Degenerate orbit transitions in a one-dimensional inelastic particle system

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Continuous transitions between different periodic orbits in a one-dimensional inelastic particle system are investigated. We show that continuous transitions that occur when adding or subtracting a single collision are, generically, of co-dimension 2. We give a full mechanical description of the system and explain why this is the case. Surprisingly, we also show that there are an infinite set of degenerate transitions of co-dimension 1. We provide a theoretical analysis that gives a simple criteria to classify which transitions are degenerate purely using the discrete set of collisions that occur in the orbits. Our analysis allows us to understand the nature of the degeneracy. We also show that higher degrees of degeneracy can occur, and provide an explanation.

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I. INTRODUCTION

Many industrial applications involve impacts between discrete masses. Typical examples include pneumatic drills, vibration hammers, and types of piercing machines. The interactions between the masses are highly dissipative and so such machines must be subjected to externally applied driving. Finding stable operating conditions, that avoid chattering and excessive wear is crucial in a number of applications. For this reason, the study of periodic behavior in such systems represents an important area of research.

Although systems that contain small numbers of particles appear to be extremely simple, they can actually exhibit numerous types of highly complicated behavior. Mehta and Luck [1] and Luck and Mehta [2] considered a single particle moving under gravity on a vibrating plate. The periodic behavior of this system has also been studied more recently by Gilet *et al.* [3]. Despite the apparent simplicity of the system, abrupt termination of period doubling can occur and arbitrarily long period sequences can be shown to exist. Two particles that experience gravity and are constrained to move in one-dimension above a stationary floor have been considered by Whelan *et al.* [4].

The maximum number of collisions that can occur in a one-dimensional unconstrained system with a finite number of particles has been studied by Murphy [5]. In two or more dimensions, the system has also been studied by Murphy and Cohen [6]. They showed that the maximum number of collisions between three rigid particles is four if the dimension of the system is two or greater. Another interesting phenomena that can occur is inelastic collapse, in which particles experience an infinite number of collisions in finite time. The nature and stability of inelastic collapse has been studied extensively [7–10]. Recently, one-dimensional particle systems have received significant attention [11–19]. At the microscopic level, Yang [17] showed that nonuniqueness of periodic orbits can exist in a driven inelastic particle system. Wylie and Zhang [18] showed that large numbers of periodic orbits in such systems can coalesce onto a single orbit at a critical parameter value. The diffusion of particles in a quasi-one-dimensional setting has also received significant attention [20–27]. In particular, the asymptotic mean squared displacement has been determined [25] and careful colloid experiments have shown excellent agreement with theoretical predictions [27].

In this paper, we will study the periodic orbits of a simple one-dimensional inelastic particle system. Determining which regions of parameter space admit periodic behavior is an important goal in understanding such systems. Moreover, a detailed knowledge of the transitions between periodic orbits is crucial for a complete understanding of the underling dynamics. In applications, knowledge of the periodic orbits and the characteristic frequencies of mechanical systems is important in understanding resonant behavior. Therefore, we will focus on the continuous transitions between different periodic orbits. We develop the analytic machinery required to determine the region of parameter space in which any given periodic orbit can exist. We use this machinery to explain the nature of the different continuous transitions that occur. We will study the case of a periodic orbit changing into another periodic orbit via the addition or subtraction of a single collision. Generically, we will show that two conditions are required for a continuous transition of this type. That is, continuous orbit transitions are, generically, of co-dimension 2. We also carefully explain why this is the case from a mechanical viewpoint. Although most continuous orbit transitions are indeed of co-dimension 2, we also show that there are an infinite set of degenerate transitions of co-dimension 1. We provide a theoretical analysis that can explain the nature of the degeneracy. Moreover, we show how one can classify continuous orbit transitions as being degenerate or nondegenerate purely using the sequences of collisions in the respective orbits. We also show that higher degrees of degeneracy can occur.

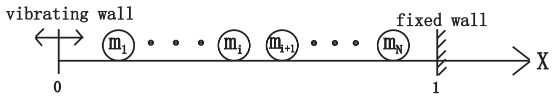


FIG. 1. Schematic of the N-particle system. The left wall vibrates and is located at $X=0$ while the right wall is fixed and located at $X=1$ in dimensionless units. The mass of the ith particle is denoted by m_i.

This paper is organized as follows. In Sec. II, we present the formulation and phenomena. In Sec. III, we present the methodology for the analytic construction of periodic orbits. In Sec. IV, we give a detailed discussion of the continuous orbit transitions. Finally, in Sec. V, we give the conclusion.

II. FORMULATION AND PHENOMENA

We consider a system of N perfectly rigid particles, whose motion is constrained on a line between two walls. We consider the case in which all of the masses of the particles can be different, and denote the mass of the ith particle as m_i (see Fig. 1).

We define a coordinate, X, to represent the location on the line. The particles interact via inelastic collisions which conserve momentum but dissipate kinetic energy. If there is no energy input, then the particles will eventually come to rest and in order to maintain a nontrivial state one must add energy to the system. We do this by vibrating one of the boundaries. We assume that the left wall is vibrating and the right wall is fixed. For simplicity we adopt a "sawtooth" motion [28] for the left wall, in which the wall moves with a constant speed V over a distance δ before executing an instantaneous jump back to its starting location. This means that the left vibrating wall is moving with speed V whenever the first particle collides with it. Furthermore, the distance δ is assumed to be much smaller than the distance between the two walls, and thus, we assume all collisions between the left vibrating wall and the first particle must occur at the same point. We define the location of the left wall to be $X=0$ (see Fig. 1).

We only consider point particles, since the physical size of the particles does not affect the motion. This is because the particles are rigid and so in a one-dimensional setting, a system with N particles of diameter γ moving between two walls seperated by a distance l is completely equivalent to a system with N point particles moving between two walls seperated by a distance $l-\gamma N$. We choose scales such that the distance between the two walls and the speed of the left wall are unity, and set m_1 to be unity by nondimensionalizing all of the masses of the particles using the mass of the first particle. Particles can only collide with their adjacent neighbors and we denote the coefficient of restitution between the ith and $(i+1)$-th particles as e_i for $i=1,2,\cdots,N-1$. Furthermore, we denote the coefficient of restitution between the left wall and the first particle as e_0 and the coefficient of restitution between the Nth particle and the right wall as e_N. In between collisions, the particle velocities remain constant.

