Minimizers with discontinuous velocities for the electromagnetic variational method

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The electromagnetic two-body problem has *neutral differential delay* equations of motion that, for generic boundary data, can have solutions with *discontinuous* derivatives. If one wants to use these neutral differential delay equations with *arbitrary* boundary data, solutions with discontinuous derivatives must be expected and allowed. Surprisingly, Wheeler-Feynman electrodynamics has a boundary value variational method for which minimizer trajectories with discontinuous derivatives are also expected, as we show here. The variational method defines continuous trajectories with piecewise defined velocities and accelerations, and electromagnetic fields defined *by* the Euler-Lagrange equations *on* trajectory points. Here we use the piecewise defined minimizers with the Liénard-Wierchert formulas to define generalized electromagnetic fields almost everywhere (but on sets of points of zero measure where the advanced/retarded velocities and/or accelerations are discontinuous). Along with this generalization we formulate the *generalized absorber hypothesis* that the far fields vanish asymptotically *almost everywhere* and show that localized orbits with far fields vanishing almost everywhere *must* have discontinuous velocities on sewing chains of breaking points. We give the general solution for localized orbits with vanishing far fields by solving a (linear) neutral differential delay equation for these far fields. We discuss the physics of orbits with discontinuous derivatives stressing the differences to the variational methods of classical mechanics and the existence of a spinorial four-current associated with the generalized variational electrodynamics.

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I. INTRODUCTION

Non-radiating motion of extended charge distributions in classical electrodynamics has been known to exist for some time (c.f [1–4] and references therein, and [5–7]). On the other hand, for systems with a few *point charges*, Larmor's radiation of energy at a rate proportional to the squared modulus of the acceleration plagues classical electrodynamics. To construct orbits that do not radiate, and hence are without acceleration, a simple option are constant velocity motions, which imply unbounded motion.

Along *bounded* two body motions supported by mutual action at a distance, we expect acceleration to be needed to change velocities, unless velocities are allowed to change discontinuously. For example, periodic polygonal orbits with piecewise constant velocity segments have vanishing radiation fields.

Here, we extend Wheeler-Feynman electrodynamics [8] to include motion with discontinuous velocities. This is a natural extension provided by the variational boundary value problem [9]. The resulting extended electrodynamics has several appealing physical features: (i) There exists a scalar function (the finite action [9]), and the condition for a minimizer demands that the partial derivatives of the action, with respect to each particle's four velocity, be continuous along minimal orbits. These continuous four-component linear currents are analogous to the Dirac-equation of quantum mechanics, thus endowing the extended Wheeler-Feynman electrodynamics with spin. This is a feature not present in any other classical electrodynamics of point charges; (ii) Besides naturally including nonradiating orbits, the extended electrodynamics can be shown to lead simply to a de Broglie length for double-slit scattering upon detailed modeling [10]; (iii) The absorber hypothesis, first idealized to hold as an average over an infinite universe [8], has no known solutions [11] for many-body motion in Wheeler-Feynman theory [11–15] with which it is consistent. Here we show that the variational electrodynamics allows a concrete realization of the absorber hypothesis for a two-particle universe, i.e., there exists a non-empty class of two-body motions with vanishing far fields, so that we do not need either large universes or randomization [16,17]; and (iv) two-body orbits with vanishing far-fields were used in Ref. [18] to predict spectroscopic lines for hydrogen with a few percent precision.

Since the speed of light is constant in inertial frames, the equations of motion for point-charges are state dependent differential delay equations. More specifically, Wheeler-Feynman electrodynamics [8,10,19] has mixed-type state-dependent neutral differential delay equations of motion for the two-body problem.

The theory of delay equations is still incomplete [20,21] but it is known that purely-retarded differential delay equations with generic C^1 initial histories have continuous solutions with a discontinuous derivative at the initial time. The derivative becomes continuous at the next breaking point [20] and progresses from C^k to C^{k+1} at successive breaking points. On the other hand, a purely retarded neutral differential delay equation with a generic C^1 initial history [20] can have continuous solutions with discontinuous derivatives at *all* breaking points.

If one wants to use the electromagnetic neutral differential delay equations with arbitrary boundary data, solutions with discontinuous derivatives must be expected and accommodated. Surprisingly, this same neutrality is compatible with

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the recently developed boundary-value-variational method for Wheeler-Feynman electrodynamics [9]. For orbits where the acceleration is not defined at a few points, the variational method offers a well-posed alternative to define trajectories beyond those satisfying a Newtonian-like neutral differential delay equation *everywhere*. The variational method involves an integral that requires only piecewise-defined velocities, generalizing naturally to continuous orbits with discontinuous derivatives at breaking points.

Our generalized electrodynamics contains the C^2 orbits of the Wheeler-Feynman theory. As shown in Ref. [9], if boundary data are such that the extremum orbit is piecewise C^2 with *continuous* velocities, the Wheeler-Feynman equations hold everywhere with the exception of a countable set of points where accelerations are discontinuous (which is a set of measure zero for the action integral). We henceforth define a breaking point as a point where velocity or acceleration are discontinuous. Here we show that continuous orbits with discontinuous velocities are possible minimizers if these satisfy further continuity conditions. These continuity conditions are non-local, unlike the conditions for an extremum of the variational methods of classical mechanics, which do *not* allow discontinuous velocities. Finally, if the extremum is not piecewise C^2, the variational method defines minimizers that are not described by piecewise-defined-Wheeler-Feynman neutral differential delay equations (which are not studied here).

To discuss the relationship to Maxwell's electrodynamics it is important to keep in mind that: (i) Wheeler-Feynman electrodynamics is a theory of *trajectories*, where fields are only *derived quantities*; and (ii) the boundary-value-variational-method defines only a *finite* segment of a trajectory, rather than a global trajectory [9]. The variational equations along piecewise C^2 orbits include the electromagnetic fields in the Euler-Lagrange equations [9], which are used here to give a derived operational meaning to the electromagnetic fields [22]. The electromagnetic fields appear as coupling terms of the variational equations and are defined *on* trajectory segments by the usual electromagnetic formulas [9].

In our generalization we use the Liénard-Wierchert electromagnetic formulas to define fields by extension at all space-time points for which future and past lightcones fall in the finite segment of the minimizer trajectory. For continuous trajectories with discontinuous velocities, and/or accelerations on sets of measure zero, we construct the electromagnetic fields only for points having a future and past lightcone, leaving the fields undefined where the past or future lightcones have a discontinuous velocity/acceleration (usually another set of measure zero). We further introduce the concept of short-range orbits as localized orbits with far-fields vanishing almost everywhere. This bears a close relation to the electromagnetic notion of radiation [22].

