Analysis of a class of granular motors in the Brownian limit

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We analyze the dynamics of heterogeneous granular particles immersed in a bath of thermalized particles, which are candidates for granular motors, with a mechanical approach. We first apply the method to the previously introduced asymmetric piston and show that it gives the exact drift velocity in the Brownian limit. We also obtain results for the efficiency of the motor and compare with numerical simulations. Finally, we introduce a chiral rotor model and discuss opportunities for observing a real granular motor.

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I. INTRODUCTION

Brownian ratchets, which are devices that extract work from a thermal bath, have fascinated scientists for nearly a century [1,2]. In order to function a ratchet must break both time and spatial symmetries. The former occurs whenever detailed balance does not apply and spatial symmetry may be broken by intrinsic asymmetry of the object or an external force. For an extensive review of recent developments and applications, see Ref. [3]. The fact that time-reversal symmetry is automatically broken in systems of inelastic particles prompted several groups to propose simple models of granular ratchets that are coupled to only one thermal bath at a given temperature.

Cleuren and Eichhorn [4], and independently Costantini *et al.* [5], showed that an asymmetric granular particle in a homogeneous bath of particles displays a drift velocity that is proportional to $1-\alpha$, where α is the coefficient of restitution characterizing the collisions between the tagged particle (of mass M) and the bath particles (of mass m). Costantini *et al.* [6] proposed possibly the simplest model of a granular motor consisting of an asymmetric piston composed of two materials with different inelasticities. The piston, which is constrained to move along a line, is immersed in a bath of elastic particles. The asymmetry leads to a violation of detailed balance and to a net drift. Starting from the Boltzmann equation, Costantini *et al.* proposed a phenomenological approach based on the evolution of the first three moments of the velocity distribution. Although their theory is in good agreement with numerical simulation results in some cases, it is somewhat unwieldy and, more importantly, it breaks down in the limit of large piston mass. Here, we propose a much simpler mechanical approach that gives the average force on the particle when it is moving at a given velocity. This leads directly to an *exact* expression for steady-state drift velocity in the limit of large piston mass.

We apply the method to the asymmetric piston as well as to a new model motor, namely, a chiral rotor [7], which is more interesting from an experimental perspective. As an application of the method, we calculate the efficiency of these motors in the Brownian limit. The efficiency at nonzero power was first considered by Curzon and Ahlborn [8] and later generalized by van den Broeck [9]. The efficiency of Brownian motors has also been discussed [10,11], but there are no published results for the efficiency of granular motors.

II. MODEL

The asymmetric piston of mass M and length L is composed of two different materials with coefficients of restitution α_L and α_R on the left- and right-hand sides, respectively, and is constrained to move along a line. The chiral rotor is also heterogeneous but is constrained to rotate about a fixed point (see Fig. 1). Both devices are immersed in a two-dimensional granular gas at a density ρ composed of structureless particles each of mass m and whose velocity distribution is given by $\phi(\mathbf{v})$. We let v_x and v_y denote the components of the gas particle's velocity perpendicular and parallel to the surface of the motor, respectively. The granular temperature of the bath is defined as $T_B=m\langle v_x^2\rangle=m\langle v_y^2\rangle$. We first focus on the piston. Following a collision with a bath particle v_y is conserved, while v_x, denoted henceforth by v, and the piston velocity V change as

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} V \\ v \end{pmatrix} + \frac{1+\alpha}{1+\mu}(v-V)\begin{pmatrix} 1 \\ -\mu \end{pmatrix}, \tag{1}$$

where α is chosen depending on the side, α_L or α_R, and $\mu=M/m$. The granular temperature of the piston is given by $T_g=M\langle(V-\langle V\rangle)^2\rangle$ where the average is over its velocity distribution $f(V)$.

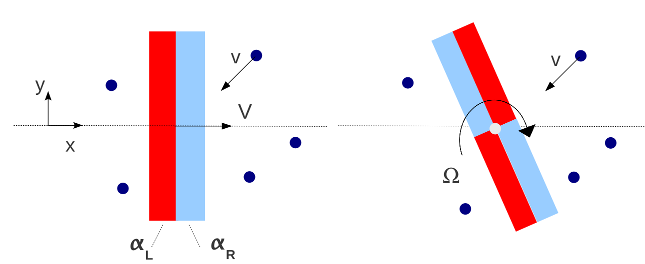

FIG. 1. (Color online) Granular motors. Left: the asymmetric piston; right: the chiral rotor. Both motors are constructed from materials with different coefficients of restitution and are immersed in a bath of thermalized particles. The asymmetric piston acquires a net drift velocity along a line, while the chiral rotor acquires a net rotation around its axis.