When the first particle hits the left vibrating wall, kinetic energy is added and velocities of the particles are updated using the operator L defined by

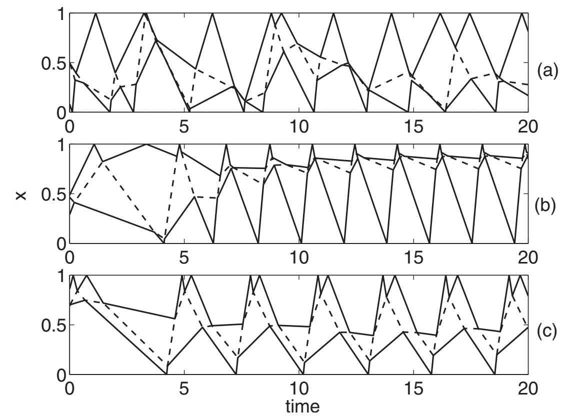


FIG. 2. The trajectories of three inelastic particles with coefficient of inelasticity $e_0=e_3=1$ and $e_1=e_2=1/2$. The mass of the first particle is unity and the mass of the second particle is 1.15. The masses of the third particle in (a), (b), and (c) are 2.1, 1.35, and 0.8, respectively. The horizontal axis represents dimensionless time while the vertical axis represents the dimensionless location in the domain. (a) does not evolve to any low-order periodic orbit during the time of simulation, (b) evolves to a periodic orbit with sequence $C_2C_1C_2RC_2C_1L$, and (c) evolves to a periodic orbit with sequence $C_2C_1RC_2RC_2C_1L$.

$$L\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} 1+e_0-e_0v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} 1+e_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} -e_0 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}, \quad (1)$$

where v_i is the velocity of the ith particle. For convenience we introduce the notation

$$T=\begin{pmatrix} -e_0 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

When the Nth particle hits the right wall, velocities of the particles are updated using the operator R defined by

$$R\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \\ -e_Nv_N \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \dots & 0 & -e_N \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}. \quad (2)$$

That is, we can consider R as a matrix. When the ith and $(i+1)$-th particles collide, velocities of the particles are updated using the operator C_i defined by

$$C_i = \left(\begin{array}{cccccccc} 1 & 0 & & & & & & \\ 0 & \ddots & \ddots & & & & & \\ & \ddots & 1 & 0 & & & & \\ & & 0 & \dfrac{m_i - m_{i+1}e_i}{m_i + m_{i+1}} & \dfrac{m_{i+1}(1+e_i)}{m_i + m_{i+1}} & & & \\ & & & \dfrac{m_i(1+e_i)}{m_i + m_{i+1}} & \dfrac{m_{i+1} - m_i e_i}{m_i + m_{i+1}} & 0 & & \\ & & & & 0 & 1 & \ddots & \\ & & & & & \ddots & \ddots & 0 \\ & & & & & & 0 & 1 \end{array}\right) . \begin{array}{l} \\ \\ \\ \leftarrow i\text{th } row \\ \leftarrow (i+1)-th\ \ row \\ \\ \\ \\ \end{array} \tag{3}$$

We define $V^{(k)}=[V_1^{(k)},V_2^{(k)},\cdots,V_N^{(k)}]^T$ to be the velocities of the N particles after the kth collision and $X^{(k)}=[X_1^{(k)},X_2^{(k)},\cdots,X_N^{(k)}]^T$ to be the locations of the N particles when the kth collision occurs. We denote the initial velocity and location vectors as $V^{(0)}$ and $X^{(0)}$, respectively. We define $t^{(k)}=[t_0^{(k)},t_1^{(k)},\cdots,t_N^{(k)}]^T$, where

$$t_j^{(k)} = \begin{cases} -\dfrac{X_{j+1}^{(k)} - X_j^{(k)}}{V_{j+1}^{(k)} - V_j^{(k)}}, & if\ \ V_{j+1}^{(k)} \neq V_j^{(k)} \\ \infty, & if\ \ V_{j+1}^{(k)} = V_j^{(k)} \end{cases} \qquad j = 0,1,\cdots,N, \tag{4}$$

and $X_0^{(k)} \equiv 0$, $X_{N+1}^{(k)} \equiv 1$, $V_0^{(k)} \equiv 0$, and $V_{N+1}^{(k)} \equiv 0$. This vector computes the times at which each pair of neighbors would collide in the absence of any other particles. For numerical simulations, we start with the initial velocities and locations of all the particles. Then we use Eq. (4) to calculate the times for each of the next $N+1$ possible collisions. We then compute

$$t_*^{(k)} = \min\{t_j^{(k)} | t_j^{(k)} \rangle 0, j = 0,1,\cdots,N\}, \tag{5}$$

which gives the minimal positive time among these $N+1$ collision times. This is the time at which the next collision occurs. We therefore update the locations of all the particles using

$$X^{(k+1)} = X^{(k)} + V^{(k)} t_*^{(k)}. \tag{6}$$

We then choose the corresponding collision operator L, C_j or R to update the velocities of all the particles. This procedure can be repeated to obtain the locations, velocities, times of collisions and the order of collisions that occur. We note that the only nonlinear operation in this procedure comes from the minimum in Eq. (5), which selects the appropriate collision. If the collision sequence is known *a priori*, then the procedure is purely linear.

Since the particles are rigid and inelastic, it is possible that inelastic collapse (in which an infinite number of collisions occur in a finite time) can occur. In this paper, we will only consider periodic orbits that contain a finite number of collisions.

We now give some examples of simulations for a three-particle system using the above procedure. In Fig. 2, we show examples of trajectories that occur for $e_0 = e_3 = 1$ and $e_1 = e_2 = 1/2$. The mass of the first particle is unity (by definition) and the mass of the second particle is 1.15, and we take various values for the mass of the third particle. If the mass of the third particle is 2.1, the motion does not appear to be periodic [see Fig. 2(a)]. If the mass of the third particle is 1.35, after a few initial collisions, the motion rapidly becomes periodic. This periodic motion is made up of seven collisions [see Fig. 2(b)]. We represent this sequence symbolically as $C_2C_1C_2RC_2C_1L$, where the sequence of collisions is read from right to left. If the mass of the third particle is 0.8, after a few initial collisions, the motion also rapidly becomes periodic. In this case, the periodic motion is made up of eight collisions [see Fig. 2(c)], represented symbolically as $C_2C_1RC_2RC_2C_1L$.

We note that the sequence $C_2C_1RC_2RC_2C_1L$ has only an extra R collision than $C_2C_1C_2RC_2C_1L$. Natural questions that arise are: as one changes the parameters, how does a given periodic orbit change into another periodic orbit? Furthermore, under what conditions do these transitions occur continuously?

Before continuing further, we introduce a naming convection that will allow us to identify periodic orbits that have similar mechanics. We begin by noting that, in terms of matrix operators, $C_iC_{i+k} \equiv C_{i+k}C_i$ for $k > 1$. This is because the C_i and C_{i+k} collisions do not involve a common particle, and applying the C_iC_{i+k} or $C_{i+k}C_i$ operators gives the same outcome velocities. So, the order in which the C_i and C_{i+k} collisions occur is not important in terms of the outcome velocities, and therefore does not affect the fundamental mechanics. Similarly, $LC_i \equiv C_iL$ for $i > 1$, $C_iR \equiv RC_i$ for $i < N-1$ and $LR \equiv RL$. For example, we first consider a two-particle system, in which there is only one interparticle collision, which we denote as C. The two sequences RLC and LRC are mechanically similar in the sense that one can swap the R and L collisions in the first sequence to obtain the second sequence. Therefore, in a sequence we always choose to represent both C_iC_{i+k} and $C_{i+k}C_i$ as C_iC_{i+k} for $k > 1$. That is, if possible, we will always commute collision operators so that the operator with the higher index is to the right. Similarly, we always choose to represent both LC_i and C_iL as LC_i

for $i>1$. We also choose to represent both C_iR and RC_i as C_iR for $i<N-1$, and represent both LR and RL as LR. For example, we always choose to represent both $RLCRC$ and $LRCRC$ as $LRCRC$.

