

Aging and fluctuation-dissipation ratio in a nonequilibrium q -state lattice model

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A generalized version of the nonequilibrium linear Glauber model with q states in d dimensions is introduced and analyzed. The model is fully symmetric, its dynamics being invariant under all permutations of the q states. Exact expressions for the two-time autocorrelation and response functions on a d -dimensional lattice are obtained. In the stationary regime, the fluctuation-dissipation theorem holds, while in the transient the aging is observed with the fluctuation-dissipation ratio leading to the value predicted for the linear Glauber model.

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I. INTRODUCTION

The model introduced originally by Glauber [1] is defined on a one dimensional lattice where a one-spin-flip Markovian stochastic process takes place. It simulates the dynamics of a ferromagnetic Ising chain with first-neighbor interaction, and the stationary state is described by the Gibbs measure associated with the Ising Hamiltonian as a consequence of obeying detailed balance. In stationary regime at nonzero temperatures, the two-time functions like the autocorrelation and response function, are time-translationally invariant and connected through the fluctuation-dissipation theorem; furthermore, the model displays a transient where aging is observed [2]. At zero temperature, when the system becomes critical, the fluctuation-dissipation ratio (which is related to the effective temperature of the system [3]) assumes a nontrivial value $X_\infty = 1/2$.

When the model is extended to higher dimensions, and called Glauber model, no analytical solution is available due to the nonlinear structure of the transition rate. Nevertheless, the linearized version of the Glauber model, proposed a few years ago [4], can be treated by analytical tools in any dimension. The linear Glauber model can be seen as the voter model with noise, which displays a disordered (paramagnetic) phase only; in the absence of noise, however, the system becomes critical. The dynamics of linear Glauber model was investigated by some of the authors in a previous paper [5]. This model is microscopically irreversible, that is, it does not obey detailed balance in the stationary state, for dimension $d > 1$. Moreover, it has both the stationary and aging regimes, with a nontrivial fluctuation-dissipation ratio $X_\infty = 1/2$ in the later regime as in the usual one-dimensional Glauber case.

The fluctuation-dissipation relation was usually conceived for systems that obey detailed balance [6,7], and has been generalized to include nonstationary regimes by the introduction of an effective temperature measuring the violation of fluctuation-dissipation theorem ([8], and references therein). Many works have confirmed this phenomenon for several models [9–15]. Recent progress has suggested that it can also be invoked for nonequilibrium models [5,16], which does not have an associated known Hamiltonian (see also this issue in the context of kinetically constrained models [17,18]). In [16], the fluctuation-dissipation relations were

analyzed in a general class of models that exhibit up-down symmetry which does not obey detailed balance.

The above mentioned works are closely related to the question of universality in out-of-equilibrium processes [19]. In equilibrium statistical mechanics, it is widely known that the critical behavior of a system is governed by the fixed point of the renormalization transformation, and it turns out that only a few characteristics of the model are relevant to determine its universality class.

In out-of-equilibrium dynamics, some of the problems related to universality may be addressed by the generalized version of the fluctuation-dissipation theorem

$$R(t, t') = X(t, t') \frac{\partial}{\partial t'} C(t, t'), \quad (1)$$

where $R(t, t')$ and $C(t, t')$ are the response function and autocorrelation, respectively (see [20] for some recent results). The typical experimental situation under consideration is a quench from a completely disordered state, which in reversible systems corresponds to a high-temperature state, to the critical point. The usual fluctuation-dissipation relation is verified when the fluctuation-dissipation ratio $X(t, t')$ equals unity. It was conjectured that $X(t, t')$ would depend functionally on $C(t, t')$ only [21], but renormalization group analysis [22] and numerical calculations [23] indicated that the fluctuation-dissipation ratio is a function of t/t' . Furthermore, scaling arguments were casted to propose the asymptotic behavior for the autocorrelation and response function [10] at the critical temperature. This result suggested that the quantity

$$X_\infty = \lim_{t' \rightarrow \infty} \lim_{t \rightarrow \infty} X(t, t') \quad (2)$$

is universal due to its dependence to dynamical exponents and the ratio of autocorrelation and response amplitudes, which are conjectured to be universal [10].

In this paper, we introduce and analyze a nonequilibrium lattice model, which is a generalization of the nonequilibrium linear Glauber model to more than two states, which we call linear q -state model. The model can be understood as the linearized version of the dynamics associated to the equilibrium q -state Potts model. The dynamics of the model, as is the case of any dynamics of the Potts model, is invariant under the permutation of any two states. The model can also

be understood as a q -state voter model with noise. In this interpretation, a group of individuals are called to vote for one of q candidates. A voter changes his opinion by choosing randomly a neighbor individual and adopting the neighbor's opinion with probability μ and remaining with his opinion with probability $1-\mu$, the noise. Without noise, it reduces to the ordinary q -state voter model [24]. Similarly to the linear Glauber model, the present linear q -state model displays a paramagnetic phase whenever $0 < \mu < 1$ and becomes critical at $\mu=1$.

The analysis of such model has two main aims. First, it addresses the question raised in a previous result [5] about the dynamical phenomena of aging and violation of fluctuation dissipation for a class of systems that does not obey detailed balance—recall that these problems were usually studied through models that are described by a Hamiltonian. Finally, there is an additional interest in considering a model with a more general symmetry in order to verify its influence on the (possibly) universal quantity cited above, since in equilibrium statistical physics, symmetry plays a major role in the critical behavior.