In their original articles, Wheeler and Feynman [8] attempted to derive an electrodynamics with retarded-only fields from the hypothesis that the universal far-fields vanish at all times (the absorber hypothesis) [8]. Here, we generalize the absorber hypothesis [8] to include fields that can be undefined on sets of measure zero, thus arriving at the *generalized absorber hypothesis* (GAH) that the far-fields vanish *almost everywhere*. We show that short-range-two-body-orbits *must* involve discontinuous derivatives on a countable set of points.

One advantage of our generalization is to include spatially bounded globally defined continuous orbits with far fields vanishing almost everywhere, which we call short-range orbits. This generalization presents itself naturally as the next option after one shows that there are no C^2 localized orbits with far fields vanishing everywhere. The short-range piecewise C^2 continuous orbits are naturally described by the variational method in the same way as the globally C^2 continuous orbits. However, the former are minimizers inside a larger family of boundary data, which is the second advantage of our generalization. This extended class of orbits includes orbits that are limits of Cauchy sequences of orbits with far fields disturbing the universe less and less, i.e., the vanishing far-field limit of the GAH.

In this paper we use the word "minimizer" meaning a generalized critical point of the variational method, that could be either a minimum or a saddle point. The paper is divided as follows: In Sec. II, we discuss the variational method for piecewise-defined continuous orbits with discontinuous derivatives. We show that the variational method prescribes a continuous momentum current at each breaking point in addition to Euler-Lagrange equations from each side of the breaking point. We discuss how the non-local-momentum currents can be conserved even in the presence of velocity discontinuities along "sewing chains." In Sec. III, we prove that globally-defined short-range bounded orbits must have discontinuous velocities on a sewing chain of breaking points by giving the general solution to a neutral differential delay equation for the far-fields. In Sec. IV, we discuss the physics of generalized minimizers along with some open questions and differences from bounded orbits to unbounded scattering orbits.

II. BOUNDARY VALUE VARIATIONAL METHOD

The variational method [9] is well defined for continuous trajectories $\boldsymbol{x}_1(t_1)$ and $\boldsymbol{x}_2(t_2) \in \mathbb{R}^3$ that are piecewise C^1. The boundary conditions for the variational method [9] are illustrated in Fig. 1, i.e., the initial point O_A for trajectory 1 plus the segment of trajectory 2 inside the lightcone of O_A, and the end point L_B for the trajectory 2 plus the segment of trajectory of particle 1 inside the lightcone of L_B. For variations of trajectory 1 the action functional [9] reduces to

$$\begin{aligned} S &\equiv K_2 + \int_0^{T_{L^-}} \mathcal{L}(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2)dt_1 \\ &= -\int_0^{T_{L^-}} m_1\sqrt{1-\mathbf{v}_1^2}dt_1 + \int_0^{T_{L^-}} \frac{(1-\mathbf{v}_1\cdot\mathbf{v}_{2+})}{2r_{12+}(1+\mathbf{n}_{12+}\cdot\mathbf{v}_{2+})}dt_1 \\ &\quad + \int_0^{T_{L^-}} \frac{(1-\mathbf{v}_1\cdot\mathbf{v}_{2-})}{2r_{12-}(1-\mathbf{n}_{12-}\cdot\mathbf{v}_{2-})}dt_1 + K_2, \end{aligned} \tag{1}$$

where K_2 depends only on trajectory 2 and quantities of particle 2 are defined at times $t_{2\pm}$ according to the implicit condition for the advanced/retarded light cones of t_1, i.e.,

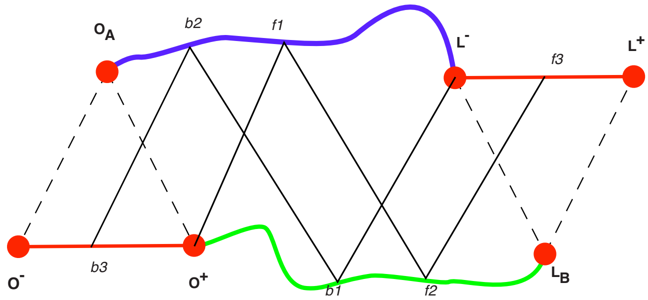

FIG. 1. (Color online) Illustrated in red is the initial point O_A of trajectory 1 plus the segment of trajectory 2 inside the lightcone of O_A, i.e., from point O^- to point O^+ and the end point L_B of trajectory 2 plus the segment of trajectory of particle 1 inside the lightcone of L_B, i.e., from point L^- to point L^+. The trajectory of particle 1 of the variational method goes from O_A to L^- (blue line) while the trajectory of particle 2 goes from O^+ to L_B (green line). The first breaking point is point O^+ which generates a forward sewing chain of breaking points f_1, f_2, f_3 while end point L^- is a breaking point generating a backward sewing chain of breaking points b_1, b_2, b_3.

$$t_{2\pm} = t_1 \pm |\mathbf{x}_1(t_1) - \mathbf{x}_2(t_{2\pm})|. \tag{2}$$

In Eq. (1) the $r_{12\pm} \equiv |\mathbf{x}_1(t_1) - \mathbf{x}_2(t_{2\pm})|$ are the distances from $\mathbf{x}_1(t_1)$ to the respective advanced/retarded position $\mathbf{x}_2(t_{2\pm})$ along trajectory 2, unit vector $\mathbf{n}_{12\pm}$ points from $\mathbf{x}_1(t_1)$ to the respective advanced/retarded position $\mathbf{x}_2(t_{2\pm})$, i.e., $\mathbf{n}_{12\pm} \equiv (\mathbf{x}_1(t_1) - \mathbf{x}_2(t_{2\pm}))/r_{12\pm}$ and last $\mathbf{v}_{2\pm} \equiv d\mathbf{x}_2/dt_2|_{t_{2\pm}}$. Notice that Eq. (1) is an integral over the velocities, and is a well-defined operation even for trajectories with discontinuous velocities (and even more general types of continuous trajectories with square-integrable velocities that are not studied here).

Here, we extend Wheeler-Feynman electrodynamics to trajectories with discontinuous velocities on a countable set of points using the boundary-value-variational method. For piecewise C^1 trajectories (*and* piecewise C^1 histories) it is possible to define disjoint intervals $t \in (l^+_{\sigma-1}, l^-_\sigma)$, with $l^-_\sigma = l^+_\sigma$, for $\sigma = 1, \ldots, N$, where the continuous trajectory $\boldsymbol{x}_1(t_1)$ and delayed arguments $t_{2\pm}(t_1)$ are piecewise C^1. The upper plus in l^+_σ indicates the right-limit of the σ^{th} breaking point while the upper minus in l^-_σ indicates the left-limit of the σ^{th} breaking point. These are not to be confused with the lower plus or the lower minus used to denote quantities evaluated on the future or past lightcones.