Furthermore, for periodic orbits, we always choose to write a L collision first. For example, the sequences $LRCRC$, $RCRCL$, $CRCLR$, $RCLRC$, and $CLRCR$ are similar because of periodicity, and we denote them all as $RCRCL$. If there is more than one L collision in a given sequence, we need to determine which L collision to put in the first (rightmost) position. To do this, we choose the L collision that has the collision with the highest index directly after it. For example, we always choose to represent both $C_2LRC_2C_1L$ and $RC_2C_1LC_2L$ as $RC_2C_1LC_2L$. In the case of a tie between two or more collisions, we consider the next collision in the sequence. These rules give us a unique naming convention for similar periodic orbits.

III. ANALYTIC CONSTRUCTION OF PERIODIC ORBITS

We consider a given periodic orbit G, that contains K collisions. For example, for the orbit shown in Fig. 2(b) we have $G=C_2C_1C_2RC_2C_1L$ with $K=7$. The periodicity of the velocities requires

$$V^{(K)} = V^{(0)}. \tag{7}$$

Since the operators L, C_i and R can be written in terms of matrices, we can use Eq. (1)–(3) to rewrite Eq. (7) as

$$(I-G')V^{(0)} = G''\begin{pmatrix} 1+e_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{8}$$

where I is the identity matrix. G' and G'' are matrices that are products of T, C_i and R, which depend only on the order of the collisions. For example, for the orbit in Fig. 2(b), we have $G'=-C_2C_1C_2RC_2C_1T$ and $G''=C_2C_1C_2RC_2C_1$. One can show that all eigenvalues of G' have magnitude less than 1 if $e_i<1$ for $i=1,2,\cdots,N-1$ [18]. Thus, the matrix $(I-G')$ is nonsingular, and Eq. (8) has a unique solution. Having obtained $V^{(0)}$ by solving Eq. (8), it is straightforward to use the operators L, C_i and R to compute $V^{(1)},\cdots,V^{(K)}$. Using these velocities and the order of collisions we can find the particle locations at the $(k+1)$-th collision as a linear function of the particle locations at the kth collision. That is,

$$X^{(k+1)} = A^{(k)}X^{(k)} + b^{(k)} \quad \mathit{for} \quad k=0,\ldots,K-1, \tag{9}$$

where $A^{(k)}$ is a matrix and $b^{(k)}$ is a vector, whose elements are functions of m_i and e_i only. The periodicity of the locations requires $X^{(K)}=X^{(0)}$. Hence, Eq. (9) gives $X^{(0)}=AX^{(0)}+b$, where $A=\Pi_{k=0}^{K-1}A^{(k)}$ and $b=\Sigma_{i=0}^{K-1}(\Pi_{k=i+1}^{K-1}A^{(k)})b^{(i)}$. This equation can be solved to obtain $X^{(0)}$. Then, the locations of all the particles at each collision can be computed using Eq. (9).

Although one can compute the velocities and locations for any given sequence, it is by no means guaranteed that such an orbit can be realized. This is because for each collision in the sequence, the time at which the collision occurs must be the minimal positive time in Eq. (5). To ensure this is satisfied, one must ensure that all particles must remain in the correct spatial order. That is,

$$0 \le X_1^{(k)} \le X_2^{(k)} \le \cdots \le X_N^{(k)} \le 1 \tag{10}$$

for $k=1,2,\cdots,K$.

Finally, even if the particles are in the correct spatial order, one still needs to ensure that the orbit is stable. If any eigenvalue of A has magnitude greater than unity, initial perturbations can grow. Therefore, the condition for the periodic orbit to be stable is that, all the eigenvalues of A must have magnitude less than unity.

Because of so many constraints, any given periodic orbit can only exist in a restricted region of the parameter space. Every inequality in Eq. (10) gives a constraint, and the combination of all these constraints gives the region of parameter space in which a given periodic orbit exists.

For any given periodic orbit in a N-particle system, we can use the above procedure to determine the region of the $2N$-dimensional $(e_0,e_1,\cdots,e_N,m_2,m_3,\cdots,m_N)$ parameter space where the orbit exists. For example, we consider a three-particle system and fix $e_0=e_3=1$ and $e_1=e_2=1/2$ (see Fig. 3). In Fig. 3(a), we plot the regions where the periodic orbits $C_2C_1C_2RC_2C_1L$ [see Fig. 2(b)] and $C_2C_1RC_2RC_2C_1L$ [see Fig. 2(c)] exist in the (m_2,m_3) parameter space. The two regions are connected at a point a. Since this is a point (having zero dimension) in a two-dimensional parameter space, there must be two conditions that are satisfied when the $C_2C_1C_2RC_2C_1L$ orbit continuously changes to the $C_2C_1RC_2RC_2C_1L$ orbit. We therefore refer to this continuous transition as being co-dimension 2. In the following sections, we will explain why, generically, continuous transitions are of co-dimension 2. Figure 3(b) shows a similar type of transition. Surprisingly, we will also show that a number of degenerate orbit transitions occur with lower co-dimension, and we determine a simple criteria that can identify degenerate orbit transitions.

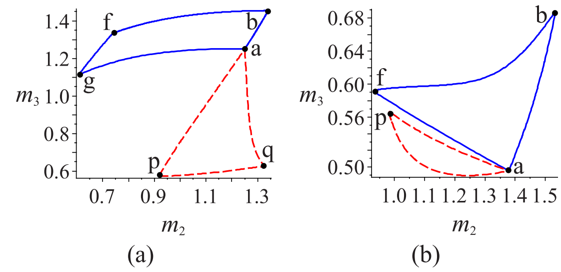


FIG. 3. (Color online) Continuous transitions that occur when adding a R collision are shown in (m_2,m_3) parameter space for $e_0=e_3=1$ and $e_1=e_2=1/2$. (a) The periodic orbit $C_2C_1C_2RC_2C_1L$ exists in the region enclosed by the solid curves $abfga$, and the periodic orbit $C_2C_1RC_2RC_2C_1L$ exists in the region enclosed by the dashed curves $apqa$. (b) The periodic orbit $C_2C_1C_2RC_2RC_2C_1L$ exists in the region enclosed by the solid curves $abfa$, and the periodic orbit $RC_2C_1C_2RC_2RC_2C_1L$ exists in the region enclosed by the dashed curves apa.