The equilibrium Potts model with q states has been used to describe experimentally systems that display a number of identical states or structures at low temperatures [25] such as the adsorption of noble gases on graphite [25]. It has also been used to describe biological cell sorting [26]. The non-equilibrium model with many equivalent states such as the one studied here may be relevant in the description of systems where microscopic reversibility is not ensured like some biological phenomena[26].

The layout of this paper is as follows. In Sec. II, the nonequilibrium linear q -state model is defined and many one-time functions are determined analytically. The two-time functions are calculated in Sec. III, where the fluctuation-dissipation relations are carefully examined, and some dynamical exponents are calculated in Sec. IV. The summary of the main results and its discussions are found in the last section.

II. LINEAR q -STATE MODEL

Consider a d -dimensional hypercubic lattice with $N=L^d$ sites and periodic boundary conditions. To each site i there is a spin variable σ_i that takes the values $0, 1, \dots, q-1$. The time evolution is governed by a one-site dynamics in which the state of a given site i changes from σ_i to $\sigma'_i = \sigma_i + \alpha$ modulo q , where α is one of the q states, and the states of the other sites remain unchanged. The possible transitions are then the ones in which the state $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_i, \dots, \sigma_N)$ changes to the state $\sigma^{i,\alpha} = (\sigma_1, \sigma_2, \dots, \sigma'_i, \dots, \sigma_N)$ where $\sigma'_i = \sigma_i + \alpha$ modulo q . The corresponding transition rate is denoted by $w_i^\alpha(\sigma)$ and, for the nonequilibrium linear q -state model, is defined by

$$w_i^\alpha(\sigma) = \frac{1-\mu}{q} + \frac{\mu}{2d} \sum_{\delta} \delta(\sigma_i + \alpha, \sigma_{i+\delta}), \quad (3)$$

where the summation is over the nearest neighbors and $\delta(x, y)$ is the Kronecker delta, which equals 1 if $x=y$ and 0

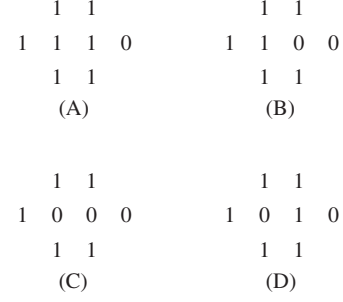


FIG. 1. A possible irreversible sequence for the bidimensional q -state model ($q \geq 2$).

otherwise and the parameter μ takes values in the interval $0 < \mu \leq 1$. The time evolution of the probability $P(\sigma, t)$ of finding the system at state σ at time t is governed by the master equation

$$\frac{d}{dt} P(\sigma, t) = \sum_i \sum_{\alpha} [w_i^\alpha(\sigma^{i,-\alpha}) P(\sigma^{i,-\alpha}, t) - w_i^\alpha(\sigma) P(\sigma, t)], \quad (4)$$

where the summation in α extends over the q states.

The probability of a spin at site j be at state, say 1, is given by $\langle \delta(\sigma_j, 1) \rangle$. Throughout this paper, the notation

$$\langle A(\sigma) \rangle = \sum_{\sigma} A(\sigma) P(\sigma, t) \quad (5)$$

will denote the average over spin configurations of the state function $A(\sigma)$. The equation of motion for $\langle \delta(\sigma_j, 1) \rangle$ can be written from the master Eq. (4) as

$$\frac{d}{dt} \langle \delta(\sigma_j, 1) \rangle = \frac{1-\mu}{q} - \langle \delta(\sigma_j, 1) \rangle + \frac{\mu}{2d} \sum_{\delta} \langle \delta(\sigma_{j+\delta}, 1) \rangle. \quad (6)$$

It is also possible to describe the time evolution of $\langle \delta(\sigma_j, 1) \delta(\sigma_k, 1) \rangle$,

$$\begin{aligned} \frac{d}{dt} \langle \delta(\sigma_j, 1) \delta(\sigma_k, 1) \rangle &= \left(\frac{1-\mu}{q} \right) [\langle \delta(\sigma_j, 1) \rangle + \langle \delta(\sigma_k, 1) \rangle] \\ &\quad - 2 \langle \delta(\sigma_j, 1) \delta(\sigma_k, 1) \rangle \\ &\quad + \frac{\mu}{2d} \sum_{\delta} [\langle \delta(\sigma_{j+\delta}, 1) \delta(\sigma_k, 1) \rangle \\ &\quad + \langle \delta(\sigma_{k+\delta}, 1) \delta(\sigma_j, 1) \rangle], \quad j \neq k, \quad (7) \end{aligned}$$

which is closely related to the pair correlation. In order to recover the results obtained by [4,5] for the linear Glauber model, it is necessary to connect $\delta(\sigma_j, 1)$ to an Ising spin variable s_j , that takes the values -1 or $+1$, through the relation

$$s_j = 2\delta(\sigma_j, 1) - 1. \quad (8)$$

As a last remark on the model, its irreversible property for $d \geq 2$ will be discussed. Consider, for instance, the four states shown in Fig. 1 on a square lattice ($d=2$). Suppose that the system follows the sequence of states A, B, C, and D and returns to the initial state A. If the interval between two suc-

cessive states is Δt , the probability of occurrence of the sequence $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ can be calculated through the transition rate w_i (which is $w_i^{\alpha=0}(\sigma)$) as

$$\begin{aligned} P(A \rightarrow B \rightarrow C \rightarrow D \rightarrow A) &= P(A|D)P(D|C)P(C|B)P(B|A)P(A) \\ &= \left(\frac{1-\mu}{q} + 3\frac{\mu}{2d}\right)^2 \left(\frac{1-\mu}{q} + 2\frac{\mu}{2d}\right) \left(\frac{1-\mu}{q}\right). \end{aligned} \quad (9)$$