Intuitively, one expects the motor effect to be greatest when $|\alpha_L-\alpha_R|$ is maximum, which corresponds to $\alpha_R=0$, $\alpha_L=1$ (or the inverse) since this results in the greatest difference in momentum transfer to each side. The parameter space of the model can be reduced by virtue of the fact that a system with $0\leq\alpha_R\leq\alpha_L\leq 1$ and mass M is equivalent to the one characterized by $\alpha_R'=0$, $0\leq\alpha_L'\leq 1$, and M', a feature occurring in several other tracer problems [12–14]. [The explicit relationships are $\mu'=(\mu-\alpha_R)/(1+\alpha_R)$, $\alpha_L'=(\alpha_L-\alpha_R)/(1+\alpha_R)$.]

III. MECHANICAL APPROACH

The instantaneous impulse exerted on the piston of velocity V by a collision with a bath particle of velocity v on the right-hand side is

$$I_R=\frac{M}{1+\mu}(1+\alpha_R)(V-v), \quad v<V, \tag{2}$$

with a similar equation involving α_L for a collision on the left-hand side. Assuming that successive collisions between bath particles and the granular tracer are uncorrelated, the pressure is obtained by averaging on different collisions at a given velocity of the granular piston V that consists of integrating the impulse times the rate that the specified face collides with a particle moving with a velocity v,

$$P(V)=\frac{M\rho}{1+\mu}\int_0^\infty dy\, y^2[(1+\alpha_L)\phi(V+y)-(1+\alpha_R)\phi(V-y)]. \tag{3}$$

For a Gaussian bath velocity distribution, $\phi(v)=(1/\sqrt{2\pi}v_{th})\exp[-v^2/(2v_{th}^2)]$, where $v_{th}=\sqrt{T_B/m}$, we obtain the explicit expression

$$P(U)=\frac{\rho M v_{th}^2}{4(1+\mu)}\Bigg\{(\alpha_L-\alpha_R)(1+U^2) - (2+\alpha_L+\alpha_R)\Bigg[(1+U^2)\mathrm{erf}\left(\frac{U}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}Ue^{-U^2/2}\Bigg]\Bigg\}, \tag{4}$$

where we have introduced the dimensionless velocity $U=V/v_{th}$. We anticipate (and justify below) that in the limit $\mu\to\infty$, where the piston velocity fluctuations go to zero, the solution of $P(U)=0$ gives the exact mean drift velocity U^*. The result of Costantini *et al.* [6], $U^*=[(\alpha_L-\alpha_R)/4(2+\alpha_L+\alpha_R)]\sqrt{2\pi}$, however, underestimates the exact solution, e.g., for $\alpha_R=0$, $\alpha_L=1$, $U^*=0.217\,02$, while their estimate is 0.2089, or a difference of 3.7%. We compare these results with numerical simulations of the model using the Gillespie approach [equivalent results can be obtained by a direct simulation Monte Carlo (DSMC) method [15]]. Figure 2, which shows in a logarithmic-linear plot the mass dependence of the mean velocity, confirms that our approach provides the correct description in the limit of large piston mass.

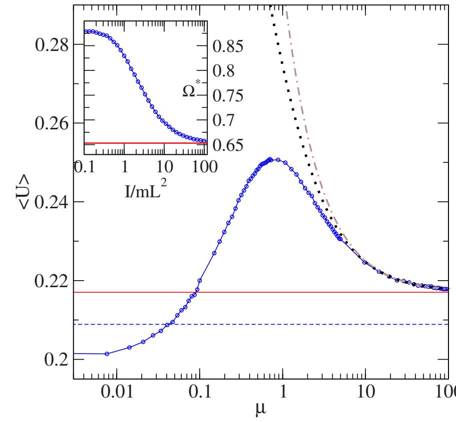

FIG. 2. (Color online) Dimensionless steady-state velocity U as a function of the mass ratio $\mu=M/m$ for $\alpha_L=1$ and $\alpha_R=0$. Circles correspond to simulation results, the solid line shows the exact solution in the Brownian limit [solution of $P(U)=0$ with $P(U)$ given by Eq. (4)], and the dashed straight line corresponds to the Brownian limit given in [6]. The dashed-dotted and dotted lines correspond to the first-order [Eq. (12)] and second-order [Eq. (13)] corrections for finite mass ratios. The inset shows results for the rotor; see text.