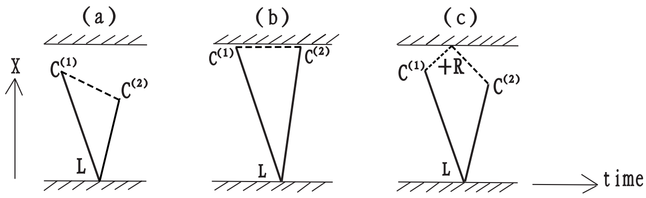


FIG. 4. The continuous transition resulting from adding a R collision to an orbit in a two-particle system. Here, we only sketch the part of the orbit where the R collision is added.

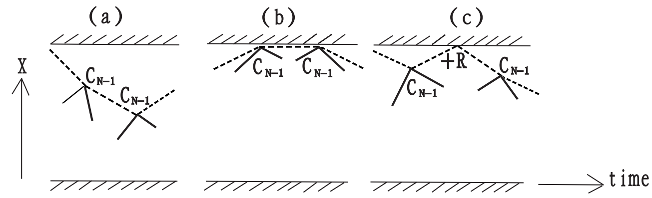


FIG. 5. The continuous transition resulting from adding a R collision to an orbit in a N-particle system. Here we only sketch the part of the orbit where the R collision is added.

IV. CONTINUOUS ORBIT TRANSITIONS

Having introduced the methodology and given some examples, we now consider a N-particle system. We consider continuous transitions that occur when adding or subtracting a single collision. In any given sequence, there are $N+1$ possible types of collisions, namely, R, C_i and L for $i=1,2,\cdots,N-1$. Therefore, there are different possible continuous transitions that occur when adding different collisions. We refer to these continuous transitions as adding R, adding C_i and adding L, respectively.

A. Continuous transitions adding *R*

In this subsection, we will consider the possible ways of adding a R collision to a given sequence. We recall that certain operators commute, in particular $RC_i \equiv C_iR$ for $i<N-1$. This means that the number of positions in which one needs to consider adding a R collision is less than one might naively imagine. For example, in a three-particle system, we consider the periodic orbit $C_2C_1C_2RC_2C_1L$. To distinguish the different collisions we denote the sequence by $C_2^{(7)}C_1^{(6)}C_2^{(5)}R^{(4)}C_2^{(3)}C_1^{(2)}L^{(1)}$. At first sight, it appears that there are five possible ways to add a R collision. These are, adding a R collision between $C_2^{(7)}$ and $C_1^{(6)}$, $C_1^{(6)}$ and $C_2^{(5)}$, $C_2^{(3)}$ and $C_1^{(2)}$, $C_1^{(2)}$ and $L^{(1)}$, and $L^{(1)}$ and $C_2^{(7)}$. This is because it is impossible to add a R collision between $C_2^{(5)}$ and $R^{(4)}$ or $R^{(4)}$ and $C_2^{(3)}$, since simple mechanics dictates that there cannot be two consecutive R collisions. However, using the fact that operators can commute, we see that adding a R collision between $C_2^{(7)}$ and $C_1^{(6)}$, and $C_1^{(6)}$ and $C_2^{(5)}$ are fundamentally similar while adding a R collision between $C_2^{(3)}$ and $C_1^{(2)}$, $C_1^{(2)}$ and $L^{(1)}$, and $L^{(1)}$ and $C_2^{(7)}$ are also fundamentally similar. Therefore, for the case of adding a R collision we only need to consider the collisions that cannot swap with R, namely, C_{N-1}. This implies that we just need to consider adjacent pairs of C_{N-1} collisions. In our example, these adjacent pairs of C_{N-1} collisions are $C_2^{(7)}$ and $C_2^{(5)}$, $C_2^{(5)}$ and $C_2^{(3)}$, and $C_2^{(3)}$ and $C_2^{(7)}$. Of these three adjacent pairs we can only add a R collision between $C_2^{(7)}$ and $C_2^{(5)}$ or $C_2^{(3)}$ and $C_2^{(7)}$.

To understand the continuous transitions when adding a R collision, we first consider the simpler case of a two-particle system. We begin by presenting the sketch of the continuous transition that occurs when adding a R collision (see Fig. 4). Figure 4(a) shows a part of a periodic orbit made up of the collisions CLC (that we refer to as a block). While Fig. 4(c) shows the block $CLRC$ which arises after adding a R collision. As we will argue below, Fig. 4(b) represents the only possible continuous transition between the CLC and $CLRC$ blocks.

For the block CLC, we need to distinguish the two different interparticle collisions. We do this by denoting them as $C^{(1)}$ and $C^{(2)}$ so that the block is written as $C^{(2)}LC^{(1)}$. In this block, the L collision occurs between the two interparticle collisions [see Fig. 4(a)]. In order to add a R collision, the right particle must collide with the right wall between the two interparticle collisions to form the new block $C^{(2)}LRC^{(1)}$ [see Fig. 4(c)]. For this change to occur in a continuous way, the new collision R must introduce no discontinuous change of momentum to the right particle when the transition occurs. This implies that, at the point of transition, the right particle must have zero velocity. Therefore, for this to occur, the two collisions $C^{(1)}$ and $C^{(2)}$ must both occur at the right wall, and hence $X_1^{(1)}=1$ and $X_1^{(3)}=1$ [see Fig. 4(b)]. These two conditions represent the requirement for a continuous transition when adding a R collision. Since, generically, two conditions must be satisfied, we refer to this transition as being co-dimension 2. The situation is obviously similar when removing the R collision from the block $CLRC$.

Having introduced the continuous transitions that occur when adding a R collision in a two-particle system, we now consider the general case of a N-particle system. For the case of adding a R collision, a two-particle system and a N-particle system are fundamentally similar, with the only difference being the following. In a two-particle system, between the two adjacent C collisions, the only possible collision is L [see Fig. 4(a)]. While in a N-particle system, many possible collisions can occur between two adjacent C_{N-1} collisions [see Fig. 5(a)]. Figure 5 shows the sketch of the continuous transition that occurs when adding a R collision in a N-particle system. Similarly to the two-particle system, at the point of transition, the Nth particle must have zero velocity. Therefore, at the point of transition, the two adjacent C_{N-1} collisions must both occur at the right wall. These two conditions represent the requirement for a continuous transition when adding a R collision. Since there are two conditions (the two adjacent C_{N-1} collisions both occur at the right wall) that must both be satisfied, this type of transition is, generically, of co-dimension 2.

