordinary differential equations,

$$\dot{X}=-\sigma X+\sigma Y-\omega_o W, \qquad (6)$$

$$\dot{Y}=-XZ+rX-Y, \qquad (7)$$

$$\dot{Z}=XY-bZ, \qquad (8)$$

$$\dot{W}=(4\pi/\mu_o)\omega_o X-\sigma_{\mathrm{m}}W, \qquad (9)$$

where an overdot denotes an ordinary derivative with respect to the normalized time $t'=(1+a^2)\kappa(\pi/h)^2t$, and $b=4/(1+a^2)$. As usual $r=R_{\mathrm{a}}/R_{\mathrm{c}}$ is a control parameter of the system proportional to the temperature gradient δT or a Rayleigh number $R_{\mathrm{a}}=g\beta h^3\delta T/(\nu\kappa)$ normalized by a critical number $R_{\mathrm{c}}=(1+a^2)^3(\pi^2/a)^2$. In addition, we now introduce another control parameter proportional to the initial magnetic field strength B_o applied to the system, which is defined here as a basic dimensionless magnetic frequency, related to Alfvén waves by $\omega_o=v_{\mathrm{Ao}}/v_o$, with $v_o=16\pi^2\kappa/(abh\mu_o)$. Naturally, besides the Prandtl number $\sigma=\nu/\kappa$, the properties of the magnetized fluid are characterized by the magnetic Prandtl number $\sigma_{\mathrm{m}}=\eta/\kappa$ resulting from the last term in Eq. (2).

More specifically, the last term in Eq. (6) comes from the anisotropic tension of the magnetic field $(\mathbf{B}\cdot\nabla)\mathbf{B}/(\mu_o\rho)$ in Eq. (1). Similarly, the first term of Eq. (9) results from $(\mathbf{B}\cdot\nabla)\mathbf{v}$ on the right-hand side of Eq. (2), taking into account changes in the velocity $\mathbf{v}$ in space at a given time along the magnetic field $\mathbf{B}$. We can argue that this term, describing how the velocities of the fluid are changed owing to the convected magnetic fields, is in our case more important than $(\mathbf{v}\cdot\nabla)\mathbf{B}$, which is responsible for advection of the magnetic field $\mathbf{B}$ in a fluid moving with velocity $\mathbf{v}$. In fact, for a constant magnetic field $\mathbf{B_o}$ the later term vanishes. Therefore, as an approximation of the convective movement of the magnetic field frozen in a fluid only a first-order term $(\mathbf{B_o}\cdot\nabla)\mathbf{v}$ is maintained [16]. Admittedly, we have also verified that in this case for the inclusion of any higher-order terms one would need to consider a wider spectrum of modes, certainly not limited to Eq. (5). It is interesting to note that Eqs. (6)–(9) are somewhat similar to those obtained for acoustic-gravity waves in the atmosphere [17], but different signs in the novel terms result in new phenomena.

Now, combining the set of the generalized Lorenz system we can write Eqs. (6) and (9) in the following way:

$$\ddot{X}+\sigma\dot{X}+[\sigma r-(4\pi/\mu_o)\omega_o^2]X=-\sigma(Y+XZ)+\sigma_{\mathrm{m}}\omega_o W, \qquad (10)$$

$$\ddot{W}+\sigma_{\mathrm{m}}\dot{W}+(4\pi/\mu_o)\omega_o^2W=(4\pi/\mu_o)\sigma\omega_o(Y-X). \qquad (11)$$

Hence, formally both variables X and W satisfy the equations of two familiar damped linear oscillators. However, the terms on the right-hand side of Eqs. (10) and (11) may be interpreted as nonlinear driving forces. Moreover, we see that the coupling between X,W and Y,Z is enhanced owing to the magnetic field $\mathbf{B}$. Obviously, when $\omega_o=0$ this coupling ceases and the variable W is damped by the magnetic viscosity [see Eqs. (6) and (9)].

For the generalized Lorenz system [Eqs. (6)–(9)] besides a zero fixed point C^0, we have two other fixed points $C^{\pm}=\{\pm d/\sqrt{1+e},\pm d\sqrt{(1+e)},r-(1+e),\pm(\sigma/\omega_o)de/\sqrt{1+e}\}$, with $d=\sqrt{b[(r-1)-e]}$ and $e=(4\pi/\mu_o)\omega_o^2/(\sigma\sigma_{\mathrm{m}})$. The zero fixed point C^0 is stable for $0\le r<r_o$, but the additional fixed points $C^{\pm}$ are stable for $r_o\le r<r_H$, where $r_o=1+e$ is the critical normalized Rayleigh number for the onset of convection and $r=r_H$ is a critical value where a Hopf bifurcation takes place. We see that here the critical number r_o for the onset of convection increases with the magnetic field; thus, the magnetic field can stabilize the convection as regards to the appearance of convective rolls. However, if we consider oscillations of the convection rolls as described by the model

