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Relativistic Weierstrass random walks

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The Weierstrass random walk is a paradigmatic Markov chain giving rise to a Lévy-type superdiffusive behavior. It is well known that special relativity prevents the arbitrarily high velocities necessary to establish a superdiffusive behavior in any process occurring in Minkowski spacetime, implying, in particular, that any relativistic Markov chain describing spacetime phenomena must be essentially Gaussian. Here, we introduce a simple relativistic extension of the Weierstrass random walk and show that there must exist a transition time t_c delimiting two qualitative distinct dynamical regimes: the (nonrelativistic) superdiffusive Lévy flights, for $t < t_c$, and the usual (relativistic) Gaussian diffusion, for $t > t_c$. Implications of this crossover between different diffusion regimes are discussed for some explicit examples. The study of such an explicit and simple Markov chain can shed some light on several results obtained in much more involved contexts.

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I. INTRODUCTION

Relativistic Brownian motion is an interesting and active area of research nowadays [1]. The dynamical behavior of any relativistic process is, of course, strongly constrained by the main properties of the relativistic kinematics, namely, the speed of light c as the maximum possible physical velocity, the invariance under Lorentz transformations, and the causal structure associate with the light cone. Some well established nonrelativistic dynamical behaviors are simply incompatible with the principle of Special Relativity. This is specifically the case of the so-called Lévy flights [2], where very rare events with arbitrarily high velocities give rise to a superdiffusive regime characterized by a power law

$$\langle X^2(t)\rangle \propto t^{\mu}, \tag{1}$$

where $\mu > 1$ is the corresponding anomalous diffusion exponent. However, Special Relativity does not allow such arbitrarily high velocities, preventing the appearance of such superdiffusive regime in Markov chains involving spacetime events. These points are studied, for instance, in [3] by using relativistic versions of the Fokker-Planck equation and of the fluctuation-dissipation theorem, leading to a fractional-derivative extension of the diffusion equation. An earlier analysis of relativistic random walks can be found in [4]. In [5], a generalized Wiener process avoiding superluminal propagation is introduced, giving rise to a non-Markovian relativistic diffusion process. The influence of the spacetime causal structure on dynamical processes, namely the implication of the presence of an event horizon, was investigated recently in [6].

We consider here a simple relativistic extension of the Weierstrass random walk in order to shed some light on several relativistic aspects of Lévy flights. We remind that the usual one-dimensional Weierstrass random walk [7] corresponds to a Markov chain governed by the following probability density function

$$\psi(x) = \frac{a-1}{2a}\sum_{n=0}^{\infty} a^{-n}[\delta(x+vJ_n) + \delta(x-vJ_n)], \tag{2}$$

with the "jump" function

$$J_n = J_n^{(\mathrm{NR})} = b^n, \tag{3}$$

where $a > 1$ and $b > 1$ are dimensionless constants and $v > 0$ gives the scale of the jumps. A particle moving according to Eqs. (2) and (3) can perform jumps, in both directions, with magnitude $v, bv, b^2v, b^3v, \ldots$ and with probability, respectively, given by $(a-1)/a, (a-1)/a^2, (a-1)/a^3, (a-1)/a^4, \ldots$. We consider here that each step ℓ lasts for a given and fixed time interval and, hence, the asymptotic dynamics for large times and for large number of steps are identical. One can think the constant v as the velocity acquired by the particle, just prior to the jump, by some unspecific microscopic mechanism. Notice that one can indeed define a continuous-time version of the Weierstrass random walk [8], but for our purposes here, the simple Markov chain governed Eq. (2) is enough.