We now see that the examples in Fig. 3 illustrate the above result. In this and all subsequent examples, we take $e_0=e_N=1$ and choose all interparticle coefficients of restitution to be the same, that is, $e\equiv e_1=e_2=\cdots=e_{N-1}$. In Fig. 3, we fix $e=1/2$ and plot the regions in which the two different periodic orbits exist in the 2-dimensional (m_2,m_3) parameter space. In Fig. 3(a), the periodic orbit $C_2C_1C_2RC_2C_1L$ exists

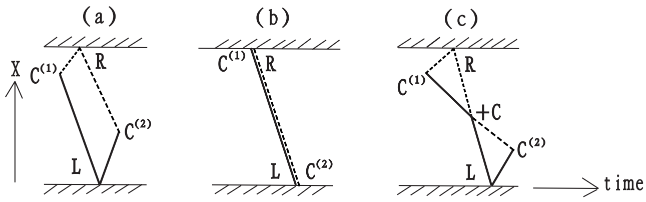


FIG. 6. The continuous transition resulting from adding a C collision to an orbit in a two-particle system. Here, we only sketch the part of the orbit where the C collision is added.

in the region enclosed by the solid curves *abfga*, and the periodic orbit $C_2C_1RC_2RC_2C_1L$ exists in the region enclosed by the dashed curves *apqa*. The continuous transition between the two orbits is co-dimension 2. Therefore, we would expect the two regions in the two-dimensional (m_2,m_3) parameter space to be connected on a zero-dimensional space, that is, a point. This is the case in this example in which the continuous transition between $C_2C_1C_2RC_2C_1L$ and $C_2C_1RC_2RC_2C_1L$ occurs at the point a. Figure 3(b) shows a similar continuous transition when the orbit $C_2C_1C_2RC_2RC_2C_1L$ changes to $RC_2C_1C_2RC_2RC_2C_1L$.

Up to this point, the continuous transitions that occur when adding a R collision require two conditions to be met. That is, they are co-dimension 2. However, surprisingly, there are also an infinite set of non-generic transitions that we will describe in detail in Sec. IV D.

B. Continuous transitions adding C_i

Similarly to the previous subsection, for the case of adding a C_i collision, we just need to consider collisions that cannot swap with the C_i collision, namely, C_{i-1} and C_{i+1}. That is, we only need to consider adding a C_i collision after (to the left of) collisions $C_{i+1}C_i$ or after collisions $C_{i-1}C_i$, where the C_0 collision represents L and the C_N collision represents R.

We also begin by first considering a two-particle system. In Fig. 6, we give the sketch of the continuous transition that occurs when adding a C collision. Figure 6(a) shows the block *CLRC*, while Fig. 6(c) shows the block *CLCRC* which arises after adding a C collision. As we will argue below, Fig. 6(b) represents the only possible continuous transition between the *CLRC* and *CLCRC* blocks.

We denote the block *CLRC* by $C^{(2)}LRC^{(1)}$ to distinguish the two different interparticle collisions. Similarly to the previous subsection, we need to determine the appropriate conditions at the point of transition. For the case of continuously adding a C collision, the new C collision must not result in a discontinuous change in the momentum of the two particles. This implies that the velocities of the two particles must be equal when the continuous transition occurs. Therefore, at the point of transition, the two lines $LC^{(1)}$ and $C^{(2)}R$ in Fig. 6(b) must coincide. That is to say, the $C^{(1)}$ collision must occur at the right wall and the $C^{(2)}$ collision must occur at the left wall, and hence $X_1^{(1)}=1$ and $X_1^{(4)}=0$ [see Fig. 6(b)]. These two conditions represent the requirement for a continuous transition when adding a C collision. Therefore, this continuous transition is also, generically, co-dimension 2.

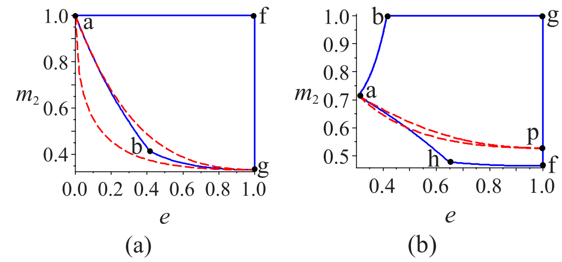


FIG. 7. (Color online) Continuous transitions that occur when adding a C collision in a two-particle system. (a) The periodic orbit *RCRCL* exists in the region enclosed by the solid curves *abgfa*, and the periodic orbit *CRCRCL* exists in the region enclosed the dashed curves *aga*. (b) The periodic orbit [*CRCL*][*RCRCL*] exists in the region enclosed by the solid curves *abgfha*, and the periodic orbit [*CRCL*][*CRCRCL*] exists in the region enclosed by the dashed curves *apa*.

In Fig. 7, we give two examples of this type of continuous transitions that occur in a two-particle system. In Fig. 7(a), the periodic orbit *RCRCL* exists in the region enclosed by the solid curves *abgfa*, and the periodic orbit *CRCRCL* exists in the region enclosed by the dashed curves *aga*. The continuous transition between them is co-dimension 2. Figure 7(b) shows a similar continuous transition, when the orbit [*CRCL*][*RCRCL*] changes to [*CRCL*][*CRCRCL*]. Here, we have grouped the collisions using brackets in order to see the sequence clearly.

In fact, one can show that there are an infinite number of continuous transitions of this type. For example, in a two-particle system, the continuous transitions between the periodic orbits $[(CR)^jCL]^M[R(CR)^jCL]$ and $[(CR)^jCL]^M[CR(CR)^jCL]$, $j=1,2,\ldots,M=0,1$, also occur in a similar way.

We next note that there are some examples that appear to differ from the above examples, but are actually fundamentally similar. For example, in a two-particle system, the periodic orbit [*CLRCL*][*CRCL*] exists in the region enclosed by the solid curves *dpqd*, and the periodic orbit [*CLCRCL*][*CRCL*] exists in the region enclosed by the dashed curves *aba* [see Fig. 8(a)]. We can get the periodic orbit [*CLCRCL*][*CRCL*] by adding a C collision to the periodic orbit [*CLRCL*][*CRCL*]. Therefore, we would expect the two regions to share a common co-dimension 2 point in the two-dimensional (e,m_2) parameter space, that is, a corner. However, in Fig. 8(a), it is clear that there is no such common corner between them. If we check the conditions $X_1^{(6)}=1$ and $X_1^{(9)}=0$ in [*CLRCL*][*CRCL*], and $X_1^{(6)}=1$ and $X_1^{(10)}=0$ in [*CLCRCL*][*CRCL*], four curves *rg*, *rh*, *rt* and *rf*, which represent the four conditions, still intersect at a point r [see Fig. 8(b)]. Thus, there is a co-dimension 2 point. However, in this case, the above four conditions are redundant in determining the regions in which the respective orbits exist. That is, for this particular continuous transition, other restrictions mean that the conditions which determine the co-dimension 2 point are redundant. We refer to this type of continuous transition as a “hidden continuous transition.” The results for the two-particle system can be generalized to the case of a N-particle system using similar ideas to those used in Sec. IV A. We omit the details for brevity.

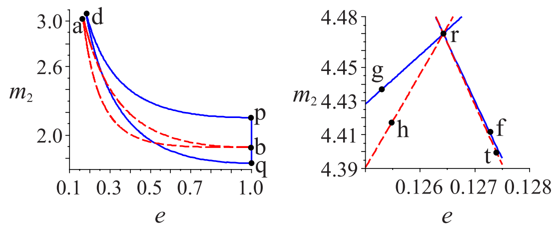


FIG. 8. (Color online) Hidden continuous transition that occurs when adding a C collision in a two-particle system for $e=1/2$. (a) The periodic orbit $[CLRCL][CRCL]$ exists in the region enclosed by the solid curves $dpqd$, and the periodic orbit $[CLCRCL][CRCL]$ exists in the region enclosed by the dashed curves aba. (b) The four curves rg, rh, rf, and rt, which represent $X_1^{(6)}=1$, $X_1^{(9)}=0$ in $[CLRCL][CRCL]$ and $X_1^{(6)}=1$, $X_1^{(10)}=0$ in $[CLCRCL][CRCL]$ intersect at a point r.