This result is not necessarily equal to the probability of observing the reversed sequence $A \rightarrow D \rightarrow C \rightarrow B \rightarrow A$, which is

$$\begin{aligned} P(A \rightarrow D \rightarrow C \rightarrow B \rightarrow A) &= P(A|B)P(B|C)P(C|D)P(D|A)P(A) \\ &= \left(\frac{1-\mu}{q} + 4\frac{\mu}{2d}\right) \left(\frac{1-\mu}{q} + 2\frac{\mu}{2d}\right) \left(\frac{1-\mu}{q} + 1\frac{\mu}{2d}\right)^2. \end{aligned} \quad (10)$$

The existence of a sequence of states that is not reversible implies that the system is irreversible. A generalization of this result to higher dimensions is obtained by, for instance, filling the sites created by the introduction of more dimensions with spins $\tilde{\sigma}$ where $\tilde{\sigma} \neq 0, 1$ (this is the situation when $q > 2$; the case $q=2$ was already discussed in [5]). The argument above can also be invoked to show that the model is reversible in the one-dimensional case.

A. Site magnetization

The definition of the site magnetization will be guided by some constraints. At a fully ordered state, where the spin is at state, say $\sigma_j=1$, one should have $\langle \delta(\sigma_j, 1) \rangle = 1$, while at disordered state, where the spin is at any one of the q state with equal probability, the condition $\langle \delta(\sigma_j, 1) \rangle = 1/q$ should be satisfied. This leads to a natural definition of an ordered parameter $\sum_j m_j / N$, where the site magnetization is defined by

$$m_j(t) = \frac{q}{q-1} \langle \delta(\sigma_j, 1) \rangle - \frac{1}{q-1}. \quad (11)$$

The above definition leads to

$$\frac{d}{dt} m_j(t) = -m_j(t) + \frac{\mu}{2d} \sum_{\delta} m_{j+\delta}(t), \quad (12)$$

which is the time evolution of the site magnetization.

The Eq. (12) can be solved by the usual methods by introducing, for instance, the Fourier transform

$$m_p^F(t) = \sum_j m_j(t) e^{-ijp}, \quad (13)$$

and its inverse

$$m_j(t) = \frac{1}{N} \sum_p m_p^F(t) e^{ijp}, \quad (14)$$

where the summation in p is over the sites of the first Brillouin zone, in which each component of the vector p takes values inside the interval between $-\pi$ and π . The solution of the differential Eq. (12) is then

$$m_j(t) = \sum_{\ell} \Gamma_{j-\ell}(t-t') m_{\ell}(t'), \quad (15)$$

where

$$\Gamma_j(t) = \frac{1}{N} \sum_p e^{ijp-f_{\mu}(p)t}, \quad (16)$$

and

$$f_{\mu}(p) = 1 - \frac{\mu}{d} \sum_{i=1}^d \cos p_i, \quad (17)$$

for a d -dimensional hypercubic lattice with coordination number $z=2d$. For a homogeneous initial condition $m_j(t') = m^0$ for any j , it is straightforward that the site magnetization is constant ($m_j(t) = m^0$) for $\mu=1$. On the other hand, the condition $\mu \neq 1$ implies $m_j(t) = m^0 e^{-(1-\mu)(t-t')}$, which means that the magnetization decays to zero for sufficiently long time. The time correlation length $\bar{\tau}$, defined by $m = m^0 e^{-t/\bar{\tau}}$, diverges as $\bar{\tau} \sim (1-\mu)^{-1/\mu}$, from which $\nu_{\parallel} = 1$. Moreover, the static magnetization $M = \lim_{t \rightarrow \infty} \sum_j m_j(t) / N$ is always zero for $\mu \neq 1$ and is a non-zero constant (if $m^0 \neq 0$) at criticality ($\mu=1$); this jump in the magnetization implies $\beta=0$.

B. Pair correlation

The definition of pair correlation $q_{j,k}(t)$ for spins at j and k will be gauged to obey the following requirements: (i) $q_{j,j}(t) = 1$ for any j , (ii) $q_{j,k}(t) = 0$, $j \neq k$, for the paramagnetic state (iii) $q_{j,k}(t) = 1$, $j \neq k$, for the ordered state. These conditions leads to a natural definition for the pair correlation, which is

$$q_{j,k}(t) = \frac{q^2}{q-1} \langle \delta(\sigma_j, 1) \delta(\sigma_k, 1) \rangle - \frac{q}{2} [m_j(t) + m_k(t)] - \frac{1}{q-1}, \quad (18)$$

and is the only one where the pair correlation $q_{j,k}(t)$ is linear to $\langle \delta(\sigma_j, 1) \delta(\sigma_k, 1) \rangle$. Its time evolution,

$$\begin{aligned} \frac{d}{dt} q_{j,k}(t) &= -2q_{j,k}(t) + \frac{\mu}{2d} \sum_{\delta} [q_{j+\delta,k}(t) + q_{k+\delta,j}(t)] \\ &\quad - \frac{1}{2} (1-\mu)(q-2) [m_j(t) + m_k(t)], \quad j \neq k, \end{aligned} \quad (19)$$

is obtained from the master Eq. (4) and from Eq. (7).