We are mainly interested in the anomalous diffusion process associated to the Weierstrass random walk. A brief review of the main results concerning this topic is necessary here. We wish to characterize the large L behavior of $\langle X^2(L)\rangle$, where

$$X(L) = x_1 + x_2 + x_3 + \cdots + x_L, \tag{4}$$

with x_ℓ being the size of the ℓ^{th} jump, with probability density function given by Eqs. (2) and (3). Since x_ℓ are independent random variables for different ℓ, we will have

$$\langle X^2(L)\rangle = L\langle x^2\rangle. \tag{5}$$

Notice that

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$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi(x) dx = \frac{a-1}{a} v^2 \sum_{n=0}^{\infty} \left(\frac{b^2}{a}\right)^n, \tag{6}$$

from where we have that, for $b^2 < a$, $\langle x^2 \rangle$ is finite, Eq. (5) reduces to the usual diffusion process with $\mu=1$, and, thanks to the central limit theorem, the total probability density function describing the passage from $\ell=1$ to $\ell=L$ will be very close to a Gaussian. For $b^2 \geq a$, on the other hand, $\langle x^2 \rangle$ diverges, the central limit theorem cannot by applied anymore and the associated distributions will not be Gaussian, leading to an anomalous diffusion exponent $\mu > 1$. We can use the heuristic approach of [9] in order to evaluate μ for this case as well, which corresponds to $\alpha \leq 2$, where

$$\alpha = \frac{\ln a}{\ln b}. \tag{7}$$

The key idea is that, according to Eqs. (2) and (3), if one considers a large number L of steps, the most frequent steps will have magnitude v. Steps with magnitude bv will be $1/a$ less frequent than the previous one. In general, steps with magnitude $b^{n+1}v$ will be $1/a$ less frequent than those ones with magnitude $b^n v$. Also, if we consider large, but finite, number of steps, the summation [Eq. (2)] will be effectively truncated at a given value $n=n_{\max}$. Let us consider the following succession of steps, which exhibits the required hierarchy of jumps,

$$L = a^{n_{\max}} + a^{n_{\max}-1} + a^{n_{\max}-2} + \cdots + 1, \tag{8}$$

corresponding to $a^{n_{\max}}$ steps with magnitude v, $a^{n_{\max}-1}$ with magnitude bv, and so on. For large $n_{\max}$, we have

$$L = \sum_{j=0}^{n_{\max}} a^{n_{\max}-j} \approx \frac{a}{a-1} a^{n_{\max}}. \tag{9}$$

Yet for large but finite $n_{\max}$, we can estimate $\langle x^2 \rangle$ as

$$\frac{\langle x^2 \rangle}{v^2} \approx \frac{a-1}{a} \sum_{j=0}^{n_{\max}} \left(\frac{b^2}{a}\right)^j. \tag{10}$$

For $\alpha > 2$, as we already know, we have the Gaussian result, since

$$\frac{\langle x^2 \rangle}{v^2} \approx \frac{a-1}{a-b^2} < \infty, \tag{11}$$

which coincides with the exacted result evaluated from Eq. (6). For $\alpha < 2$, on the other hand, Eq. (10) implies that

$$\frac{\langle x^2 \rangle}{v^2} \approx \left(\frac{a-1}{a}\right)^{2/\alpha} \frac{a}{b^2-a} L^{2/\alpha-1}, \tag{12}$$

where Eq. (9) was used, leading to a superdiffusive ($\mu=2/\alpha>1$) behavior characterized by

$$\frac{\langle X^2(L) \rangle}{v^2} \approx \left(\frac{a-1}{a}\right)^{2/\alpha} \frac{a}{b^2-a} L^{2/\alpha}. \tag{13}$$

The relativistic kinematics, however, will change dramatically this scenario.

II. RELATIVISTIC WALK

The microscopical origin of the jumps in a nonrelativistic random walk is not relevant from the dynamical point of view. Provided that the Markov property holds, i.e., the position $X(\ell)$ of the system at a given step ℓ depends only on the position at the previous step $X(\ell-1)$, and that successive jumps are independent random variables, the standard approach to lead with random walks [1] can be applied. Let us, nevertheless, suppose that the jumps are due to microscopical collisions, as in the historical example of Brownian motion. The hierarchy of jumps $v, bv, b^2v, b^3v, \ldots$ is, in the nonrelativistic case, associated with a similar hierarchy of momentum transfers in the collisions $p, bp, b^2p, b^3p, \ldots$, with $p=m_0 v$, where m_0 is the particle rest mass. However, according to special relativity, the velocity and the momentum of a particle should obey