C. Continuous transitions adding L

For the case of adding a L collision to a sequence, because of the left vibrating wall, there must be a discontinuous change in momentum of the left particle. Therefore, no continuous transition of this type can occur.

D. Degenerate transitions

From the above analysis and examples, it seems clear that all continuous transitions should be co-dimension 2. However, surprisingly, one can also find degenerate transitions that require only a single condition to be met, that is, transitions that are co-dimension 1.

In Fig. 9, we give two examples of degenerate transitions that occur when continuously adding a R collision in a two-particle system. In Fig. 9(a), the periodic orbit $CRCL$ exists in the region enclosed by the curves $dabt$, and the periodic orbit $RCRCL$ exists in the region enclosed by the curves $abpqa$. The continuous transition between them occurs at the one-dimensional curve ab in the two-dimensional (e,m_2) parameter space and is therefore co-dimension 1. This is surprising because in Sec. IV A we have shown that continuous transitions require two conditions to be met and should therefore be co-dimension 2. Figure 9(b) shows a similar type of degenerate transition. In fact, one can show that there are an infinite number of this type of degenerate transitions between the periodic orbits $[(CR)^jCL]^M$ and $[(CR)^jCL]^{M-1}[R(CR)^jCL]$ for $j=1,2,\cdots,M=1,2,\cdots$.

We note that Wylie and Zhang [18] have shown that large numbers of periodic orbits can collapse onto a simple orbit at certain critical mass ratios. In fact, this is one type of degenerate transitions between the periodic orbits $[CRCL]^M$ and $[CRCL]^{M-1}[RCRCL]$. These degenerate transitions occur when $m=1$. We further note that the operators R, C, and L are all linear, therefore the dynamics of the orbit $[CRCL]^M$ must exhibit the repeated dynamics of the orbit $CRCL$ [18]. Thus, all of the periodic orbits $[CRCL]^{M-1}[RCRCL]$ collapse onto a single orbit $CRCL$ when $m=1$ (see Fig. 9).

In the following, we give the explanation why degeneracy of this type occurs. We first consider orbits of the type

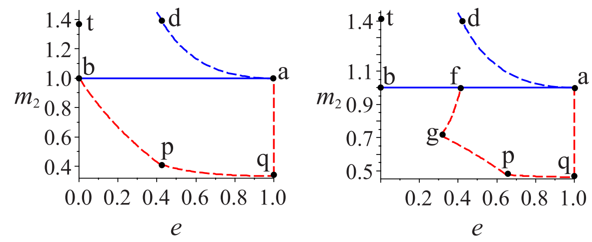


FIG. 9. (Color online) Continuous transitions that occur adding a R collision in a two-particle system for $e=1/2$. In both (a) and (b), the periodic orbit $CRCL$ exits in the region enclosed by the curves $dabt$. In (a) the periodic orbit $RCRCL$ exits in the region enclosed by the curves $abpqa$. In (b) the periodic orbit $[CRCL][RCRCL]$ exits in the region made up of the curves $afgpqa$.

$(CR)^jCL$. The time that the left particle needs to travel from the $2i$-th collision to the $(2i+2)$-th collision is $(X_1^{(2i+2)}-X_1^{(2i)})/V_1^{(2i)}$. The time that the right particle needs to travel from the $2i$-th collision to the $(2i+2)$-th collision is $(1-X_2^{(2i)})/V_2^{(2i)}+(1-X_2^{(2i+2)})/V_2^{(2i)}$. By definition, the two times must be equal. Furthermore, since the $2i$-th and the $(2i+2)$-th collisions both are interparticle collisions, we have $X_1^{(2i)}\equiv X_2^{(2i)}$ and $X_1^{(2i+2)}\equiv X_2^{(2i+2)}$. Equating the two transition times and solving for $X_1^{(2i+2)}$, we obtain,

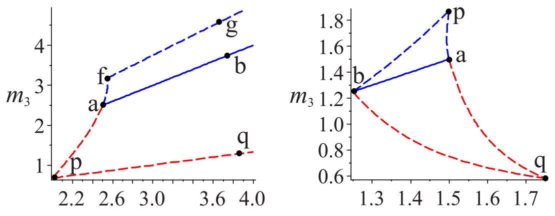
$$X_1^{(2i+2)}=\frac{V_2^{(2i)}-V_1^{(2i)}}{V_2^{(2i)}+V_1^{(2i)}}X_1^{(2i)}+\frac{2V_1^{(2i)}}{V_2^{(2i)}+V_1^{(2i)}}.$$

If the parameters are chosen such that the system is at a transition at which a R collision is continuously added, then, by the argument given in Sec. IV A, we know that $X_1^{(2i)}$ must be 1. In this case,

$$X_1^{(2i+2)}=\frac{V_2^{(2i)}-V_1^{(2i)}}{V_2^{(2i)}+V_1^{(2i)}}+\frac{2V_1^{(2i)}}{V_2^{(2i)}+V_1^{(2i)}}\equiv 1.$$

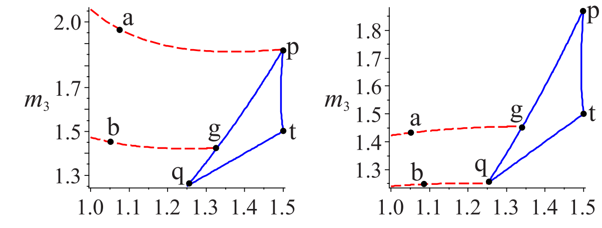
The above calculation shows that for the orbit $(CR)^jCL$, if $X_1^{(2)}=1$ then $X_1^{(2i+2)}=1$ for all i. This implies that once the first interparticle collision occurs at the right wall, all the other interparticle collisions must occur at the right wall. Due to the linearity of the operators R, C_i, and L, the dynamics of the orbit $[(CR)^jCL]^M$ must exhibit the repeated dynamics of the orbit $(CR)^jCL$ [18]. That is to say, $X_1^{(2j+2+i)}\equiv X_1^{(i)}$ in the orbit $[(CR)^jCL]^M$, and hence all the interparticle collisions must occur at the right wall as long as one interparticle collision occurs at the right wall. Therefore, for a continuous transition between the orbits $[(CR)^jCL]^M$ and $[(CR)^jCL]^{M-1}[R(CR)^jCL]$, the two conditions $X_1^{(2j+2)}=1$ and $X_1^{(2j+4)}=1$ are degenerate. In fact, all of the $2^{M-1}(j+1)^M$ equations $X_1^{(2)}=X_1^{(4)}=\cdots=X_1^{(2j+2)^M}=1$ are identical. That is, only one condition is required when continuously adding a R collision and the co-dimension of the continuous transition is therefore 1.

