Phase transition in a class of nonlinear random networks

M. Andrecut[1] and S. A. Kauffman[2]

[1]*Institute for Space Imaging Science, University of Calgary, Calgary, Alberta, Canada T2N 1N4*
[2]*Institute for Biocomplexity and Informatics, University of Calgary, Calgary, Alberta, Canada T2N 1N4*

We discuss the complex dynamics of a nonlinear random networks model as a function of the connectivity k between the elements of the network. We show that this class of networks exhibits an order-chaos phase transition for a critical connectivity $k_c=2$. Also, we show that both pairwise correlation and complexity measures are maximized in dynamically critical networks. These results are in good agreement with the previously reported studies on random Boolean networks and random threshold networks, and show once again that critical networks provide an optimal coordination of diverse behavior.

I. INTRODUCTION

Random Boolean networks (RBNs) are a class of complex systems that show a well-studied transition between ordered and disordered phases. The RBN model was initially introduced as an idealization of genetic regulatory networks. Since then, the RBN model has attracted much interest in a wide variety of fields, ranging from cell differentiation and evolution to social and physical spin systems (for a review of the RBN model, see [1,2], and the references within). The dynamics of RBNs can be classified as ordered, disordered, or critical, as a function of the average connectivity k, between the elements of the network, and the bias p in the choice of Boolean functions. For equiprobable Boolean functions, $p=1/2$, the critical connectivity is $k_c=2$. The RBNs operating in the ordered regime ($k<k_c$) exhibit simple dynamics and are intrinsically robust under structural and transient perturbations. In contrast, the RBNs in the disordered regime ($k>k_c$) are extremely sensitive to small perturbations, which rapidly propagate throughout the entire system. Recently, it has been shown that the pairwise mutual information exhibits a jump discontinuity at the critical value k_c of the RBN model [3]. More recently, similar results have been reported for a related class of discrete dynamical networks, called random threshold networks (RTNs) [4].

In this Brief Report we consider a nonlinear random network (NLRN) model, which represents a departure from the discrete valued state representation, corresponding to the RBN and RTN models, to a continuous valued state representation. We discuss the complex dynamics of the NLRN model as a function of the average connectivity (in-degree) k. We show that the NLRN model exhibits an order-chaos phase transition, for the same critical connectivity value $k_c=2$, as the RBN and RTN models. Also, we show that both pairwise correlation and complexity measures are maximized in dynamically critical networks. These results are in very good agreement with the previously reported studies on the RBN and RTN models, and show once again that critical networks provide an optimal coordination of diverse behavior.

II. NLRN MODEL

The NLRN model consists of N randomly interconnected variables, with continuously valued states $-1 \le x_n \le +1$,

where $n=1,\ldots,N$. At time t the state of the network is described by an N-dimensional vector

$$\mathbf{x}(t) = [x_1(t), \ldots, x_N(t)]^T, \tag{1}$$

which is updated at time $t+1$ using the following map:

$$\mathbf{x}(t + 1) = f(\mathbf{w}, \mathbf{x}(t)), \tag{2}$$

where

$$f(\mathbf{w}, \mathbf{x}(t)) = [f_1(\mathbf{w}, \mathbf{x}(t)), \ldots, f_N(\mathbf{w}, \mathbf{x}(t))]^T, \tag{3}$$

$$f_n(\mathbf{w}, \mathbf{x}(t)) = \tanh\left(\sum_{m=1}^{N} w_{nm} x_m(t) + x_0\right), \quad n = 1, \ldots, N. \tag{4}$$

Here, $\mathbf{w}$ is an $N \times N$ interaction matrix, with the following randomly assigned elements:

$$w_{nm} = \begin{cases} -1 & \text{with probability} \quad \dfrac{k}{2N} \\[2mm] 0 & \text{with probability} \quad \dfrac{N-k}{N} \\[2mm] +1 & \text{with probability} \quad \dfrac{k}{2N}, \end{cases} \tag{5}$$

and k is the average in-degree of the network.

The interaction weights can be interpreted as excitatory, if $w_{nm}=1$, and inhibitory, if $w_{nm}=-1$. Also, we have $w_{nm}=0$ if x_m is not an input to x_n. Obviously, the threshold x_0 can be considered as a constant input, with a fixed weight $w_{n0}=1$, to each variable x_n. Therefore, in the following discussion we do not lose generality by assuming that the threshold parameter is always set to $x_0=0$.