The variations of trajectory 1 for the action Eq. (1) are defined piecewise C^1 with fixed end points, i.e.,

$$\mathbf{u}_1(t_1) = \mathbf{x}_1(t_1) + \mathbf{b}_1(t_1),$$

$$\dot{\mathbf{u}}_1(t_1) = \dot{\mathbf{x}}(t_1) + \dot{\mathbf{b}}_1(t_1), \tag{3}$$

where and overdot denotes a time derivative and the boundary conditions are

$$\mathbf{b}_1(l^+_o = 0) = 0,$$

$$\mathbf{b}_1(l^-_N = T_{L^-}) = 0. \tag{4}$$

If the continuous and piecewise C^1 perturbation $\mathbf{b}_1(t_1)$ has a discontinuous derivative in another set of intervals $t \in (h^+_{\mu-1}, h^-_\mu)$, then the perturbed trajectory $\mathbf{u}_1(t_1)$ is continuous and piecewise C^1 in the extended set of intervals defined by all intersections of the sets $(h^+_{\mu-1}, h^-_\mu)$ and $(l^+_{\sigma-1}, l^-_\sigma)$. This simply increases the number of piecewise intervals $(l^+_{\sigma-1}, l^-_\sigma)$ up to $\sigma = M \geq N$ and the boundary condition for $\mathbf{b}_1(T_{L^-})$ of Eq. (4) reads

$$\mathbf{b}_1(l^-_M = T_{L^-}) = 0. \tag{5}$$

Substituting the perturbed trajectory Eq. (3) into the action Eq. (1) and making a linear expansion about the orbit defines the Frechét derivative, i.e.,

$$\delta S = \int_0^{T_{L^-}} \left[\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}_1} \cdot \mathbf{b}_1 \right) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} \cdot \dot{\mathbf{b}}_1 \right) \right] dt_1 + O(|\boldsymbol{b}_1|^2), \tag{6}$$

where a "·" indicates the scalar product in $\mathbb{R}^3$ and $|\boldsymbol{b}_1|$ is the sup norm for the Banach space of piecewise C^1 variations [9]. In particular if the orbit $\boldsymbol{x}_1(t_1)$: $[0, T_{L^-}] \to \mathbb{R}^3$ is piecewise C^2 then $\mathbf{u}_1(t_1)$ is continuous and piecewise C^1 on the same extended set of intervals.

If the orbit is piecewise C^2, then we can integrate Eq. (6) by parts in each interval, yielding

$$\delta S = \int_0^{T_{L^-}} \left(\mathbf{b}_1 \cdot \left[\frac{\partial \mathcal{L}}{\partial \mathbf{x}_1} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} \right) \right] \right) dt_1 + \sum_{\sigma=1}^{\sigma=M} \int_{l^+_{\sigma-1}}^{l^-_\sigma} \frac{d}{dt_1} \left(\mathbf{b}_1(t_1) \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} \right) dt_1. \tag{7}$$

Since $\mathbf{b}_1(t_1)$ is continuous we can rearrange the second term of the right-hand-side of Eq. (7) to give

$$\delta S = \int_0^{T_{L^-}} \left(\mathbf{b}_1 \cdot \left[\frac{\partial \mathcal{L}}{\partial \mathbf{x}_1} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} \right) \right] \right) dt_1 - \sum_{\sigma=1}^{\sigma=M-1} \left(\mathbf{b}_1(l^-_\sigma) \cdot \left. \frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} \right|_{l^-_\sigma}^{l^+_\sigma} \right), \tag{8}$$

where

$$\delta J_1 \equiv \left. \frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} \right|_{l^-_\sigma}^{l^+_\sigma} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}_1}(l^+_\sigma) - \frac{\partial \mathcal{L}}{\partial \mathbf{v}_1}(l^-_\sigma). \tag{9}$$

Equation (9) defines the momentum jump at $t = l^-_\sigma$, i.e., the first (second) term on the right-hand side of Eq. (9) is the momentum evaluated from the right (left) of $t = l^-_\sigma$.

The conditions for a critical point in the class of continuous piecewise C^1 orbital variations of the piecewise C^2 continuous orbit are: (i) satisfy the Euler-Lagrange equations piecewise, to make the first term on the right-hand side of Eq. (8) vanish; and (ii) have a continuous momentum $\partial\mathcal{L}/\partial\mathbf{v}_1$ at the breaking points so each term of the sum of the right-hand side of Eq. (8) vanishes for arbitrary $\mathbf{b}_1(l^-_\sigma)$, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1}(l_\sigma^-) = \frac{\partial \mathcal{L}}{\partial \mathbf{v}_1}(l_\sigma^+). \quad (10)$$

As is usual in the neighborhood of breaking points one defines derivatives from the left-hand side and from the right-hand side [20]. For the local Lagrangians of classical mechanics one usually has $\partial\mathcal{L}/\partial\mathbf{v}_1 = G_1(\mathbf{x}_1,\mathbf{v}_1)$, which combined with Eq. (10) along a continuous trajectory would imply that *each velocity* is continuous. Continuity of velocity along a piecewise C^2 continuous orbit combined with the Euler-Lagrange equations from each side of the breaking point further determine a continuous acceleration, so that the orbit is actually C^2 at the breaking point. Therefore, in classical mechanics the restriction to piecewise C^2 orbits *implies* globally C^2 orbits.

However, for Eq. (1), or the Lagrangian given in Eq. (14) of Ref. [9], the continuous momentum term is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_1} = \frac{m_1\mathbf{v}_1}{\sqrt{1-\mathbf{v}_1^2}} - \frac{\mathbf{v}_{2-}}{2r_{12-}(1-\mathbf{n}_{12-}\cdot\mathbf{v}_{2-})} - \frac{\mathbf{v}_{2+}}{2r_{12+}(1+\mathbf{n}_{12+}\cdot\mathbf{v}_{2+})}, \quad (11)$$

which displays a surprising difference compared with the result obtained from variational methods in classical mechanics.