A similar calculation for the block $(C_{N-1}R)^MC_{N-1}$ in a N-particle system shows that, if any one of the C_{N-1} collisions in the block occurs at the right wall, then all the other C_{N-1} collisions in the block must also occur at the right wall. This result allows us to identify exactly the types of orbits



(a) (b)

FIG. 10. (Color online) Degenerate transitions that occur when adding a R collision in (m_2, m_3) parameter space for $e=1/2$. (a) The periodic orbit $C_2RC_2RC_2C_1L$ exists in the region enclosed by the curves *bafg*, and the periodic orbit $RC_2RC_2RC_2C_1L$ exists in the region enclosed by the curves *bapq*. (b) The periodic orbit $C_1C_2RC_2C_1L$ exists in the region enclosed by the curves *abpa*, and the periodic orbit $C_1RC_2RC_2C_1L$ exists in the region enclosed by the curves *abqa*.



(a) (b)

FIG. 11. (Color online) Degenerate transitions that occur when adding a C_i collision in (m_2, m_3) parameter space for $e=1/2$. In both (a) and (b), the periodic orbit $C_1C_2RC_2C_1L$ exists in the region enclosed by the curves *pqtp*. In (a) the periodic orbit $C_1C_2C_1RC_2C_1L$ exists in the region enclosed by the curves *apgb*. In (b) the periodic orbit $C_2C_1C_2RC_2C_1L$ exists in the region enclosed by the curves *agqb*.

that have degenerate transitions. We illustrate this with the following two examples. We first consider an example of a degenerate transition from the periodic orbits $C_2RC_2RC_2C_1L$ to $RC_2RC_2RC_2C_1L$ [see Fig. 10(a)]. To explain the degeneracy, it is convenient to label the collisions as $C_2^{(7)}R^{(6)}C_2^{(5)}R^{(4)}C_2^{(3)}C_1^{(2)}L^{(1)}$. We wish to add a R collision in the eighth position. So the C_{N-1} collisions adjacent to the new R collision are $C_2^{(7)}$ and (by periodicity) $C_2^{(3)}$. The $C_2^{(7)}$ and $C_2^{(3)}$ collisions appear in the block $C_2^{(7)}R^{(6)}C_2^{(5)}R^{(4)}C_2^{(3)}$ that has the form $(C_{N-1}R)^M C_{N-1}$. Hence, if the $C_2^{(7)}$ collision occurs at the right wall, then the $C_2^{(3)}$ collision automatically occurs at the right wall. Therefore, this transition is degenerate. In fact, one can show that there are an infinite number of degenerate transitions of a similar type between the periodic orbits $(C_2R)^M C_2C_1L$ and $R(C_2R)^M C_2C_1L$ for $M=1,2,\cdots$. Another infinite family of degenerate transitions are between the periodic orbits $C_1(C_2R)^M C_2C_1L$ and $C_1R(C_2R)^M C_2C_1L$, which also have co-dimension 1. Figure 10(b) shows the degenerate transition for $M=1$ in the second infinite family.

We now give an example of a nondegenerate transition from the periodic orbits $C_2^{(7)}C_1^{(6)}C_2^{(5)}R^{(4)}C_2^{(3)}C_1^{(2)}L^{(1)}$ to $C_2C_1RC_2RC_2C_1L$ [see Fig. 3(a)]. We wish to add a R collision between the fifth and sixth collisions. So the C_{N-1} collisions adjacent to the new R collision are $C_2^{(7)}$ and $C_2^{(5)}$. The $C_2^{(5)}$ collision appears in the block $C_2^{(5)}R^{(4)}C_2^{(3)}$, and so if the $C_2^{(5)}$ collision occurs at the right wall, then the $C_2^{(3)}$ collision must also occur at the right wall. However, the $C_2^{(7)}$ collision is not in this type of block and so it will not automatically occur at the right wall. So in this case, the two conditions do not reduce to a single condition and so the transition is co-dimension 2.

From the above analysis, it is clear how to classify degenerate or nondegenerate transitions that occur when adding a R collision, by only using the sequence of the collisions. That is, the continuous transition is degenerate with co-dimension 1 if and only if all of the C_{N-1} collisions in a given sequence appear in a single block of the form $(C_{N-1}R)^M C_{N-1}$.

Besides the degenerate transition that occurs when continuously adding a R collision, there are other types of degenerate transitions. When continuously adding a C_i collision, the transition can also be degenerate. For example, in Fig. 11, we give two examples of degenerate transitions that occur when adding a C_i collision in a three-particle system. In Fig. 11(a), the periodic orbit $C_1C_2RC_2C_1L$ exists in the region enclosed by the curves *pqtp*, and the periodic orbit $C_1C_2C_1RC_2C_1L$ exists in the region enclosed by the curves *apgb*. The continuous transition between the two orbits occurs at the one-dimensional curve *pg* in the two-dimensional (m_2, m_3) parameter space and is therefore co-dimension 1. Figure 11(b) shows a similar degenerate transition when the orbit $C_1C_2RC_2C_1L$ changes to $C_2C_1C_2RC_2C_1L$.

We now explain why the degenerate transitions in Fig. 11 occurs. To distinguish the different interparticle collisions, we denote the periodic orbit $C_1C_2RC_2C_1L$ by $C_1^{(4)}C_2^{(3)}RC_2^{(2)}C_1^{(1)}L$. When the continuous transition between $C_1C_2RC_2C_1L$ and $C_1C_2C_1RC_2C_1L$ occurs, according the argument in Sec. IV B, the following two conditions should be satisfied: the velocities of the left particle and the middle particle should be equal after the $C_2^{(2)}$ collision and the locations of the two interparticle collisions $C_1^{(1)}$ and $C_2^{(2)}$ should also be equal. For the orbit $C_1C_2RC_2C_1L$, one can solve to find the velocities of the left and middle particles after the $C_2^{(2)}$ collision using Eq. (8). Equating the locations of the two interparticle collisions $C_1^{(1)}$ and $C_2^{(2)}$ using Eq. (9) and solving for m_3, we obtain,

$$m_3 = \frac{2m_2(m_2+1)}{7-2m_2}. \tag{11}$$

If the parameters are chosen such that the system is at a transition at which a C_1 collision is continuously added, then, by the argument given in Sec. IV B the two interparticle $C_1^{(1)}$ and $C_2^{(2)}$ collisions must occur at the same location, which implies that Eq. (11) must hold. When Eq. (11) holds, one can show that $V_1^{(3)} \equiv V_2^{(3)}$. Thus, when the locations of the two interparticle collisions $C_1^{(1)}$ and $C_2^{(2)}$ are equal the velocities of the left particle and the middle particle are automatically equal after the $C_2^{(2)}$ collision. Therefore, the two conditions for the continuous transition between $C_1C_2RC_2C_1L$ and $C_1C_2C_1RC_2C_1L$ are both met as long as Eq. (11) holds. That