From now on, it will be assumed that the pair correlation $q_{j,k}(t)$ depends on sites j and k through their difference $r=j-k$ only [note that $q_r(t) = q_{-r}(t)$]. Moreover, the system will be assumed to be in a random initial state (see [27–29] for other possibilities), such that the evolution process can be understood as a quench from $\mu=1/q$ to a $\mu \neq 1/q$, which implies $m_j(t) = 0$ for any t . This condition leads to

$$\frac{d}{dt} q_r(t) = -2q_r(t) + \frac{\mu}{d} \sum_{\delta} q_{r+\delta}(t), \quad r \neq 0. \quad (20)$$

The above equation, which is valid for $r \neq 0$ only, should be modified to comprise the case $r=0$, for which $q_0(t) = 1$. By

using a previously introduced method [4], we write the Eq. (20) in the form

$$\frac{d}{dt}q_r(t) = -2q_r(t) + \frac{\mu}{d} \sum_{\delta} q_{r+\delta}(t) + b(t) \delta_{r,0}, \quad (21)$$

which is now valid for any r , including the case $r=0$, provided $b(t)$ be chosen to ensure $q_0(t)=1$ [note that if $r \neq 0$, then Eq. (21) recovers Eq. (20)]. Formally, this means that the function $b(t)$ should satisfy

$$b(t) = \frac{d}{dt}q_0(t) + 2q_0(t) - \frac{\mu}{d} \sum_{\delta} q_{\delta}(t) = 2 - \frac{\mu}{d} \sum_{\delta} q_{\delta}(t). \quad (22)$$

Since the system was assumed to be in a completed disordered initial condition, then $q_r(0) = \delta_{r,0}$ and the equation of motion for the pair correlation Eq. (21) can be written as

$$s q_r^L(s) + 2q_r^L(s) - \frac{\mu}{d} \sum_{\delta} q_{r+\delta}^L(s) = [b^L(s) + 1] \delta_{r,0}, \quad (23)$$

where

$$q_r^L(s) = \int_0^{\infty} dt e^{-st} q_r(t) \quad (24)$$

is the Laplace transform of $q_r(t)$ (similar formula connects $b^L(s)$ and $b(t)$).

The Eq. (23) can be solved by introducing the Green function

$$G_r^L(s, \mu) = \frac{1}{N} \sum_p \frac{e^{irp}}{s + 2f_{\mu}(p)}, \quad (25)$$

where f is defined in Eq. (17), that satisfies

$$s G_r^L(s, \mu) + 2G_r^L(s, \mu) - \frac{\mu}{d} \sum_{\delta} G_{r+\delta}^L(s, \mu) = \delta_{r,0}. \quad (26)$$

Hence, the solution of the nonhomogeneous differential Eq. (23) is computed as

$$q_r^L(s) = \sum_{r'} G_{r-r'}^L(s, \mu) [1 + b^L(s)] \delta_{r',0} = [1 + b^L(s)] G_r^L(s, \mu). \quad (27)$$

The function $b^L(s)$ is fixed remembering that the condition $q_0(t)=1$, or $q_0(s)=1/s$, should be satisfied. It is easy to see that

$$b^L(s) = \frac{1}{s G_0^L(s, \mu)} - 1, \quad (28)$$

which implies

$$q_r^L(s) = \frac{1}{s} \frac{G_r^L(s, \mu)}{G_0^L(s, \mu)}. \quad (29)$$

The stationary value for the pair correlation is obtained through the Laplace final value theorem

$$q_r(\infty) = \lim_{t \rightarrow \infty} q_r(t) = \lim_{s \rightarrow 0} s q_r^L(s) = \frac{G_r^L(0, \mu)}{G_0^L(0, \mu)}. \quad (30)$$

In the same fashion, the stationary value for $b(t)$ can be calculated as

$$b(\infty) = \lim_{t \rightarrow \infty} b(t) = \lim_{s \rightarrow 0} s b^L(s) = \frac{1}{G_0^L(0, \mu)}. \quad (31)$$