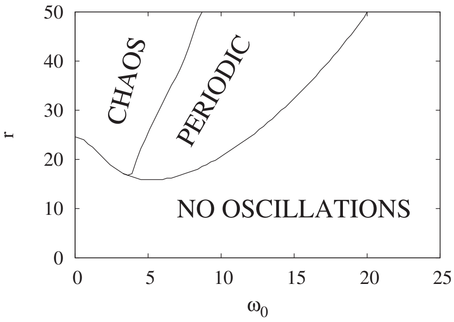

FIG. 1. Long-term behavior of the dynamical system of Eqs. (6)–(9) in the space of dimensionless control parameters ω_o and r for fixed values of other parameters of the system: $\sigma=10$, $b=8/3$, $\sigma_{\mathrm{m}}=1$. Solid lines separate regions of different dynamical behaviors.

of Eqs. (6)–(9), the influence of the magnetic field is more intricate.

This is illustrated in Fig. 1 using the plane spanned by two dimensionless control parameters r and ω_o (note that in Gauss units $4\pi/\mu_o=1$), where we can distinguish the following three regions of different dynamical behaviors of convective rolls: one without possibility of long-term oscillations, another with periodic oscillations, and third with chaotic dynamics. From the point of view of the theory of dynamical systems these three possibilities correspond to the situations where the trajectories of the dynamical system described by Eqs. (6)–(9) are attracted by fixed points (equilibrium), limit cycles, and chaotic attractors, correspondingly. For example, given $\omega_o=2$ and increasing r we observe a direct transition from a fixed point to a chaotic attractor similar to that observed for the classical Lorenz system. However, for higher values, e.g., for $\omega_o=5$, the model predicts periodic oscillations for some intermediate values of r between the state without oscillations (for small r) and chaotic dynamics (for large r). On the other hand for fixed $r=30$ when increasing ω_o we observe the transition from chaotic to periodic oscillations and then from periodic oscillations to nonoscillating rolls, which exhibits stabilizing influence of the magnetic field on the dynamics. Surprisingly, for $r=20$ by increasing magnetic field (ω_o) one may induce chaotic oscillations for some range of ω_o, which are then damped for still increasing magnetic field. One should notice that the control parameter ω_o depends on both the strength of the magnetic field and the average density; thus, the transitions described above may be obtained by changes in one of these quantities.

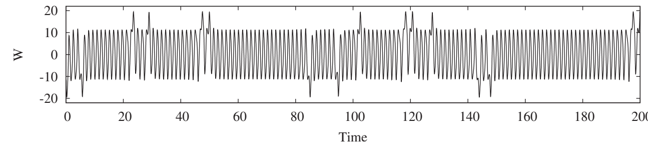

FIG. 3. The intermittent behavior of the generalized Lorenz model as a function of normalized time identified here for the variable W with the control parameters $\omega_o=4.8$ and $\sigma_{\mathrm{m}}=1$.

Some of many interesting new types of hydromagnetic strange attractors of the system projected onto the three-dimensional subspace spanned by X, Y, and W axes are now illustrated in Fig. 2. We take the standard values of the Lorenz model parameters: $r=28$, $\sigma=10$, and $b=8/3$. With a magnetic field, $\omega_o=1$, and a small magnetic viscosity, $\sigma_{\mathrm{m}}\approx 0$, a familiar butterfly-shaped set strongly wanders along the W axis [case (a)]. Naturally, the system is dissipative and the volume in phase space shrinks rapidly due to kinematic and magnetic viscosities, $\dot{V}=-(\sigma+\sigma_{\mathrm{m}}+b+1)V$; the attractor is strange and has a measure of zero. Because of that the trajectories only appear to merge, but they actually remain distinct.

In particular, it is worth noting a structure for $\omega_o=6$ in the presence of some magnetic viscosity, $\sigma_{\mathrm{m}}=2$, presented in Fig. 2(b). When changing the magnetic control parameter in some narrow range near $\omega_o=6$ trajectories in the phase space describing the perturbed magnetic vector potential merge and separate again resulting in irregularly reappearing "islands." This merging is related to a special (hyper)surface, which separates small oscillations around one of the two fixed points, $C^{\pm}$, from large oscillations that encircle all fixed points, including a zero unstable fixed point, C^0. Therefore, in the vicinity of this value a periodic motion is interrupted with chaotic bursts as shown in Fig. 3. Finally, if the magnetic field strength is further increased, so that the last term in the left-hand side of Eq. (10) changes sign, $\omega_o^2>\sigma r$, the oscillations are depressed and the system will tend to a fixed point as shown in Fig. 1.