$$p = \frac{m_0 v}{\sqrt{1-\beta^2}}, \tag{14}$$

where $\beta=v/c$, with profound implications for the jump hierarchy. Assuming that the same hierarchy of momentum transfers is present in the relativistic case, we will have the following jump function

$$J_n = J_n^{(\mathrm{R})} = \frac{b^n}{\sqrt{1+\beta^2 b^{2n}}}, \tag{15}$$

instead of Eq. (3). The hierarchy of relativistic jumps will be $\{J_n v\}$, occurring in the dynamics, respectively, with probability $\{(a-1)/a^{n+1}\}$, with $n=0,1,2,\ldots$. Notice that, in frank contrast with the nonrelativistic case, for large n, one has $J_n v \approx c$, meaning that there will be no arbitrarily large jumps in the relativistic case, in agreement with the fact that no acquired velocity by any microscopical mechanism can exceed c. The first conclusion we can draw from this relativistic extension of the Weierstrass random walk is that the whole process must be Gaussian, since, in this case, we have

$$\frac{\langle x^2 \rangle}{v^2} = \frac{a-1}{a} \sum_{n=0}^{\infty} \frac{(b^2/a)^n}{1+\beta^2 b^{2n}} < \infty, \tag{16}$$

for any $\beta > 0$, implying the usual diffusion with $\mu=1$, irrespective of the value of α.

Typical nonrelativistic situations are characterized by a small β. For such cases, from Eq. (16), one realizes that there should exist a critical value n_c such that, for $n \ll n_c$, the summand of Eq. (16) is essentially the same one of the nonrelativistic case. If it is possible to choose a large $n_{\max} \ll n_c$, one could apply the same heuristic approach of last section, implying in a superdiffusive behavior with $\mu=2/\alpha>1$. (We assume hereafter that $0<\alpha<2$.) This occurs because L is large enough to justify the averages of last section, but it is still small enough to guarantee that the large n events that would imply the relativistic regime are extremely rare and will not contribute effectively to the averages. In other words, the system needs some time to realize that it is indeed relativistic. If we allow for $n_{\max} \gg n_c$, we will have in Eq. (16) the convergent relativistic summation, implying the usual diffusion with exponent $\mu=1$. It is clear that

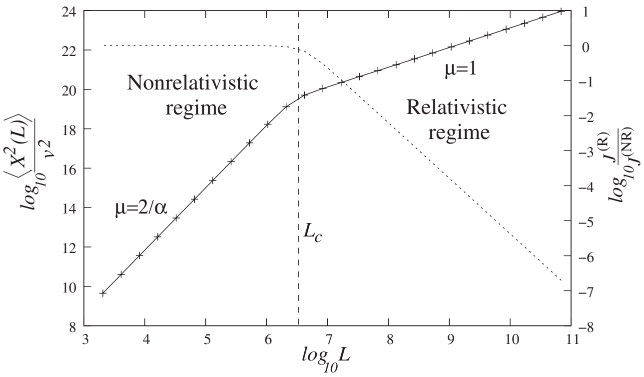

FIG. 1. Diffusion in the relativistic Weierstrass random walk. The solid line (left scale) corresponds to Eq. (19), which, for large n_{max}, is well approximated by Eq. (20). The dotted line (right scale) is the ratio between the nonrelativistic [Eq. (3)] and relativistic [Eq. (15)] jump functions. It is clear the appearance of a crossover between the different diffusion regimes near L_c given by Eq. (18). For $L < L_c$, we have Lévy-type superdiffusive behavior, while for $L > L_c$ the dynamics settle into a Gaussian diffusion. This plot corresponds to the case where $a=2$, $b=3$, and $\beta=10^{-10}$.