As illustrated in Fig. 1, a simple piecewise-defined orbit has the "sewing chain" of breaking points ($f_1, f_2, \ldots$ and $b_1, b_2, \ldots$), where one velocity can jump *if* the other velocity has jumped at either the past or future breaking point. The two-body-Noether-momentum [formula (A23) of Ref. [9]], involves an integral that is insensitive to velocity jumps plus two non-local momentum terms given by Eq. (11) (see Eqs. (A25) and (A26) of Ref. [9]), that *are* sensitive to jumps. Therefore, the two-body-Noether momentum is conserved as long as Eq. (11) is continuous across the jumps. The first term on the right-hand side of Eq. (8) is the Wheeler-Feynman equation of motion for particle 1 [9] (i.e., the usual Euler-Lagrange equation restricted here to piecewise segments). Minimization respect to variations of trajectory 2 yields the neutral differential delay equation of motion for particle 2, and an analogous continuity with indices 1 and 2 exchanged. Notice that this surprising difference compared with the results from variational principles in classical mechanics requires a *minimum of two bodies and a non-local Lagrangian*.

We note that the electromagnetic variational method has a parametrization-invariance-symmetry that allows the action Eq. (1) to be expressed in Minkowski-four-space using four-velocities respect to an arbitrary evolution parameter [9]. The derivative of the parametrization-invariant Lagrangian with respect to the first component of the four-velocity is,

$$\frac{\partial \mathcal{L}}{\partial v_1^o} = \frac{m_1}{\sqrt{1-\mathbf{v}_1^2}} - \frac{1}{2r_{12-}(1-\mathbf{n}_{12-}\cdot\mathbf{v}_{2-})} - \frac{1}{2r_{12+}(1+\mathbf{n}_{12+}\cdot\mathbf{v}_{2+})}. \quad (12)$$

Equation (12) represents the time-component of the four-momentum which must be continuous at the breaking points of a minimizing trajectory. This is a generalization of the argument leading to Eq. (11). There is a four current associated with the minimization respect to each particle's trajectory.

Last, after solving for velocity discontinuities along sewing chains of breaking points, the second condition for a minimizer are the piecewise-restricted Wheeler-Feynman equations of motion. These hold at each side of a breaking point and involve the limiting accelerations from the two sides of that breaking point. The discontinuous velocities satisfying Eqs. (11) and (12), when substituted into the Wheeler-Feynman equations at each side of a breaking point, then define a condition to be satisfied by the acceleration discontinuities. Since this condition involves a singular matrix [30], acceleration discontinuities are not fully determined by velocity discontinuities. In general, velocity discontinuities cause acceleration discontinuities, even though there can be special orbits with continuous velocities and discontinuous accelerations along the null direction of the singular matrix [30]. Only in that case our generalization is equivalent to piecewise restricted Wheeler-Feynman equations, otherwise completion by a finite action [9] includes other types of trajectories. In either case, along a piecewise defined orbit, continuity of the spinor currents Eqs. (11) and (12) ensure the well-posed continuation of the minimizer across each breaking point [31]. For example, solutions with continuous velocities yield trivially continuous momenta Eqs. (11) and (12), so that any finite portion of a global solution of the piecewise-restricted Wheeler-Feynman equations with continuous velocities is a minimizer of the finite variational method with suitable boundaries. For solutions with discontinuous velocities, continuation to a global trajectory is possible using the (discontinuous) velocity determined by solving conditions Eqs. (11) and (12) for the most advanced velocity (to be shown elsewhere).

III. SHORT-RANGE ORBITS

Wheeler-Feynman electrodynamics is a theory of direct interaction between charges [8,22]. The boundary-value-variational method (previous section and Fig. 1) defines minimizers with a vanishing Frechét derivative Eq. (6) between the time spans of Fig. 1, rather than globally defined trajectories. As shown in Ref. [9], the two-body-Euler-Lagrange equations can be cast in the form of Newtonian equations of motion with each acceleration multiplied by the mass on the left-hand side, while the right-hand side has the form of a Lorentz-force law. It is precisely these Euler-Lagrange equations that define the electromagnetic fields of Wheeler-Feynman theory as *derived* quantities evaluated *on* trajectories.

Extending these fields defined by the Lorentz-sector-of-Euler-Lagrange equations to fields on positions outside trajectories is tricky, because in a theory of trajectories one should: (i) add a third particle to the variational problem; and (ii) arrange things such that the third trajectory passes by the desired point. Obviously, a third charge changes the minimization problem and perturbs the original two-body-orbit, unless it can be placed so far that its couplings to the original two-body-orbit are small. A bounded GAH two-body-orbit is special because its far-fields vanish almost everywhere and a third trajectory can be placed reasonably near without disturbing the two-body-orbit. Keeping in mind that the far fields are the strongest couplings to a third charge, we investigate the existence of such (localized) short-range-two-body-orbits (GAH).

We now adopt a unit system in which the speed of light is $c=1$ and apply the usual formulas of electrodynamics to piecewise-defined trajectories with the exception of points where past/future velocities/accelerations are undefined, i.e., the fields are undefined on a set of measure zero. We consider continuous piecewise C^2 trajectories $\mathbf{x}_k(t_k)$ enclosed by a sphere of radius R in an inertial frame. We specify space-time points $(t,R\mathbf{n})$ on the sphere by a time t and unit vector $\mathbf{n}$ normal to the surface of the sphere, and introduce an index $k=1,2$ to label the charges.

The far-electric field of a point charge in the Wheeler-Feynman electrodynamics is the sum of the half-advanced/half-retarded fields [8],

$$\boldsymbol{E}(t,R\mathbf{n}) = \frac{1}{2}\mathbf{E}^{adv} + \frac{1}{2}\mathbf{E}^{ret}, \tag{13}$$

while the far-magnetic field is given by

$$\boldsymbol{B}(t,R\mathbf{n}) = \frac{1}{2}\boldsymbol{n}_+ \times \mathbf{E}^{adv} - \frac{1}{2}\boldsymbol{n}_- \times \mathbf{E}^{ret}. \tag{14}$$

The unit vectors $\boldsymbol{n}_\pm$ point respectively from the charge's advanced/retarded position to the position $R\mathbf{n}$ on the sphere [23]. Trajectories are assumed to be bounded such that $|\mathbf{x}_k(t_k)| \ll R$, so that for each charge we have $\boldsymbol{n}_+ \simeq \boldsymbol{n}_- \equiv \boldsymbol{n}$.