C. Susceptibility

In this work, the susceptibility is defined through the (spatial) variance

$$\begin{aligned} \chi(t) &= \sum_r [\langle \delta(\sigma_0, 1) \delta(\sigma_r, 1) \rangle - \langle \delta(\sigma_0, 1) \rangle \langle \delta(\sigma_r, 1) \rangle] \\ &= \left(\frac{q-1}{q^2} \right) \sum_r q_r(t), \end{aligned} \quad (32)$$

where the random initial condition is being assumed. Starting from a disordered state [$m_j(0)=0$ for any j] and assuming $\mu \neq 1$, the stationary susceptibility,

$$\chi(\infty) = \lim_{t \rightarrow \infty} \chi(t) = \frac{q-1}{2q^2(1-\mu)} \frac{1}{G_0^L(0, \mu)}, \quad (33)$$

is obtained by invoking the previous result Eq. (30). In a hypercubic lattice, one has

$$G_0^L(0, \mu) \sim \begin{cases} \frac{1}{2} \left(\frac{d}{2\pi\mu} \right)^{d/2} \Gamma\left(1 - \frac{d}{2}\right) (1-\mu)^{d-2/2}, & 0 < d < 2, \\ -\frac{1}{2\pi\mu} \ln(1-\mu), & d = 2, \end{cases} \quad (34)$$

as $\mu \sim 1$, and $\lim_{\mu \rightarrow 1} G_0^L(0, \mu) < \infty$ for $d > 2$. Therefore, the stationary susceptibility is

$$\chi(\infty) \sim \begin{cases} \left(\frac{q-1}{q^2} \right) \left(\frac{2\pi}{d} \right)^{d/2} \frac{1}{\Gamma\left(1 - \frac{d}{2}\right)} (1-\mu)^{-d/2}, & 0 < d < 2, \\ \pi\mu \left(\frac{q-1}{q^2} \right) \frac{(1-\mu)^{-1}}{[-\ln(1-\mu)]}, & d = 2, \\ \left[\left(\frac{q-1}{2q^2} \right) \frac{1}{G_0^L(0, \mu)} \right] (1-\mu)^{-1}, & d > 2, \end{cases} \quad (35)$$

from which the exponent γ is obtained: the susceptibility diverges algebraically with exponent $d/2$ for $0 < d < 2$ and 1 for $d \geq 2$ with logarithmic corrections for $d=2$.

III. TWO-TIME AUTOCORRELATION AND RESPONSE FUNCTIONS

A. Two-time functions

The analytical form for the autocorrelation and response function will be determined in this subsection in order to analyze the stationary and aging dynamical regimes. The two-time autocorrelation is defined as

$$C(t, t') = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_j \{ \langle \delta[\sigma_j(t), 1] \delta[\sigma_j(t'), 1] \rangle - \langle \delta[\sigma_j(t), 1] \rangle \langle \delta[\sigma_j(t'), 1] \rangle \}, \quad (36)$$

with the two-time correlation

$$\begin{aligned} & \langle \delta[\sigma_j(t), 1] \delta[\sigma_j(t'), 1] \rangle \\ &= \sum_{\sigma} \sum_{\sigma'} \delta[\sigma_j(t), 1] P(\sigma, t | \sigma', t') \delta[\sigma_j(t'), 1] P(\sigma', t'), \end{aligned} \quad (37)$$

where $P(\sigma, t | \sigma', t')$ is the conditional probability of finding the configuration σ at time t given the configuration σ' at an earlier time t' . Noting that

$$\langle \delta[\sigma_j(t), 1] \rangle = \sum_{\sigma} \delta[\sigma_j(t), 1] P(\sigma, t | \sigma', t') \quad (38)$$

with the condition at time t' being $\langle \delta[\sigma_j(t'), 1] \rangle = \delta[\sigma_j(t'), 1]$, and invoking the definition Eq. (11) and Eq. (15), it is possible to show that

$$C(t, t') = \lim_{N \rightarrow \infty} \left(\frac{q-1}{q^2} \right) \sum_j \Gamma_j(t-t') q_j(t'), \quad (39)$$

for the disordered initial condition.

On the other hand, if one assumes an arbitrary initial condition, one has

$$\begin{aligned} C(t, t') = \lim_{N \rightarrow \infty} & \left[\left(\frac{q-1}{q^2} \right) \sum_j \Gamma_j(t-t') q_j(t') \right. \\ & + \frac{(q-1)(q-2)}{q^2} \frac{1}{N} \sum_j m_j(t) \\ & \left. - \left(\frac{q-1}{q} \right)^2 \frac{1}{N} \sum_j m_j(t) m_j(t') \right]. \end{aligned} \quad (40)$$

The evaluation of response function requires the presence of a (small) perturbation on the system. In the analysis of stochastic models, the introduction of an external field modifies the one-spin-flip rate to

$$\begin{aligned} w_j^h(\sigma) &= w_j(\sigma) e^{h_j \delta[\sigma_j, 1]} \\ &= \left(\frac{1-\mu}{q} \right) \delta[\sigma_j, 1] + \frac{h_j \mu}{2d} \sum_{\delta} \delta[\sigma_{j+\delta}, 1] \delta[\sigma_j, 1] + \mathcal{O}(h_j^2), \end{aligned} \quad (41)$$

where a Taylor's expansion was performed in the last step.