We now demonstrate that the result for the mean drift velocity is consistent with the Boltzmann equation for the asymmetric piston. This may be written as

$$\int_0^\infty dy\, yf\left(V-\frac{1+\alpha_L}{1+\mu}y\right)\phi\left(V+\frac{\mu-\alpha_L}{1+\mu}y\right) + \int_0^\infty dy\, yf\left(V+\frac{1+\alpha_R}{1+\mu}y\right)\phi\left(V-\frac{\mu-\alpha_R}{1+\mu}y\right) = f(V)\int_0^\infty dy\, y[\phi(V+y)+\phi(V-y)]. \tag{5}$$

Introducing the drift velocity in the Brownian limit, V^*, and the variable transformation $z=(V-V^*)/\epsilon$ and expanding the distribution function $F(z)$ in powers of ϵ, we obtain at first order

$$\int_0^\infty dy\, y^2[(1+\alpha_L)\phi(V^*+y)-(1+\alpha_R)\phi(V^*-y)]=0, \tag{6}$$

which is consistent with the mechanical approach [Eq. (3)], while at second order we obtain a Fokker-Planck equation for the piston velocity distribution,

$$\frac{T_g}{m}\frac{d^2F(z)}{dz^2}+z\frac{dF(z)}{dz}+F(z)=0, \tag{7}$$

where T_g/m is given by

$$\frac{T_g}{m} = \frac{\int_0^\infty dy\, y^3[(1+\alpha_R)^2\phi(V^*-y)+(1+\alpha_L)^2\phi(V^*+y)]}{4\int_0^\infty dy\, y[(1+\alpha_R)\phi(V^*-y)+(1+\alpha_L)\phi(V^*+y)]}. \tag{8}$$

Reverting to the original variables, one obtains $f(V) \propto e^{-M(V-V^*)^2/2T_g}$. Equation (8) shows that in the Brownian limit the temperature of heterogeneous granular particles depends on the moments of bath velocity function, whereas the temperature does not depend on the bath for homogeneous granular particles [4,5,16,17]. It gives $T_g/T_B=0.734\,60\ldots$ for a Gaussian bath.

For the piston, the dependence of the mean drift velocity on the mass ratio is nontrivial. As the mass ratio decreases, the drift velocity first *increases* and then decreases after reaching a maximum for $\mu \approx 0.7$. This implies that the fluctuating force first makes a positive contribution, but then starts to oppose the motion. It may be more difficult to confirm the behavior for $\mu < 1$ experimentally. First, experimental realizations of light pistons in a bath of heavy particles are likely to be much more difficult to achieve than the inverse. Second, it may be difficult to minimize the effect of recollisions, which are not accounted for in the theory.

We now show how to obtain systematic corrections to our Brownian limit prediction for finite mass ratios. We suppose that in, addition to the fluctuating force resulting from collisions with the bath particles, the motor is subject to a constant external pressure P_{ext}. The average of the net pressure performed over all possible velocities of the granular piston is equal to zero in the stationary state. Since the successive collisions are assumed uncorrelated, this condition can be expressed as

$$\langle P_{\text{net}}(V)\rangle = \int_{-\infty}^{\infty} dV f(V)P(V) + P_{\text{ext}} = 0. \tag{9}$$

It is convenient to rewrite this as $\langle P_{\text{net}}(V)\rangle = P(\langle V\rangle) + P_f + P_{\text{ext}} = 0$, where $P(\langle V\rangle)$ is the mean pressure when the piston is moving with a constant velocity, $\langle V\rangle$, and P_f is the mean pressure resulting from the fluctuations of the velocity around the mean value, $\langle V\rangle$. Now by expanding $P(V)$ about V^*, where $P(V^*)=0$, one builds an approximate expression for $\langle V\rangle$. Hence, to lowest order the drift velocity is given by

$$\langle V\rangle = V^* - \frac{\frac{1}{2}P''(V^*)\langle (V-V^*)^2\rangle + P_{\text{ext}}}{P'(V^*)}. \tag{10}$$