III. PHASE TRANSITION

In order to illustrate the complex dynamics of the NLRN system, we consider the results of the simulation of three networks, each containing $N=128$ variables, and having different average in-degrees: $k=1$, $k=2$, and $k=4$. Also, the continuous values of the variables $x_n(t)$ are encoded in
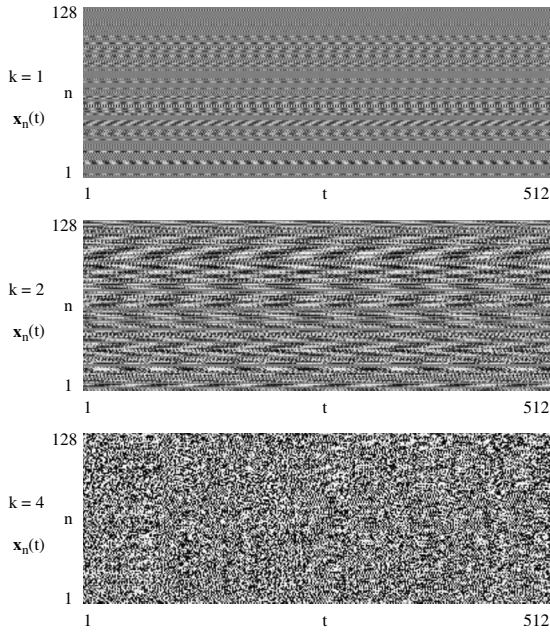
FIG. 1. Three qualitatively different types of behavior or the NLRN model: ordered ($k=1$), critical ($k=2$), and chaotic ($k=4$) (the transient has been removed).

shades of gray, with black and white corresponding to the extreme values of ± 1. In Fig. 1, one can easily see the three qualitatively different types of behavior: ordered ($k=1$), critical ($k=2$), and chaotic ($k=4$).

A quantitative characterization of the transition from the ordered phase to the chaotic phase is given by the Lyapunov exponents [5], which measure the rate of separation of infinitesimally close trajectories of a dynamical system. The linearized dynamics in tangent space is given by

$$\delta \mathbf{x}(t+1) = \mathbf{J}(\mathbf{w}, \mathbf{x}(t)) \delta \mathbf{x}(t), \tag{6}$$

where $\mathbf{J}$ is the Jacobian of the map f, with the elements

$$J_{nm} = \frac{\partial f_n}{\partial x_m} = w_{nm}\left[1 - \tanh^2\left(\sum_{m=1}^{N} w_{nm} x_m(t) \right) \right], \tag{7}$$

and $\delta \mathbf{x}(t)$ is the separation vector. The dynamics of $\delta \mathbf{x}(t)$ is typically very complex, involving rotation and stretching. Therefore, the rate of separation can be different for different orientations of the initial separation vector, such that one obtains a whole spectrum of Lyapunov exponents. In general, there are N possible values, which can be ordered: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. These Lyapunov exponents are associated with the Lyapunov vectors, $v_1, v_2, \ldots, v_N$, which form a basis in the tangent space. A perturbation along v_n will grow exponentially with a rate λ_n. Oseledec's theorem [6] proves that the following limit exists:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\delta \mathbf{x}(t)\|}{\|\delta \mathbf{x}(0)\|}. \tag{8}$$

We should note that Oseledec's limit will always correspond to λ_1, because an initial random perturbation will always have a component along the most unstable direction, v_1, and
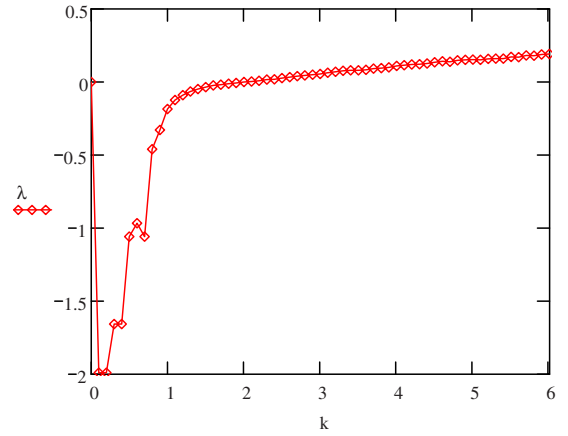