The retarded far-electric and far-magnetic fields of a charge q_k at the space-time point $(t,R\mathbf{n})$ are piecewise defined by the Liénard-Wiechert formulas [23]

$$\mathbf{E}_k^{ret}(t,\mathbf{n}) = \frac{q_k}{R}\frac{\mathbf{n} \times [(\mathbf{n}-\mathbf{v}_k) \times \mathbf{a}_k]}{(1-\mathbf{n}\cdot\mathbf{v}_k)^3}, \tag{15}$$

and

$$\mathbf{B}_k^{ret}(t,\mathbf{n}) = \mathbf{n} \times \mathbf{E}_k^{ret}(t,\mathbf{n}). \tag{16}$$

In Eq. (15) we have used the far-field limit in which the light cone distance

$$r_k(t_k) \equiv |\mathbf{x}_k(t_k) - R\mathbf{n}|, \tag{17}$$

is equal to R since $|\mathbf{x}_k(t_k)| \ll R$. In Eq. (15) $\mathbf{v}_k \equiv d\mathbf{x}_k/dt_k|_{t_k}$ and $\mathbf{a}_k \equiv d^2\mathbf{x}_k/dt_k^2|_{t_k}$ are respectively the charge's velocity and charge's acceleration at the retarded time t_k defined implicitly and piecewise by the *retardation condition*

$$t_k = t - |\mathbf{x}_k(t_k) - R\mathbf{n}|, \tag{18}$$

where $|\cdot|$ denotes Cartesian distance. Equation (25) defines t_k as an implicit function of time t with a piecewise defined derivative

$$\frac{dt_k}{dt} = \frac{1}{(1-\mathbf{n}\cdot\mathbf{v}_k)}. \tag{19}$$

Using Eq. (15) to evaluate the far-magnetic field Eq. (16) yields

$$\mathbf{B}_k^{ret}(t,\mathbf{n}) = -\frac{q_k\mathbf{n}}{R} \times \left[\frac{\mathbf{a}_k}{(1-\mathbf{n}\cdot\mathbf{v}_k)^2} + \frac{(\mathbf{n}\cdot\mathbf{a}_k)\mathbf{v}_k}{(1-\mathbf{n}\cdot\mathbf{v}_k)^3}\right]. \tag{20}$$

The trajectory $\mathbf{x}_k(t_k)$ is a function of t from Eq. (18) so using the chain rule and Eq. (19) twice we can re-write the far-magnetic field Eq. (20) as

$$\mathbf{B}_k^{ret}(t,\mathbf{n}) = -\frac{q_k\mathbf{n}}{R} \times \frac{d^2}{dt^2}[\mathbf{x}_k(t_k)]. \tag{21}$$

The far-electric field is a linear function of the far-magnetic field obtained using Eq. (16) and the transversality property $\mathbf{n}\cdot\mathbf{E}_k^{ret}(t,\mathbf{n})=0$ of the far-electric field Eq. (15), i.e.,

$$\mathbf{E}_k^{ret}(t,\mathbf{n}) = -\mathbf{n} \times \mathbf{B}_k^{ret}(t,\mathbf{n}). \tag{22}$$

In view of Eq. (22), it suffices to study the vanishing of the retarded-far-magnetic fields. We further assume a symmetry that the time-reversed orbit yields the same orbit rotated about an axis. For these reverse-rotate-symmetric orbits the vanishing of the retarded far fields implies the vanishing of the advanced far fields.

From now on charge 1 is taken to be positive and equal to q while charge 2 is negative and equal to $-q$. The GAH along a bounded piecewise C^2 orbit is then expressed almost everywhere by

$$\mathbf{B}^{ret} = \mathbf{B}_1^{ret} + \mathbf{B}_2^{ret} = -\frac{q\mathbf{n}}{R} \times \frac{d^2}{dt^2}(\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2)) = 0. \tag{23}$$

In the family of orbits with discontinuous velocities one can readily construct bounded orbits with vanishing far fields; e.g., piecewise-constant-velocity orbits with trajectories consisting of polygonal lines. These are bounded orbits with each acceleration vanishing piecewise, so that the radiation fields vanish. The question that needs to be answered is "do we need these velocity discontinuities?"

Equation (23) is a (linear) *neutral differential delay equation* with piecewise-linear continuous solutions defined on the intervals $t \in (t_{\sigma-1}^+, t_\sigma^-)$, with $\sigma \in \mathbb{Z}$ by

$$\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2) = \mathbf{D}_\sigma(\mathbf{n}) + \mathbf{n}f_\sigma(t,\mathbf{n}) + (t - t_\sigma^-)\boldsymbol{V}_\sigma(\mathbf{n}), \tag{24}$$

where the $\mathbf{D}_\sigma(\mathbf{n})$ and $\boldsymbol{V}_\sigma(\mathbf{n})$ are arbitrary bounded functions and the $f_\sigma(t,\mathbf{n})$ are bounded and piecewise C^2. It is possible to choose $\mathbf{n}\cdot\mathbf{D}_\sigma(\mathbf{n})=\mathbf{0}$ and adjust $\mathbf{D}_\sigma(\mathbf{n})$ in each interval to make the left-hand side of Eq. (24) continuous.

Along a spatially bounded orbit, Eq. (18) is approximated for large values of R by

$$t_k = t - R + \mathbf{n}\cdot\mathbf{x}_k(t_k). \tag{25}$$

Notice that Eqs. (25) yield an implicit relation between t_1 and t_2,

$$t_1 - t_2 = \mathbf{n}\cdot(\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2)). \tag{26}$$

Given the trajectories $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_2)$, Eq. (26) and the implicit function theorem yield t_1 as a function of t_2 and $\mathbf{n}$. Define the *influence interval* of point $(t_2,\mathbf{x}_2(t_2))$ by the interval containing t_1 when $\mathbf{n}$ varies arbitrarily in Eq. (26), i.e.,

$$t_2 - |\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2)| < t_1 < t_2 + |\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2)|. \quad (27)$$

The time span Eq. (27) is from the retarded lightcone time of $(t_2, \mathbf{x}_2(t_2))$ to the advanced lightcone time of $(t_2, \mathbf{x}_2(t_2))$, as along the sewing chain illustrated in Fig. 1. Notice that the future lightcone appeared naturally in the two-particle problem, even though we were dealing only with the retardation conditions Eq. (25). It follows from Eqs. (26) and (24) that

$$f_\sigma(t,\mathbf{n}) = (t_1 - t_2) - (t - t_\sigma^-)\mathbf{n} \cdot V_\sigma(\mathbf{n}). \quad (28)$$

and we can therefore re-write Eq. (24) as

$$\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2) = \mathbf{D}_\sigma(\mathbf{n}) + (t_1 - t_2)\mathbf{n} - (t - t_\sigma^-)\mathbf{n} \times \mathbf{L}_\sigma(\mathbf{n}). \quad (29)$$

where $\mathbf{L}_\sigma(\mathbf{n}) \equiv \mathbf{n} \times V_\sigma(\mathbf{n})$. Since linear growth in a constant direction is unbounded, the only globally C^2 orbit must have $\boldsymbol{L}_\sigma(\mathbf{n})=0$ $\forall \sigma$, and it follows from Eq. (29) with $\boldsymbol{L}_\sigma(\mathbf{n})=0$ that

$$\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2) = \mathbf{D}_\sigma(\mathbf{n}) + (t_1 - t_2)\mathbf{n}. \quad (30)$$