Since $M\langle (V-V^*)^2\rangle \approx T_g$, we have that in the Brownian limit, $\mu \to \infty$,

$$\langle V\rangle = V^* - P_{\text{ext}}/P'(V^*) + O(1/\mu). \tag{11}$$

To obtain the first-order correction to the Brownian limit, we use Eq. (10) to obtain

$$\langle V\rangle \simeq V^* - \frac{1}{2}\frac{P''(V^*)}{P'(V^*)}\frac{T_g}{M}, \tag{12}$$

where T_g, $P'(V^*)$, and $P''(V^*)$ are functions of α_L and α_R. Figure 2 shows that, for $\alpha_R=0$, $\alpha_L=1$, the first-order correction provides a good description for $\mu > 5$. Unfortunately, it predicts a rapid increase in $\langle V\rangle$ as μ decreases below this value. We attempt to improve the correction by truncating the expansion of P_f at third order. Then by approximating $\langle (V-V^*)^3\rangle \simeq 3\langle (V-V^*)^2\rangle\langle (V-V^*)\rangle$, we obtain

$$\langle V\rangle \simeq V^* - \frac{1}{2}\frac{P''(V^*)\frac{T_g}{M}}{P'(V^*) + \frac{1}{2}P'''(V^*)\frac{T_g}{M}}. \tag{13}$$

This approximation no longer diverges in the limit of small mass ratio, but saturates at a finite mean velocity and provides an accurate analytical expression when compared with the simulation data down to moderate mass ratio. The decrease in the mean velocity for small mass ratio is not obtained to this approximation, and it is unlikely that this regime can be captured with a perturbative expansion in mass ratio since, in the small mass ratio limit, the velocity distribution becomes strongly non-Gaussian. In any case, as noted above, the model neglecting recollisions probably provides a poor physical description in this regime.

The behavior shown in Fig. 2 is qualitatively similar for other values of the coefficients of restitution. The position of the maximum shifts slightly, but not much. For example, for $\alpha_L=0.5$, $\alpha_R=0$ it occurs at $\mu=0.4$. The quality of the perturbative expansions is also similar to the one shown.

We now exploit the above development to obtain the efficiency of the motor. If the external pressure in Eq. (9) is nonzero, work is being done by, or on, the piston. The power is given by

$$\dot{W} = P_{\text{ext}}L\langle V\rangle. \tag{14}$$

In the Brownian limit we can estimate the average drift velocity using Eq. (11). Alternatively, we can take $\langle V\rangle$ as the independent variable,

$$\dot{W} = P(\langle V\rangle)L\langle V\rangle, \tag{15}$$

with $P(V)$ given by Eq. (4). Of course, this approach is only valid in the Brownian limit. In general this has a parabolic form and is maximum for $0 < V < V^*$. When the velocity is greater than V^* the power is negative, implying that it is necessary to drive the piston externally in order to maintain the velocity. The energy dissipated when a bath particle collides with the piston is $\Delta E = -\frac{1}{2}\frac{M}{1+\mu}(1-\alpha^2)(V-v)^2$, where $\alpha = \alpha_R, \alpha_L$ depending on the side. So the rate of energy dissipation due to collisions on, e.g., the right-hand side is

$$\dot{E}_R(V) = \rho L \int_{-\infty}^{V} \Delta E(V,v)(V-v)\phi(v)dv. \tag{16}$$

The efficiency, or the fraction of the dissipated energy that is converted into work, is