FIG. 2. (Color online) The maximal Lyapunov exponent of the NLRN model as a function of the connectivity: $\lambda(k)$.

because the exponential growth rate the effect of the other exponents will be obliterated over time. Thus, in general, it is enough to consider only the maximal Lyapunov exponent (MLE), which is enough to characterize the behavior of the dynamical system [5]. A negative MLE corresponds to an ordered system (fixed points and periodic dynamics), while a positive MLE is an indication that the system is chaotic. A zero MLE is associated with quasiperiodic dynamics and corresponds to the critical transition. Figure 2 shows the MLE as a function of the average in-degree, $\lambda(k)$. One can see that the critical in-degree is $k_c=2$, such that for $k<k_c$ the NLRNs are ordered, and for $k>k_c$ the NLRNs become chaotic. The numerical results were obtained by averaging over the NLRN ensemble for each k, using $M=256$ NLRNs with $N=256$ elements. Also, for each time series we have discarded the first 1024 steps, in order to eliminate the transient, and the MLE was calculated from the next 1024 steps.

In order to provide a more detailed characterization of the order-chaos phase transition we introduce the following spectral complexity measure:

$$Q_\omega = H_\omega D_\omega, \tag{9}$$

where H_ω is the spectral entropy and D_ω is the spectral disequilibrium. The complexity is defined by the interplay of two antagonistic behaviors: the increase in entropy as the system becomes more and more disordered and the decrease in the disequilibrium as the system approaches chaos (equiprobability). A similar complexity measure, evaluated in the direct (time) space, was introduced in [7], for discrete state systems. In contrast, our complexity measure is defined for continuous state systems, and it is evaluated in the inverse (frequency) space.

In order to define the spectral entropy [8], we consider the discrete Fourier transform (DFT)

$$\mathbf{X}_n(\omega) = F_\omega[\mathbf{x}_n(t)] = [X_n(1), \ldots, X_n(\Omega)]^T, \tag{10}$$

$$X_n(\omega) = \sum_{t=1}^{T} x_n(t) \exp(-2\pi i \omega t / T), \quad \omega = 1, \ldots, \Omega, \tag{11}$$

and the power spectrum

$$\mathbf{Y}_n(\omega) = [Y_n(1), \ldots, Y_n(\Omega)]^T, \tag{12}$$

$$Y_n(\omega) = X_n^*(\omega)X_n(\omega) = |X_n(\omega)|^2, \quad \omega = 1, \ldots, \Omega, \tag{13}$$

of the time series

$$\mathbf{x}_n(t) = [x_n(1), \ldots, x_n(T)]^T, \tag{14}$$

corresponding to the attractor of the variable n of a given NLRN. Here, X_n^* stands for the complex conjugate value. Since the variables $x_n(t)$ are real, the DFT result has the following symmetry:

$$X_n(\omega) = X_n^*(T - \omega), \tag{15}$$

and therefore the power spectrum $\mathbf{Y}_n(\omega)$ has only $\Omega = T/2$ positive values.

One can normalize the power spectrum such that

$$p_n(\omega) = \frac{Y_n(\omega)}{\sum\limits_{\omega=1}^{\Omega} Y_n(\omega)}, \quad \omega = 1, \ldots, \Omega, \tag{16}$$

$$\sum_{\omega=1}^{\Omega} p_n(\omega) = 1. \tag{17}$$

The new variable $p_n(\omega)$ can be interpreted as the probability of having the frequency ω *embedded* in the time series $\mathbf{x}_n(t)$. Thus, using the spectral probability vector

$$\mathbf{p}_n(\omega) = [p_n(1), \ldots, p_n(\Omega)]^T, \tag{18}$$

one can define the spectral entropy of the time series $\mathbf{x}_n(t)$ as follows:

$$H_\omega[\mathbf{p}_n(\omega)] = -\frac{1}{\log_2 \Omega} \sum_{\omega=1}^{\Omega} p_n(\omega)\log_2 p_n(\omega), \tag{19}$$

where $\log_2 \Omega$ is the normalization constant, such that $0 \leq H_\omega \leq 1$.