$$n_c = -\frac{\ln \beta}{\ln b} = \alpha \frac{\ln \beta^{-1}}{\ln a}, \tag{17}$$

leading to

$$L_c = \frac{a}{a-1}\frac{1}{\beta^{\alpha}}, \tag{18}$$

where Eq. (9) was used. If β is small, as one expects in the typical nonrelativistic problems, L_c will be large. For $L \ll L_c$, the system behaves as in the nonrelativistic regime and exhibits the properties of a Lévy flight with $\mu = 2/\alpha$. On the other hand, for $L \gg L_c$, the system change its behavior to the relativistic regime, characterized by ordinary diffusion with $\mu = 1$. Such crossover between different diffusion regimes, depicted in Fig. 1, is compatible with the results of [10]. In fact, our model can be viewed as a simple microscopical realization for the generalized Fokker-Planck equation considered there.

We can apply the same heuristic approach of last section to the relativistic case. In particular, we have for a finite and large n_{max}

$$\frac{\langle X^2(L)\rangle}{v^2} \approx \frac{a-1}{a} L \sum_{n=0}^{n_{max}} \frac{(b^2/a)^n}{1+\beta^2 b^{2n}}, \tag{19}$$

leading to the following generalization of Eq. (13) for $\beta \neq 0$ (see the Appendix for details)

$$\frac{\langle X^2(L)\rangle}{v^2} \approx \frac{1}{\ln(b^2/a)}\Bigg[\left(\frac{a-1}{a}\right)^{2/\alpha} L^{2/\alpha} \times F\left(1,1-\frac{\alpha}{2};2-\frac{\alpha}{2};-\left(\frac{L}{L_c}\right)^{2/\alpha}\right) - \frac{a-1}{a}L \times F\left(1,1-\frac{\alpha}{2};2-\frac{\alpha}{2};-\beta^2\right)\Bigg], \tag{20}$$

where $F(a,b;c;z)$ stands for the standard hypergeometric function [11]. Using that $F(a,b;c;0)=1$, we have from Eq. (20) an anomalous diffusion process with $\mu = 2/\alpha$. for large L obeying $L \ll L_c$. On the other hand, for $L \gg L_c$, we have (see the Appendix for details)

$$F\left(1,1-\frac{\alpha}{2};2-\frac{\alpha}{2};-\left(\frac{L}{L_c}\right)^{2/\alpha}\right) \propto \left(\frac{L}{L_c}\right)^{1-2/\alpha}, \tag{21}$$

leading to the usual Gaussian diffusion

$$\frac{\langle X^2(L)\rangle}{v^2} \propto L, \tag{22}$$

for $L \gg L_c$.

III. DISCUSSION

The results of the preceding sections can be summarized as follows. Suppose we have a Weierstrass random walk model with typical velocity v, implying in Lévy flights characterized by an anomalous diffusion exponent $\mu > 1$. Then, relativistic effects imply that, after a certain critical number of steps $L_c \approx (c^2/v^2)^{1/\mu}$, the system loses it anomalous diffusion properties and the dynamics necessarily settle into a Gaussian diffusion. In order to estimate the order of magnitude of these relativistic effects, let us associate the kinetic energy of the walking particle, and thus v, with the typical thermal energy $k_B T$. For nonrelativistic situations, where $k_B T$ is small if compared with $m_0 c^2$, we will have $m_0 v^2/2 \approx k_B T$, leading to

$$L_c \approx \left(\frac{m_0 c^2}{2 k_B T}\right)^{1/\mu}. \tag{23}$$

As our first explicit example, let us consider a system composed by helium atoms, for which $m_0 c^2/k_B \approx 4.36 \times 10^{13}$ K. For such a system at room temperature ($T \approx 300$ K), a ballistic ($\mu=2$) Lévy flight originated in a Weierstrass random walk, will become Gaussian due to relativistic effects after $L_c \approx 2.7 \times 10^5$ steps. Helium atoms at the surface of the sun ($T \approx 5 \times 10^3$ K) can experiment ballistic Lévy flights in a Weierstrass random walk for no more than $L_c \approx 6.6 \times 10^4$ steps. In the interior of the sun ($T \approx 5 \times 10^6$ K), only $L_c \approx 2000$ steps will be enough for the dynamics settle into a essentially Gaussian regime.