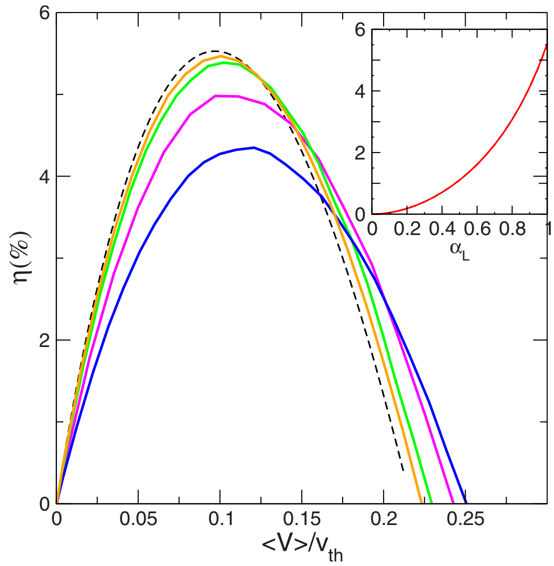

FIG. 3. (Color online) The efficiency as a function of the mean drift velocity for a system with $\alpha_R=0$, $\alpha_L=1$. The solid curves show simulation results for $\mu=1,2,5,10$ (from bottom to top), while the dashed line shows the theoretical prediction in the Brownian limit. The inset shows the maximum efficiency and efficiency at maximum power, both in the Brownian limit, as a function of α_L for a system with $\alpha_R=0$. The maximum efficiency is slightly larger but is indistinguishable from the efficiency at maximum power on the scale of the plot.

$$\eta(\alpha_L,\alpha_R,V)=\frac{\dot{W}}{|\dot{E}|}, \tag{17}$$

where $\dot{E}=\dot{E}_R+\dot{E}_L$. For an arbitrary bath distribution in the Brownian limit we find that

$$\frac{\eta}{2V}=\frac{\int_0^\infty dy\, y^2[(1+\alpha_L)\phi(V+y)-(1+\alpha_R)\phi(V-y)]}{\int_0^\infty dy\, y^3[(1-\alpha_L^2)\phi(V+y)+(1-\alpha_R^2)\phi(V-y)]}. \tag{18}$$

Figure 3 shows the efficiency calculated from the simulations. The results indicate that the efficiency increases with increasing piston mass and in the Brownian limit approaches that computed from Eq. (18) with a Gaussian distribution. We conclude that the efficiency is maximum in the Brownian limit. The inset shows the maximum efficiency as a function of α_L in this limit. This is virtually indistinguishable from the efficiency at maximum power. The exact maximum value, which occurs for $\alpha_L=1$ is 5.528%, while the efficiency at maximum power is $100/(3+48/\pi)=5.471\%$.

IV. CHIRAL ROTOR

Let us now consider the chiral rotor (see Fig. 1). The collision equations are

$$\begin{pmatrix}\Omega'\\ v'\end{pmatrix}=\begin{pmatrix}\Omega\\ v\end{pmatrix}+\frac{1+\alpha}{I+mx^2}(v-\Omega x)\begin{pmatrix}mx\\ -I\end{pmatrix}, \tag{19}$$

where Ω is the angular velocity, I is the moment of inertia, and $-L/2\le x\le L/2$ is the algebraic distance of impact from the center. The granular temperature of the rotor is given by $T_g=I\langle(\Omega-\langle\Omega\rangle)^2\rangle$.

Evaluation of the torque requires an integral over x, in addition to the bath particle velocity distribution, i.e.,

$$\Gamma(\Omega)=2M\rho\int_0^{L/2}dx\int_0^\infty dy\frac{xy^2}{1+\frac{mx^2}{I}}\times[(1+\alpha_L)\phi(x\Omega+y)-(1+\alpha_R)\phi(x\Omega-y)]. \tag{20}$$

As with the piston, we obtain the mean angular velocity in the Brownian limit, $mL^2/I\to\infty$, by setting the torque equal to zero. The result, as well as simulations, for finite ratios is shown in the inset in Fig. 2. Note that, unlike the piston, we do not observe a maximum as the moment of inertia decreases. As for the piston, this region is less interesting physically for the reasons discussed above.

For the asymmetric rotor in the Brownian limit, we find a maximum efficiency of 4.32%. This mechanical analysis indicates that heterogeneous granular particles are able to produce significant noise rectification, even in the Brownian limit. Provided that the solid friction about the axis can be controlled, the chiral rotor should be a good candidate for an experimental realization of a Brownian granular motor. Finally, strong inelasticity may induce some inhomogeneities in the bath particle distribution. While this effect is not easy to treat analytically, it is likely to be weak at low bath densities. Preliminary molecular-dynamics simulations of this system seem to confirm this expectation and the validity of the Boltzmann equation approach for rotors heavier than a bath particle [7]. After completing this work, we became aware of an experimental realization of a granular rotor similar to the one studied here [18].

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