Obviously, the spectral entropy of the ordered systems will be low, $H_\omega \sim 0$, since only a very small number of frequencies are present, while the spectral entropy of chaotic systems will be high, $H_\omega \sim 1$, since a large number of frequencies are present. The spectral entropy takes the maximum value, $H_\omega = 1$, for the equilibrium state, which is defined deep in the chaotic regime, where all frequencies become equiprobable: $p(\omega) = \Omega^{-1}$, with $\omega = 1, \ldots, \Omega$.

The spectral disequilibrium of the time series $\mathbf{x}_n(t)$ measures the displacement of the corresponding probability vector $\mathbf{p}_n(\omega)$ from the equilibrium state, and it is defined as follows:

$$D_\omega[\mathbf{p}_n(\omega), \Omega^{-1}] = \sum_{\omega=1}^{\Omega} [p_n(\omega) - \Omega^{-1}]^2. \tag{20}$$

Special attention is necessary in the case when the attractor is zero: $\mathbf{x}_n(t) = 0$. In this particular case, the power spectrum is also zero, $\mathbf{Y}_n(\omega) = 0$, and the probability vector $\mathbf{p}_n(\omega)$ is undetermined. In order to overcome this difficulty, we define $H_\omega = 0$ and $D_\omega = 1$ for this particular attractor, such that it has the lowest entropy and the largest displacement from equilibrium.

Since the spectral disequilibrium measures the distance between two distributions, one may consider also the spectral Kullback-Leibler divergence [9] as an alternative. However, for the considered NLRN model, the Kullback-Leibler divergence is simply given by

$$D_\omega^{KL}[\mathbf{p}_n(\omega) \| \Omega^{-1}] = \frac{1}{\log_2 \Omega} \sum_{\omega=1}^{\Omega} p_n(\omega)\log_2\left(\frac{p_n(\omega)}{\Omega^{-1}}\right)$$

$$= 1 - H_\omega[\mathbf{p}_n(\omega)]. \tag{21}$$

Similarly, one can show that the symmetrical Kullback-Leibler divergence is given by

$$D_\omega^{KL}[\mathbf{p}_n(\omega) \| \Omega^{-1}] + D_\omega^{KL}[\Omega^{-1} \| \mathbf{p}_n(\omega)]$$

$$= -H_\omega[\mathbf{p}_n(\omega)] - \frac{1}{\Omega \log_2 \Omega} \sum_{\omega=1}^{\Omega} \log_2 p_n(\omega). \tag{22}$$

Therefore, in this case, the Kullback-Leibler divergence (or its symmetrical version) can be expressed in terms of entropy. Thus, the spectral disequilibrium seems to be a more appropriate distance measure since it cannot be expressed in terms of entropy.

Another quantity of interest is the pairwise spectral correlation between the power spectra of two network variables n and m, which is defined as

$$C_\omega[\mathbf{Y}_n, \mathbf{Y}_m] = \frac{(\mathbf{Y}_n - \bar{\mathbf{Y}}_n)^T(\mathbf{Y}_m - \bar{\mathbf{Y}}_m)}{\|\mathbf{Y}_n - \bar{\mathbf{Y}}_n\|\|\mathbf{Y}_m - \bar{\mathbf{Y}}_m\|}, \tag{23}$$

where $\bar{\mathbf{Y}}_n$ and $\bar{\mathbf{Y}}_m$ represent the mean values. The average correlation for a given NLRN is

$$C_\omega = \frac{1}{N(N-1)} \sum_{n=1}^{N} \sum_{m=1}^{N} [1 - \delta(n,m)]C_\omega[\mathbf{Y}_n, \mathbf{Y}_m], \tag{24}$$

where we have excluded the self-correlation terms [$\delta(n,m) = 1$ if $m = n$, and $\delta(n,m) = 0$ if $m \neq n$].

In Fig. 3 we give the numerical results for the above spectral measures (entropy, disequilibrium, complexity, and correlation), obtained by averaging over the NLRN ensemble ($M = 256$ networks with $N = 256$ elements and $T = 1024$). One can see that both the complexity and the correlation measures are maximized by the critical NLRNs with $k_c = 2$.