Performing similar calculations of Sec. II A, it is possible to show that

$$\frac{dm_j(t)}{dt} = -m_j(t) + \frac{\mu}{2d} \sum_{\delta} m_{j+\delta}(t) + \frac{1}{2q} b(t) h_j(t), \quad (42)$$

assuming again disordered initial condition, when $m_j(t) = \mathcal{O}(h_j)$. The solution of this differential equation, which can be obtained following the same previous ideas, is

$$m_j(t) = \frac{1}{2q} \sum_k \int_0^t dt' \Gamma_{j-k}(t-t') h_k(t') b(t'). \quad (43)$$

The above result, Eq. (43), is sufficient to evaluate the two-time response function

$$\begin{aligned} R(t, t') &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_j \left. \frac{\delta \langle \delta[\sigma_j(t), 1] \rangle}{\delta h_j(t')} \right|_{h \downarrow 0} \\ &= \left(\frac{q-1}{q} \right) \lim_{N \rightarrow \infty} \frac{1}{N} \sum_j \left. \frac{\delta m_j(t)}{\delta h_j(t')} \right|_{h \downarrow 0} \\ &= \left(\frac{q-1}{2q^2} \right) \Gamma_0(t-t') b(t'). \end{aligned} \quad (44)$$

It is worth to stress that the formula (44) for the autoreponse function is obtained even assuming an arbitrary initial condition.

B. Stationary regime

The stationary regime can be realized when both the waiting time (t') and observational time (t) grow with the constraint that $\tau = t - t' \geq 0$ is fixed. In this limit, and assuming disordered initial condition, the autocorrelation,

$$C(t, t') = C(\tau) = \frac{1}{G_0^L(0, \mu)} \int \frac{d^d p}{(2\pi)^d} \frac{e^{-f_{\mu}(p)\tau}}{2f_{\mu}(p)}, \quad (45)$$

and the response function,

$$R(t, t') = R(\tau) = \left(\frac{q-1}{2q^2} \right) \frac{\Gamma_0(\tau)}{G_0^L(0, \mu)}, \quad (46)$$

are functions of the time difference τ only, and they are related to the usual form of the fluctuation-dissipation relation $R(\tau) = \partial_{t'} C(\tau)$, as expected in a stationary regime.

C. Aging regime

The aging scenario can be seen when both the observational time (t) and waiting time (t') are made large without the difference $\tau = t - t'$ being fixed. In the stationary regime, where the difference $\tau = t - t'$ was fixed, the limit $t' \rightarrow \infty$ made the function $q_r(t')$ in Eq. (39) and $b(t')$ in Eq. (44) time independent. This is not the case in the aging regime, where both autocorrelation and response function depend on t and t' independently. More precisely, the transient is observed if $t \gg t'$, and this condition can be realized if the limit $t \rightarrow \infty$ is taken before the limit $t' \rightarrow \infty$. Assuming disordered initial condition, and from previous results, it can be shown at criticality $\mu=1$ that in this regime the autocorrelation function behaves as

$$C(t, t') \sim \begin{cases} \left(\frac{q-1}{q^2}\right) \frac{2^{d/2+1}}{d} \frac{\sin\left(\frac{\pi d}{2}\right)}{\pi} (t-t')^{-d/2} t'^{d/2}, & 0 < d < 2 \\ 2\left(\frac{q-1}{q^2}\right) \frac{t'}{(t-t') \ln t'}, & d = 2 \\ \left(\frac{q-1}{q^2}\right) \left(\frac{d}{2\pi}\right)^{d/2} \frac{1}{G_0^L(0,1)} (t-t')^{-d/2} t', & d > 2 \end{cases} \quad (47)$$

and the response function is asymptotically equal to

$$R(t, t') \sim \begin{cases} \left(\frac{q-1}{2q^2}\right) 2^{d/2} \frac{\sin\left(\frac{\pi d}{2}\right)}{\pi} (t-t')^{-d/2} t'^{d/2-1}, & 0 < d < 2 \\ 2\left(\frac{q-1}{2q^2}\right) \frac{1}{(t-t') \ln t'}, & d = 2 \\ \left(\frac{q-1}{2q^2}\right) \left(\frac{d}{2\pi}\right)^{d/2} \frac{1}{G_0^L(0,1)} (t-t')^{-d/2}, & d > 2 \end{cases} \quad (48)$$

The above results agree with the scaling $C(t, t') \sim t'^{-b} f_C(t/t')$ and $R(t, t') \sim t'^{-1-a} f_R(t/t')$ [30], where $a=b=(d-2+\eta)/z$ (see Table I), and f_C and f_R are scaling functions that behave as $f_{C/R}(t/t') \sim A_{C/R}(t/t')^{-\lambda/z}$ for $t/t' \sim \infty$.

In the aging regime, the fluctuation-dissipation theorem is not expected to hold anymore. The fluctuation-dissipation ratio

$$X(t, t') = \frac{R(t, t')}{\partial_{t'} C(t, t')}, \quad (49)$$

which measures the distance of the model to the stationary state [when $X(t, t')=1$], has the following limit:

TABLE I. Critical exponents for the nonequilibrium linear q -state model.

Exponent	$0 < d \leq 2$	$d > 2$
ζ	$d/2$	1
z	2	2
λ	d	d
θ	0	0

Exponent	$0 < d \leq 2$	$d > 2$
β	0	0
ν_{\parallel}	1	1
ν_{\perp}	1/2	1/2
γ	$d/2$	1
η	$2-d$	0

$$X(\infty, t') = \lim_{t \rightarrow \infty} X(t, t') = \frac{b(t')/2}{b(t') - (1-\mu)\chi(t')}, \quad (50)$$

where $b(t)$ and $\chi(t)$ are given, respectively, by Eqs. (31) and (33). This result implies

$$X_{\infty} = \lim_{t' \rightarrow \infty} [\lim_{t \rightarrow \infty} X(t, t')] = \begin{cases} 1, & \mu \neq 1, \\ \frac{1}{2}, & \mu = 1, \end{cases} \quad (51)$$

which is identical to the Ising case [5].