The derivative of Eq. (30) respect to time yields

$$\frac{\mathbf{v}_1}{(1 - \mathbf{n} \cdot \mathbf{v}_1)} - \frac{\mathbf{v}_2}{(1 - \mathbf{n} \cdot \mathbf{v}_2)} = K_{12}\mathbf{n}, \quad (31)$$

where

$$K_{12} = \frac{1}{(1 - \mathbf{n} \cdot \mathbf{v}_1)} - \frac{1}{(1 - \mathbf{n} \cdot \mathbf{v}_2)}. \quad (32)$$

Equation (26) allow us to move $\mathbf{n}$ in a cone with axis along $\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2) \neq 0$ in a way that fixes t_1 and t_2 while changing t with Eq. (25). On the other hand, for fixed t_1 and t_2 the left-hand-side of Eq. (31) spans a plane of the *fixed* vectors $\mathbf{v}_1(t_1)$ and $\mathbf{v}_2(t_2)$, so that Eq. (31) can hold only if $K_{12}=0$, which combined with Eqs. (31) and (32) yields

$$\mathbf{v}_1(t_1) = \mathbf{v}_2(t_2). \quad (33)$$

Equation (33) defines globally constant velocities along a fixed direction, which in turn implies unbounded motion unless $\mathbf{v}_1 = \mathbf{v}_2 = 0$, as discussed in Ref. [12]. This impossibility follows if velocities are to be continuous.

Nontrivial alternatives to this unsatisfactory conclusion necessitate the introduction of discontinuities by varying the direction of the piecewise-velocity-like term $\boldsymbol{L}_\sigma(\mathbf{n}) \neq 0$ of Eq. (29) in each interval. The piecewise derivative of Eq. (29) respect to time yields

$$\frac{\mathbf{v}_1}{(1 - \mathbf{n} \cdot \mathbf{v}_1)} - \frac{\mathbf{v}_2}{(1 - \mathbf{n} \cdot \mathbf{v}_2)} = K_{12}\mathbf{n} - \mathbf{n} \times \mathbf{L}_\sigma(\mathbf{n}). \quad (34)$$

Notice that K_{12} is still given by Eq. (32) and with nonzero $\boldsymbol{L}_\sigma(\mathbf{n})$ the right-hand-side of Eq. (34) forms a complete three-dimensional basis to express any vector (inside or outside the plane of $\mathbf{v}_1(t_1)$ and $\mathbf{v}_2(t_2)$). Equation (26) still allows one to move $\mathbf{n}$ in a cone with axis along $\mathbf{x}_1(t_1) - \mathbf{x}_2(t_2) \neq 0$ in a way that fixes t_1 and t_2 while t changes with Eq. (25). By choosing K_{12} and a nonzero $\boldsymbol{L}_\sigma(\mathbf{n})$ for each $t \in (t_{\sigma-1}^+, t_\sigma^-)$ we can describe any vector on the left-hand side of Eq. (34), so that there is no inconsistency.

As an example, time-reversible orbits satisfying Eq. (34) are piecewise-constant-velocity orbits generated by having one velocity jump at a given time while the other velocity jumps either in the backward or forward lightcone-times symmetrically, as well as at every time in the forward and backward light-cones of a discontinuity time (the sewing chain illustrated in Fig. 1). These piecewise-linear polygonal orbits can be shown to satisfy Eq. (23) by direct substitution and use of Eq. (19). In the following we show that Eq. (29) and the implicit function theorem yield a consistent piecewise-defined trajectory $\mathbf{x}_1(t_1)$ from a given piecewise-defined trajectory $\mathbf{x}_2(t_2)$.

Notice that for given continuous and piecewise C^1 $\mathbf{x}_2(t_2)$, $\mathbf{D}_\sigma(\mathbf{n})$ and $\boldsymbol{L}_\sigma(\mathbf{n})$, in general Eq. (29) determines only a function $\mathbf{x}_1(t_1, \mathbf{n})$ of the *two variables* $(t_1, \mathbf{n})$ through

$$\mathbf{x}_1(t_1,\mathbf{n}) = \mathbf{x}_2(t_2) + \mathbf{D}_\sigma(\mathbf{n}) + (t_1 - t_2)\mathbf{n} - (t - t_\sigma^-)\mathbf{n} \times \mathbf{L}_\sigma(\mathbf{n}). \quad (35)$$

The implicit function theorem further determines t_2 and t as functions of t_1 and $\mathbf{n}$ from Eqs. (25), (26), and (35). For the implicit function theorem to yield a consistent trajectory, we must satisfy the consistency requirement that $\mathbf{x}_1(t_1, \mathbf{n})$ determined by Eq. (35) is a function of t_1 only, i.e.,

$$\frac{\partial \mathbf{x}_1(t_1,\mathbf{n})}{\partial \mathbf{n}} = 0. \quad (36)$$

Condition Eq. (36) applied to the right-hand side of Eq. (35) is the extra condition determining a consistent trajectory. Since condition Eq. (36) must hold for all values of t_1 in each piecewise interval of the orbit, we must also have inside each piecewise interval that

$$\frac{\partial^2 \mathbf{x}_1(t_1,\mathbf{n})}{\partial t_1 \, \partial \mathbf{n}} = \frac{\partial}{\partial \mathbf{n}}\left(\frac{\partial \mathbf{x}_1(t_1,\mathbf{n})}{\partial t_1}\right) = 0, \quad (37)$$

which can be expressed as

$$\frac{\partial}{\partial \mathbf{n}}\left[(\mathbf{v}_2 - \mathbf{n})\frac{\partial t_2(t_1,\mathbf{n})}{\partial t_1} + \mathbf{n} - \frac{\partial t(t_1,\mathbf{n})}{\partial t_1}\mathbf{n} \times \boldsymbol{L}_\sigma(\mathbf{n})\right] = 0. \quad (38)$$