As mentioned at the beginning of this Brief Report, the continuous NLRN model is directly related to the binary RTN model, which has been extensively studied recently [4,10]. Recently, we have investigated the binary RTN model, using similar quantities—complexity, entropy, and the mutual information—which are well defined in the time domain. The obtained results for both NLRN and RTN models are in very good agreement, showing a phase transition for the same critical connectivity $k_c = 2$. Also, for the RTN model, we have shown that the mutual information, which is the binary counterpart of the spectral correlation, is maximized for $k_c = 2$. Similar results have also been previously reported for the RBN model [3].
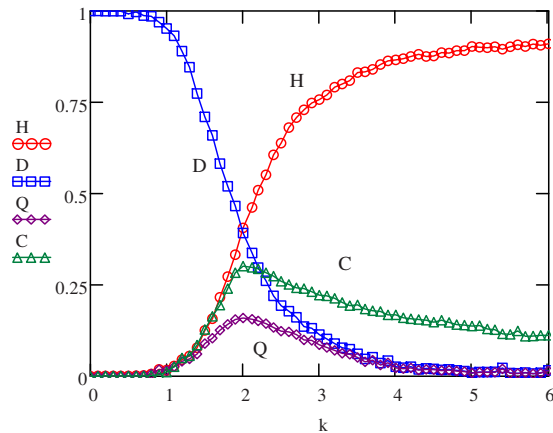
FIG. 3. (Color online) The spectral measures of the NLRN model as functions of connectivity: entropy $H_\omega(k)$, disequilibrium $D_\omega(k)$, complexity $Q_\omega(k)$, and correlation $C_\omega(k)$.

IV. CONCLUSION

We have shown numerically that the NLRN model exhibits an order-chaos phase transition, for the same critical connectivity value $k_c = 2$, as the RBN and RTN models. Also, we have shown that both the pairwise correlation and the complexity measures are maximized in dynamically critical networks. These results are in very good agreement with the previously reported studies on the RBN and RTN models, and show once again that critical networks provide an optimal coordination of diverse behavior. We would like also to note that these optimal properties of critical networks are likely to play a major role in biological systems, perhaps serving as important selective traits. Given the potential biological implications, it is of interest that recent data suggest that genetic regulatory networks in eukaryotic cells are dynamically critical [11]. Also, recent experiments conducted on rat brain slices show that these neural tissues are critical [12]. Thus, it seems plausible that in cells, neural systems, and other tissues, natural selection will have acted to maximize both the correlation across the network and the diversity of complex behaviors that can be coordinated within a causal network. Ordered networks have convergent trajectories, and hence forget their past. Chaotic networks show sensitivity to initial conditions, and thus they too forget their past and are unable to act reliably. On the other hand, critical networks, with trajectories that on average neither diverge nor converge (quasiperiodic dynamics), seem best able to bind past to future and therefore to maximize the correlated complex behavior.

[1] S. A. Kauffman, *The Origins of Order: Self-Organization and Selection in Evolution* (Oxford University Press, New York, 1993).

[2] M. Aldana, S. Coopersmith, and L. P. Kadanoff, in *Perspectives and Problems in Nonlinear Science*, edited by Ehud Kaplan, Jerrold E. Marsden, and Katepalli R. Sreenivasan, Springer Applied Mathematical Sciences Series (Springer, New York, 2003), pp. 23–89.

[3] A. S. Ribeiro, S. A. Kauffman, J. Lloyd-Price, B. Samuelsson, and J. E. S. Socolar, Phys. Rev. E **77**, 011901 (2008).

[4] M. Andrecut, D. Foster, H. Carteret, and S. A. Kauffman, J. Comput. Biol. **16**, 909 (2009).

[5] K. T. Alligood, T. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems* (Springer-Verlag, New York, 1997).

[6] V. I. Oseledec, Trudy Moskow Mat. Obshch. **19**, 179 (1968) (in Russian).

[7] R. López-Ruiz, H. L. Mancini, and X. Calbet, Phys. Lett. A **209**, 321 (1995).

[8] G. E. Powell, J. Phys. A **12**, 2053 (1979).

[9] S. Kullback and R. A. Leibler, Ann. Math. Stat. **22**, 79 (1951).

[10] T. Rohlf, Phys. Rev. E **78**, 066118 (2008).

[11] I. Shmulevich, S. A. Kauffman, and M. Aldana, Proc. Natl. Acad. Sci. U.S.A. **102**, 13439 (2005).

[12] D. Hsu and J. M. Beggs, Neurocomputing **69**, 1134 (2006).