The previous result has considered disordered initial condition, which assumes $m_j(t=0)=0$ for every site j . If one starts from an arbitrary initial condition, it is possible to show that [now using Eqs. (40) and (44)]

$$X(\infty, t') = \lim_{N \rightarrow \infty} \left[\frac{b(t')/2}{b(t') - (1-\mu) \sum_j q_j(t')} \right], \quad (52)$$

which shows that a similar formula for $X(\infty, t')$ is obtained even for an arbitrary initial condition. The fluctuation-dissipation ratio X_{∞} is identical to Eq. (51); other non-trivial values for this ratio (for nonzero magnetization as initial condition) can be found, for instance, in [27–29].

IV. DYNAMICAL EXPONENTS

A. Dynamical exponent θ

From the solution of Eq. (12),

$$m_j(t) = m^0 e^{-(1-\mu)t}, \quad (53)$$

one sees that at the critical point $\mu=1$ the magnetization is constant and does not vary with time. This implies the exponent θ , defined through $m_j(t) \sim m^0 t^{\theta}$ [31] in the short-time

regime, to be zero. It is possible also to calculate this exponent by means of the time correlation of the total magnetization [32].

B. Dynamical exponent λ/z

At the critical point $\mu=1$, one may calculate the dynamical exponents λ and z , defined through $C(t,0) \sim t^{-\lambda/z}$. From Eq. (39), it is immediate that

$$C(t,0) = \left(\frac{q-1}{q^2}\right) \sum_j \Gamma_j(t) q_j(0). \quad (54)$$

The $q_r(0)$ can be evaluated by invoking Eq. (29) and the Laplace initial value theorem

$$q_r(0) = \lim_{t' \rightarrow 0^+} q_r(t') = \lim_{s \rightarrow \infty} s q_r^L(s) = \delta_{r,0}. \quad (55)$$

In the thermodynamic limit, this result implies

$$C(t,0) = \left(\frac{q-1}{q^2}\right) e^{-t} \left[I_0\left(\frac{\mu}{d}t\right) \right]^d \sim \left(\frac{q-1}{q^2}\right) \frac{e^{-(1-\mu)t}}{(2\pi\mu/d)^{d/2}} t^{-d/2}, \quad (56)$$

where $I_0(x)$ is the modified Bessel function of order 0 and the last passage is obtained in the asymptotic limit $t \gg 1$. The autocorrelation decays exponentially for $\mu \neq 1$; nevertheless, if $\mu=1$, one sees that

$$\frac{\lambda}{z} = \frac{d}{2}. \quad (57)$$

C. Dynamical exponent ζ

Another dynamical exponent of interest is ζ , defined through $\chi(t) \sim t^\zeta$. From the results obtained in Sec. II B, the Laplace transform of the susceptibility can be written as

$$\chi^L(s) = \left(\frac{q-1}{q^2}\right) \frac{1}{s^2 G_0^L(s, \mu)}. \quad (58)$$

The asymptotic behavior of the (dynamical) susceptibility $\chi(t)$ for large times corresponds to the Laplace antitransform of $\chi^L(s)$ when $s \sim 0$. In this regime, one can evaluate the Green function at criticality as

$$G_0^L(s,1) \sim \begin{cases} \left(\frac{d}{4\pi}\right)^{d/2} \Gamma\left(1 - \frac{d}{2}\right) s^{d-2/2}, & 0 < d < 2, \\ -\frac{1}{2\pi} \ln s, & d = 2, \\ G_0^L(0,1), & d > 2, \end{cases} \quad (59)$$

which yields

$$\chi(t) \sim \begin{cases} \frac{2}{d} \left(\frac{q-1}{q^2}\right) \left(\frac{4\pi}{d}\right)^{d-2} \frac{1}{\Gamma\left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right)} t^{d-2}, & 0 < d < 2, \\ 2\pi \left(\frac{q-1}{q^2}\right) \frac{t}{\ln t}, & d = 2, \\ \left(\frac{q-1}{q^2}\right) \frac{1}{G_0^L(0,1)} t, & d > 2, \end{cases} \quad (60)$$

showing that

$$\zeta = \begin{cases} \frac{d}{2}, & 0 < d < 2, \\ 1, & d \geq 2, \end{cases} \quad (61)$$

with logarithmic corrections for $d=2$.