The general solution to Eq. (38) involves an arbitrary piecewise-defined function $A_\sigma(t_1)$, i.e.,

$$\mathbf{n} + \frac{\partial t_2(t_1,\mathbf{n})}{\partial t_1}(\mathbf{v}_2 - \mathbf{n}) - \frac{\partial t(t_1,\mathbf{n})}{\partial t_1}\mathbf{n} \times \boldsymbol{L}_\sigma(\mathbf{n}) = A_\sigma(t_1). \quad (39)$$

A symmetric condition follows by exchanging indices 1 and 2 in Eq. (38), introducing an arbitrary $B_\sigma(t_2)$ and changing the sign of $\boldsymbol{L}_\sigma(\mathbf{n})$, yielding

$$\mathbf{n} + \frac{\partial t_1(t_2,\mathbf{n})}{\partial t_2}(\mathbf{v}_1 - \mathbf{n}) + \frac{\partial t(t_2,\mathbf{n})}{\partial t_2}\mathbf{n} \times \boldsymbol{L}_\sigma(\mathbf{n}) = B_\sigma(t_2). \quad (40)$$

The partial derivatives in Eqs. (39) and (40) can be evaluated using Eqs. (25), yielding

$$\frac{A_\sigma(t_1)}{(1-\mathbf{n}\cdot\mathbf{v}_1)} - \frac{\mathbf{v}_2(t_2)}{(1-\mathbf{n}\cdot\mathbf{v}_2)} = K_{12}\mathbf{n} - \mathbf{n}\times\mathbf{L}_\sigma(\mathbf{n}), \quad (41)$$

$$\frac{\mathbf{v}_1(t_1)}{(1-\mathbf{n}\cdot\mathbf{v}_1)} - \frac{B_\sigma(t_2)}{(1-\mathbf{n}\cdot\mathbf{v}_2)} = K_{12}\mathbf{n} - \mathbf{n}\times\mathbf{L}_\sigma(\mathbf{n}), \quad (42)$$

where again K_{12} is defined by Eq. (32).

Notice that Eqs. (41) and (42) with $A_\sigma(t_1)=\mathbf{v}_1(t_1)$ and $B_\sigma(t_2)=\mathbf{v}_2(t_2)$ bring back Eq. (34) as the single necessary condition to construct a consistent piecewise C^1 continuous trajectory $\mathbf{x}_1(t_1)$ from a given piecewise C^1 continuous trajectory $\mathbf{x}_2(t_2)$ by the implicit function theorem. We stress that velocity discontinuities are *absolutely necessary*. The discontinuities introduced by the nonzero $\boldsymbol{L}_\sigma(\mathbf{n})$ in each piecewise interval play an essential role in the solution as discussed below Eq. (34). Trying to solve either Eq. (41) or Eq. (42) with $\boldsymbol{L}_\sigma(\mathbf{n})=0$ stumbles with the former obstruction that rotation of $\mathbf{n}$ with fixed t_1 and t_2 places $\mathbf{n}$ either outside the plane of $A_\sigma(t_1)$ and $\mathbf{v}_2(t_2)$ or outside the plane of $B_\sigma(t_2)$ and $\mathbf{v}_1(t_1)$. The nonzero $\boldsymbol{L}_\sigma(\mathbf{n})$ provides the third linearly independent direction forming a complete basis to express $\mathbf{n}$ in Eqs. (41) and (42).

It can be seen that piecewise-constant-velocity polygonal orbits can be constructed using the suitable $\mathbf{L}_\sigma(\mathbf{n})$ defined by Eq. (34), after which the implicit function theorem constructs consistent continuous piecewise C^1 trajectories. It would be desirable to find bounded minimizer orbits of the variational method [9] satisfying the vanishing far-field conditions Eqs. (41) and (42), as first conjectured for the orbits studied in Ref. [18]. The justification to generalize to trajectories with discontinuous derivatives is to include short-range bounded GAH orbits in the family of physically possible orbits. This was used in Ref. [18] to predict spectroscopic lines of hydrogen within a few percent agreement with the predictions of quantum mechanics.

IV. DISCUSSION AND CONCLUSION

The fact that accelerations are discontinuous is expected because the Wheeler-Feynman equations of motion are explicitly neutral for the accelerations. Consequently, it could seem that a theory of piecewise-restricted Wheeler-Feynman equations of motion should have only acceleration discontinuities, a fact that already introduces discontinuous fields *and* demands a generalization of electrodynamics. We have seen that generalizing to trajectories with discontinuous accelerations is not sufficient to include bounded two-body orbits with vanishing far-fields. Our analysis starting from the variational method as the fundamental principle has shown that, in general, the velocities are also expected to be discontinuous at the same "generalized breaking points" along the minimizer orbits. Our analysis, using the variational method as a boundary-value problem, shows that the most general solution of the Wheeler-Feynman neutral differential delay equations has discontinuous accelerations *and* velocities.

The form of Eq. (12) is reminiscent of the energy operator used formally in quantum mechanics. Since the evolution parameter is arbitrary, the same parameter can be used in a Lorentz-transformed frame, such that the momentum currents transform by like a four-vector. The existence of four components that must be continuous and given by the partial derivatives of a scalar invariant is again analogous to the quantum Dirac equation and suggests a property analogous to spin for the point charges. It is remarkable that Wheeler-Feynman electrodynamics completed with a finite action endows the point charges with a spinlike property. The existence of a spinorial four-component momentum current [Eqs. (11) and (12)] is due to the parametrization-invariance symmetry of the electromagnetic variational method [9]. Otherwise a generic action with delayed interaction has only three momentum currents continuous at breaking points of minimizer orbits.

In Ref. [29] only globally C^2 solutions were sought for the seemingly non-neutral one dimensional motion, so that piecewise-defined solutions with discontinuous velocities awaited study. In considering this, it is important that the electromagnetic-action-functional Eq. (1) of Ref. [9] yields a *neutral-boundary-value-variational-method*, as opposed to the non-neutral variational methods of classical mechanics.

The variational principles of classical mechanics are two-point boundary value problems that are equivalent to an initial-value problem with initial velocity chosen to hit the final trajectory point. Moreover, in the classical problem there is no issue of velocity continuity, because the "history" for a finite-dimensional ODE is a point. On the contrary, for the electromagnetic variational method, choosing the "initial velocity" to shoot the final point either requires a velocity discontinuous with the past boundary history *or* the trajectory arrives at the final point with the wrong velocity. The generalization to discontinuous velocities extends the solvability of the electromagnetic-boundary-value problem to a larger class of boundary value data, which is the second advantage of our extension of Wheeler-Feynman electrodynamics.