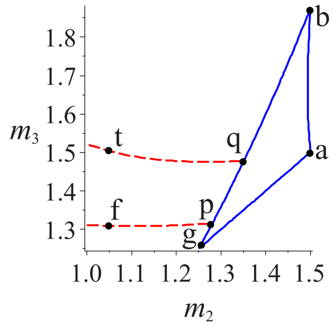


FIG. 12. (Color online) Higher degrees of degeneracy that occurs when adding a C_i collision in (m_2,m_3) parameter space for $e=1/2$. The periodic orbit $C_1C_2RC_2C_1L$ exists in the region enclosed by the curves *abga*, and the periodic orbit $C_2C_1C_2C_1RC_2C_1L$ exists in the region enclosed by curves *fpqt*.

is to say, the co-dimension of the continuous transition is 1. A similar calculation shows that, when the continuous transition between the periodic orbits $C_1C_2RC_2C_1L$ and $C_2C_1C_2RC_2C_1L$ occurs, the two conditions also reduce to a single condition,

$$m_3 = \frac{2m_2(m_2+1)}{7-2m_2}, \tag{12}$$

which is exactly the same as Eq. (11). Therefore, the continuous transition between $C_1C_2RC_2C_1L$ and $C_2C_1C_2RC_2C_1L$ is also co-dimension 1.

E. Higher degrees of degeneracy

From the above examples in Fig. 11, we can see the surprising result that even higher degrees of degeneracy can occur. In Fig. 12, the periodic orbit $C_1C_2RC_2C_1L$ exists in the region enclosed by the curves *abga*, and the periodic orbit $C_2C_1C_2C_1RC_2C_1L$ exists in the region enclosed by the dashed curves *fpqt*. The continuous transition between them occurs at the curve *pq* in the 2-dimensional (m_2,m_3) parameter space and is therefore co-dimension 1. Note that this continuous transition involves the addition of two collisions (C_1 and C_2) at the same time. According to the argument in Sec. IV B, four conditions should be satisfied at the point of transition. That is to say, we would expect the continuous transition between periodic orbits $C_1C_2RC_2C_1L$ and $C_2C_1C_2C_1RC_2C_1L$ to be co-dimension 4. However, surprisingly, Fig. 12 shows that the co-dimension is 1. This implies that only a single condition must be met for a continuous transition rather than the four conditions one should generically expect.

We now briefly explain the physical mechanism that underlies this higher degree of degeneracy. To understand the mechanics of the degeneracy, we show two periods of the trajectories of the particles with $e=0.5$, $m_2=1.4$ and $m_3=1.59$ (see Fig. 13). In the first period we label the interparticle collisions $C_1^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, and $C_1^{(4)}$. In the second period we label the first two interparticle collisions as $C_1^{(1*)}$ and $C_2^{(2*)}$. If the locations of the two interparticle collisions $C_1^{(1)}$ and $C_2^{(2)}$ are equal (that is, the two points labeled $C_1^{(1)}$ and $C_2^{(2)}$ coincide), according the argument in Sec. IV B, the velocity of the left particle (given by the slope of the line $C_1^{(4)}C_1^{(1)}$) and the velocity of the middle particle (given by the slope of the line $C_2^{(3)}C_2^{(2)}$) must be equal after the $C_2^{(2)}$ collision. This implies that the two lines $C_1^{(4)}C_1^{(1)}$ and $C_2^{(3)}C_2^{(2)}$ must coincide. That is to say, the left particle and the middle particle coincide and move with the same velocity. We now turn our attention to the next pair of collisions $C_2^{(3)}$ and $C_1^{(4)}$. Before the $C_2^{(3)}$ collision occurs, the left particle and the right particle have the same locations and the same velocities. Therefore, the $C_1^{(4)}$ collision occurs immediately after the $C_2^{(3)}$ collision. That is to say, the locations of the two interparticle collisions $C_2^{(3)}$ and $C_1^{(4)}$ must occur at the same location, which implies that the two points $C_2^{(3)}$ and $C_1^{(4)}$ coincide. Because of periodicity, the two points $C_1^{(1*)}$ and $C_2^{(2*)}$ have the same locations as the two points $C_1^{(1)}$ and $C_2^{(2)}$. Since we are considering the case in which $C_1^{(1)}$ and $C_2^{(2)}$ coincide, $C_1^{(1*)}$ and $C_2^{(2*)}$ must also coincide. Therefore, the velocity of the middle particle (given by the slope of the line $C_1^{(1*)}C_1^{(4)}$) and the velocity of the right particle (given by the slope of the line $C_2^{(2*)}C_2^{(3)}$) must also be equal, which implies that the two lines $C_1^{(1*)}C_1^{(4)}$ and $C_2^{(2*)}C_2^{(3)}$ coincide. Therefore, the continuous transition between the periodic orbits $C_1C_2RC_2C_1L$ and $C_2C_1C_2C_1RC_2C_1L$ requires only one condition that the two interparticle collisions $C_1^{(1)}$ and $C_2^{(2)}$ occur at the same location.

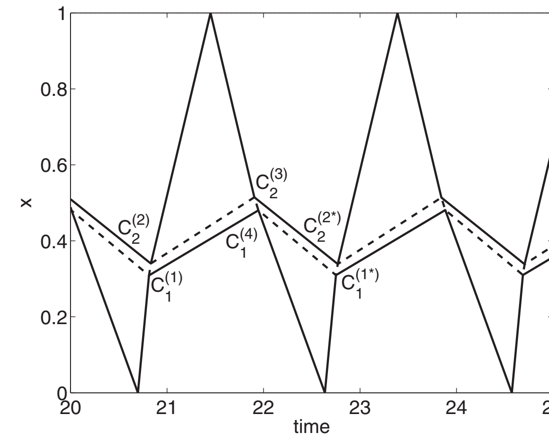


FIG. 13. Trajectory of particles in a three-particle system with $e=0.5$, $m_2=1.4$ and $m_3=1.59$. The periodic orbit is $C_1C_2RC_2C_1L$.

V. CONCLUSION

In this paper, we have considered the continuous transitions between different periodic orbits in a one-dimensional inelastic particle system with particles traveling between two walls. We have found that different transitions can occur when adding different collisions. We explained the nature of the different transitions using an analytic machinery that can determine the region of parameter space in which any given periodic orbit can exist.

For a continuous transition, we have shown that, generically, two conditions are required when a periodic orbit changes to another by adding or subtracting a single collision. That is, continuous orbit transitions are, generically, of co-dimension 2.

Surprisingly, we have also shown that there are infinite set of continuous transitions for which only one condition is required. That is, the degenerate transitions are of co-dimension 1. Moreover, we have demonstrated how one can classify orbit transitions as being degenerate or nondegenerate purely using the sequences of collisions in the two orbits. We have also shown that higher degrees of degeneracy can occur. Wylie and Zhang [18] found that large numbers of periodic orbits can coalesce onto a single orbit at certain critical mass ratios. This behavior is a direct result of the degeneracy explained in this paper.

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