It is worth noting that in the proximity of the boundary between chaotic and periodic regions in Fig. 1 we have identified intermittent behavior of the system illustrated in Fig. 3, where almost periodic oscillations are interrupted by bursts of irregular behavior. This phenomenon of intermittency can be observed as bursts of increased energy dissipation, defined here as $\nu|\mathbf{v}|^2+\eta|\mathbf{B}|^2/(\mu_o\rho)$. By the analysis of a

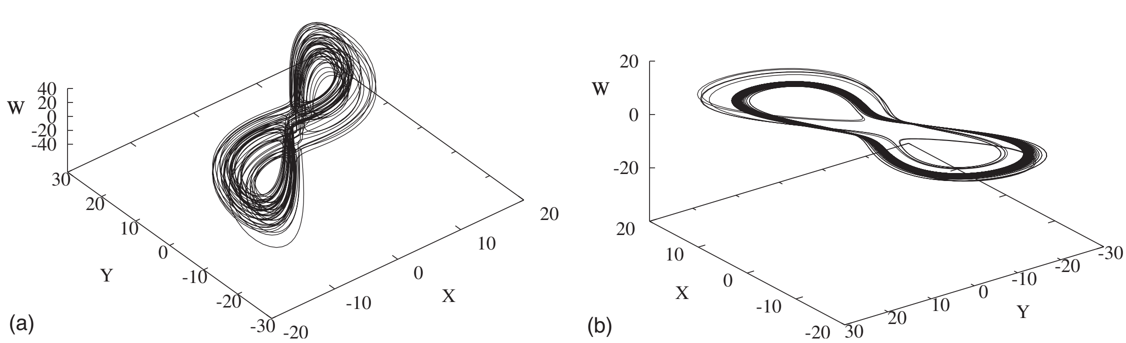

FIG. 2. The three-dimensional projection of the attractor for (a) $\omega_o=1$, $\sigma_{\mathrm{m}}\approx 0$ and (b) $\omega_o=6$, $\sigma_{\mathrm{m}}=2$, correspondingly.

Poincaré map (constructed from the values of Y variable taken for $X=0$ plane crossings) we have identified this intermittency as type III (see Ref. [18]). The intermittency of this type displays characteristic behavior of the signal, distribution of lengths of laminar intervals, and dependence of the mean length of laminar interval on bifurcation parameter as described thoroughly, e.g., in Ref. [12]. Here, in Fig. 4 we show the probability distribution of the laminar time intervals τ for our model of Eqs. (6)–(9), where a nontrivial nonlinear dependence is well approximated by the theoretical formula for type III intermittency [12],

$$P(\tau) \sim \frac{\varepsilon^{3/2} e^{4\varepsilon\tau}}{(e^{4\varepsilon\tau} - 1)^{3/2}}, \tag{12}$$

where ε is the difference between the actual value of the control parameter and its critical value for the onset of intermittency. One should note that this functional dependence is different from purely exponential behavior, predicted by self-organized criticality models, as well as from power-law dependence observed for fully developed turbulence [19].

In conclusion, we propose a new low-dimensional model describing self-consistently convective transport of magnetized fluids. It is clearly shown that the influence of the magnetic field is more intricate than purely stabilizing effect predicted by simple analytical models [16]. Intermittent behavior of the model is identified as important from the experimental point of view. One should note that, as is essential for intermittency, this transition from regular to irregular behavior results from the appearance and disappearance of fixed points or limit cycles and not from any stochastic forces. In our experience, we have identified here a fundamental mechanism of intermittent release of energy bursts, which is often observed in space and laboratory plasmas. Hence, we hope that our simple but still a more general nonlinear model could shed light on the nature of hydromagnetic turbulent convection, helping one to identify chaotic and intermittent behavior in various environments [3,9,19]. We propose this model as a useful tool for the analysis of intermittent convection in nonlinear complex systems, such as planetary and stellar interiors, including massive stars with heavy elements, which are important for the evolution of the universe, also in view of many space missions.

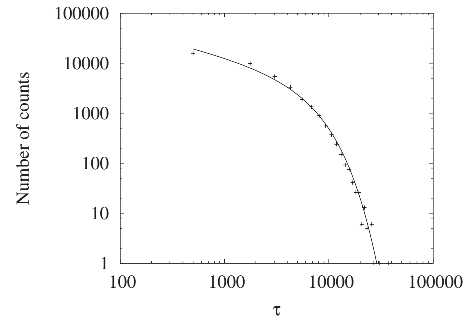

FIG. 4. Distribution of the lengths of laminar phases for $\omega_o=5.3$, $r=28$, $\sigma=10$, $b=8/3$, and $\sigma_{\mathrm{m}}=1$. Numerically obtained distribution (plus signs) is compared with theoretically predicted dependence on normalized time intervals (solid line) as given by Eq. (12).

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