Electromagnetism was originally formulated with the integral laws of Ampere, Gauss and Faraday, and only much later differential equations holding everywhere were introduced by Maxwell. The requirement of a second derivative existing everywhere is actually not needed for particle dynamics, where one is concerned only with the integral of the force along the trajectory. The variational method is a step back from Maxwell's equations in the sense of weak solutions. Replacing Maxwell's equations by the vanishing of the Frechét derivative Eq. (6) along a continuous trajectory with boundaries in future and past yields solutions defined only on bounded time-intervals. From these segments of orbits one can construct the fields as *derived quantities* [8,22]. Since fields constructed in this fashion involve a retarded and a advanced position, before defining fields everywhere in space we need to extend to a global trajectory. Extension is possible using conditions Eqs. (11) and (12) and in general involves velocity and acceleration discontinuities.

After extending to a global trajectory, it is then tempting to translate our generalized electrodynamic quantities into the concepts of Maxwell's electrodynamics, but it must be done carefully; The derivation of the differential form of Maxwell's equations given in [8] holds only in regions where the extended fields are C^1. Fields of continuous trajectories with a countable number of breaking points should satisfy

Gauss's and Ampere's law on most Gaussian surfaces and Ampere's circuits, apart from surfaces and circuits having a portion of nonzero measure equidistant from a breaking point, on which fields are undefined. Consequently, Poynting's theorem in integral form holds in regions where the fields are C^1. Nevertheless, quantities like energy flux and field energy in a volume have a meaning even for discontinuous fields, so that a statistical interpretation could be sought in the case of discontinuous fields; For example, the generalized flux of the Poynting vector is an integral that can be evaluated even when fields are undefined on sets of measure zero. The Poynting vector $\boldsymbol{P}=\boldsymbol{E}\times\boldsymbol{B}$ evaluated with Eqs. (13) and (14) at $(t,R\mathbf{n})$ becomes

$$\boldsymbol{P}=\frac{1}{4}\{|\mathbf{E}^{adv}|^2-|\mathbf{E}^{ret}|^2\}\boldsymbol{n} \tag{43}$$

where single bars denote the Euclidean modulus.

We stress that a condition of nonradiation is weaker than our GAH, as follows; The GAH *implies* the vanishing of the flux Eq. (43) because $|\mathbf{E}^{ret}|^2=|\mathbf{E}^{adv}|^2=0$ *almost everywhere*, while the converse is not true, i.e., the vanishing of the flux integral alone does not imply the GAH. For example, the circular orbits of Refs. [13,14] do not satisfy the GAH, and even though these orbits do not radiate on average, circular orbits have non-vanishing far-fields to disturb a "third charge of the variational method" (i.e., are not short-range). In Ref. [14] a model for the neutron was attempted, and even if it had not failed for other reasons, it would yield a neutron with far-fields. As regards non-vanishing far-fields, a first attempt to overcame the GAH-deficiency of circular orbits [14] was the perturbation theory of Ref. [18] that added high-frequency modes of the tangent dynamics to enforce Eq. (23) at the frequency of the circular orbit.

We have shown that our variational method yields a dynamical system even for limiting orbits with discontinuous velocities. For example, along piecewise-constant-velocity polygonal orbits the variational equations of motion would be applied as follows; On discontinuity corner points with no acceleration defined, one enforces continuity of momentum only, Eq. (11). At other points, wherever accelerations *are* defined, one uses the usual Wheeler-Feynman equations of motion. (Notice that piecewise-constant-velocity polygonal orbits have vanishing far-fields but obviously do not satisfy the equations of motion, unless charges are far apart).

We have demonstrated five different and important reasons to study orbits with discontinuous derivatives: (i) inclusion of bounded GAH orbits as short-range orbits; (ii) compatibility with the conservation of Noethers momentum; (iii) compatibility with the neutrality of the equations of motion of the Wheeler-Feynman electrodynamics; (iv) the fact that the variational method is natural in a space completed to contain orbits with discontinuous velocities; and (v) inclusion of limits of sequences of nonradiating orbits.

The physical need for trajectories with discontinuous velocities is justified as limiting orbits defined by Cauchy sequences of bounded orbits which *must* develop kinks in the short-range limit (i.e., the GAH). In Ref. [18] the GAH deficiency of circular orbits was removed with a perturbative Fourier series that solved Eq. (23) at the first harmonic frequency only. As we have shown here, the perturbative series of Ref. [18] should converge to an orbit with *discontinuous velocities*. The short-range condition of Ref. [18] predicted orbits and spectral lines in the atomic magnitude with a surprising precision, so that we can claim agreement with experiment and quantum mechanics. Piecewise-defined minimizers have also been used successfully to explain double-slit diffraction in Ref. [10].

From our generalized electrodynamics with discontinuous derivatives it should be possible to derive a generalized electrodynamics with delayed-only interactions and self-interaction, using the GAH in close analogy with the derivation of Wheeler and Feynman [8]. Since we expect solutions with velocity discontinuities, Taylor expansions of deviating arguments should be avoided or piecewise restricted. It is known that many-component delay differential equations behave like neutral differential delay equations when some solution components are discontinuous at breaking points, in the sense that the discontinuous derivatives never smooth out [31]. Therefore, the third derivative should be generalized by restricting it to a left derivative and a right derivative at breaking points. Also, the generalized absorber hypothesis with discontinuous fields no longer implies the vanishing of the difference of retarded and advanced universal fields everywhere, as used by Wheeler and Feynman [8,32]. Our Eq. (34) and its advanced version give the corresponding weaker generalization to this former stronger condition of a vanishing difference of retarded and advanced fields everywhere. We speculate that Eq. (34) should be the starting point for a generalized theory of self-interaction free of the pervasive runaways of the two-body problem with the usual self-interaction (Refs. [24–28]).

Last, we speculate that *unbounded* scattering orbits are different from the bounded orbits studied here. Along unbounded orbits Eq. (24) contains the extra secular term with a constant $\boldsymbol{V}_{\sigma}(\mathbf{n})\neq 0$. The dependence on boundary segments and time separation must be investigated for scattering trajectories with discontinuous velocities and accelerations at the boundaries; For example, even if the history segment (O^-,O^+) of Fig. 1 is assumed C^∞, the forward sewing chain of O^+ places a breaking point f_3 in the history segment (L^-,L^+) of particle 1. Unless histories are very special so that derivatives are continuous at O^+, in general the history (L^-,L^+) should involve a discontinuous derivative at point f_3. Scattering trajectories are likely to have future continuations involving stiffer jumps at later times, so that particles collide with laboratory boundaries, which can be regarded as a generalized type of radiative loss.

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