Heavier particles or bodies will, naturally, lead to larger values for L_c. Let us take, for instance, the case of an *Escherichia coli* bacterium, for which $m_0 = 665$ femtograms [12], leading to $m_0 c^2/k_B \approx 4.33 \times 10^{24}$ K. At room temperature, $L_c \approx 4.1 \times 10^{15}$ for an *Escherichia coli* undergoing a Lévy flight originated in a Weierstrass random walk with diffusion exponent close to those ones observed experimentally [13] in systems of breakable micelles ($\mu \approx 1.4$). One realizes that such microscopic bodies can indeed experience much longer anomalous diffusion process than atomic scale particles.

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APPENDIX

We can approximate Eq. (19) by an integral by using

$$\sum_{n=0}^{n_{\max}} \frac{(b^2/a)^n}{1+\beta^2 b^{2n}} \approx \frac{1}{2\ln b}\int_1^{b^{2n_{\max}}} \frac{w^{-\alpha/2}}{1+\beta^2 w}dw, \quad \text{(A1)}$$

where variable $w=b^{2n}$ was introduced and α is, in general, an irrational. For $|\beta^2 w|<1$, we can introduce the following series expansion:

$$\frac{1}{1+\beta^2 w} = \sum_{n=0}^{\infty}(-\beta^2 w)^n, \quad \text{(A2)}$$

and the integral Eq. (A1) will be written as

$$\int \frac{w^{-\alpha/2}}{1+\beta^2 w}dw = w^{1-\alpha/2}\sum_{n=0}^{\infty}\frac{(-\beta^2 w)^n}{n+1-\frac{\alpha}{2}} = \frac{2}{2-\alpha}w^{1-\alpha/2}F\left(1,1-\frac{\alpha}{2};2-\frac{\alpha}{2};-\beta^2 w\right), \quad \text{(A3)}$$

where $F(a,b;c;z)$ is the standard hypergeometric function [11]. Notice that the hypergeometric functions have a single valued analytical extension over the entire complex plane, with the only exception of the positive real axis for $z\geq 1$ [11], justifying the use of Eq. (A3) for the evaluation of the integral Eq. (A1), which limits, in fact, do not belong to the region where the expansion [Eq. (A2)] converges. The integral Eq. (A1) may also be evaluated by exploring the hypergeometric function identities [11]

$$\frac{d}{dz}(z^{c-1}F(1,b;c;z)) = (c-1)z^{c-2}F(1,b;c-1;z), \quad \text{(A4)}$$

and $F(1,b;b;z)=(1-z)^{-1}$, valid for any b and c. Equation (20) follows straightforwardly from Eq. (A3).

The evaluation of Eq. (20) for $L\gg L_c$ requires an asymptotic analysis for the hypergeometric function. By using, for instance, the identity 15.3.8 of [11], we have

$$F\left(1,1-\frac{\alpha}{2};2-\frac{\alpha}{2};1-z\right) = \frac{2-\alpha}{\alpha}z^{-1}F\left(1,1;\frac{\alpha}{2}-1;z^{-1}\right) + \frac{\pi\left(1-\frac{\alpha}{2}\right)}{\sin\pi\frac{\alpha}{2}}z^{\alpha/2-1} \times F\left(1-\frac{\alpha}{2},1-\frac{\alpha}{2};1-\frac{\alpha}{2};z^{-1}\right), \quad \text{(A5)}$$

implying that, for large z and $0<\alpha<2$,

$$F\left(1,1-\frac{\alpha}{2};2-\frac{\alpha}{2};-z\right) \approx \frac{\pi\left(1-\frac{\alpha}{2}\right)}{\sin\pi\frac{\alpha}{2}}z^{\alpha/2-1}, \quad \text{(A6)}$$

leading finally to Eq. (21).

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