D. Dynamical exponent z

The exponent z , defined by the behavior of the correlation length $\xi \sim (1-\mu)^{-\nu/z}$ will be estimated through the spatial correlation, which can be casted as

$$q_r(t=\infty) = \frac{G_r^L(0, \mu)}{G_0^L(0, \mu)} = \left(\frac{d}{2\pi\mu}\right)^{d-2} \left(\frac{\mu}{d}\right)^{d-2/4} \frac{2^{d-2/4} (1-\mu)^{d-2/4}}{G_0(0, \mu) r^{d-2/2}} \times K_{d-2/2}\left(r \sqrt{\frac{2(1-\mu)d}{\mu}}\right), \quad (62)$$

where $K_\nu(z)$ is the Macdonald's function, which behaves as

$$K_\nu(z) \sim \begin{cases} \frac{2^{\nu-1} \Gamma(\nu)}{z^\nu}, & |z| \ll 1 \text{ and } \nu \neq 0, \\ \ln\left(\frac{2}{z}\right), & |z| \ll 1 \text{ and } \nu = 0, \\ \frac{e^{-z}}{\sqrt{2\pi z}}, & |z| \gg 1. \end{cases} \quad (63)$$

Therefore, for large distances, the correlation decays exponentially as $e^{-r/\xi}$, where

$$\xi = \sqrt{\frac{\mu}{2d}} (1-\mu)^{-1/2} \quad (64)$$

is associated to the correlation length. On the other hand, when the system is near criticality in the sense that $r \ll \xi$, one has

$$q_r(t = \infty) \sim \begin{cases} \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{2-d}{2}\right)} \frac{1}{(r/2\xi)^{d-2}}, & 0 < d < 2, \\ \frac{\ln(r/2\xi)}{\ln(\sqrt{2\mu/2\xi})}, & d = 2, \\ \frac{d}{4\pi^{d/2}\mu} \frac{\Gamma\left(\frac{d-2}{2}\right)}{G_0(0,\mu)} \frac{1}{r^{d-2}}, & d > 2, \end{cases} \quad (65)$$

from which it is possible to see that the exponent η [33] is equal to zero for $d > 2$ and is equal to $2-d$ if $0 < d \leq 2$. From Eq. (64), the exponent ν_\perp [defined by $\xi \sim (1-\mu)^{-\nu_\perp}$] is equal to $1/2$, while the dynamical exponent $z = \nu_\parallel / \nu_\perp$ is then $z = 2$ (since $\nu_\parallel = 1$, as seen in Sec. II A).

Collecting the previous results (see Secs. II A, II C, and IV B–IV D), the Table I is obtained. One should recall that for $d > 2$, one has $\zeta = 1$ and $\gamma = 1$ (the other exponents remain unchanged). These values for the exponents satisfy the relations $\zeta z = (d - 2\beta/\nu)$ and $\theta z = (d - \lambda)$. It is worth mentioning that these exponents are in agreement with the universality class of the voter model [34,35].

V. CONCLUSIONS

This paper has established a number of exact calculations for the dynamical and static behavior of the d -dimensional nonequilibrium linear q -state lattice model, which is a generalization of the nonequilibrium linear Glauber model. This model is fully symmetric in the sense that it is invariant under the permutation among all the q states, having the same symmetry of the equilibrium Potts model. Although the analytical form of many functions are now distinct from the linear Glauber model, many similarities were reported in this paper. The stationary and aging regimes were both characterized, with the usual fluctuation-dissipation relation satisfied in the former regime and violated in the later regime at the criticality $\mu = 1$. The fluctuation-dissipation ratio X_∞ indicates that the dynamical behavior of the present model is similar to the linear Glauber model with Ising spins, which is just a particular case ($q = 2$), and thus independent of the number of states q .

When $\mu = 1$, we recover the voter model with q state and in this case the system finds itself in the critical state. From the results for the correlation functions it is possible to generalize a result already known for the case $q = 2$ about the stationary states. In one and two dimensions the only possible stationary states are the ones in which all the sites of the lattice are in one of the q absorbing states, which one depends on the initial configuration. This statements seems to be in contradiction with the result Eq. (53) which says that the magnetization remains constant. To understand this it suffices to remember the meaning of choosing an initial condition with magnetization m_0 . This means that one should consider several initial configurations whose average gives the magnetization m_0 . Each one of these configuration will reach

one of the q absorbing states. The averages over these absorbing states will give an average m which according to Eq. (53) should equal m_0 . In three or more dimensions there are other states stationary states besides the q absorbing states.

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APPENDIX

In this appendix (and also in the main text), the Landau notation was adopted:

- (i) if $f(x) = \mathcal{O}[g(x)]$ [assuming $g(x) > 0$], then there exists x_0 such that $|f(x)| < Ag(x)$ for some constant A if $x > x_0$.
- (ii) if $f(x) = o[g(x)]$ [assuming $g(x) > 0$], then $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$.

1. Dynamical susceptibility (case $d = 2$)

The asymptotic behavior of the dynamical susceptibility is calculated through the Laplace antitransform

$$\chi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{st} \chi^L(s), \quad (66)$$

where c is real and larger than the real part of any pole of $\chi^L(s)$, given by Eq. (58). Since the $0 < d < 2$ and $d > 2$ cases are simpler, the evaluation of dynamical susceptibility will be presented for $d = 2$ only, which implies

$$\chi^L(s) \sim 2\pi \left(\frac{q-1}{q^2} \right) \frac{1}{s^2(-\ln s)}. \quad (67)$$

One should first consider the integral

$$B(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{e^{st}}{s(-\ln s)}, \quad (68)$$

where c and \tilde{c} are real and larger than the real part of any pole of the integrand. The function $B(t)$ relates to $\chi(t)$ through

$$\frac{d}{dt} \chi(t) = 2\pi \left(\frac{q-1}{q^2} \right) B(t). \quad (69)$$

By using the contour shown in Fig. 2 and invoking the residue theorem, the integral Eq. (68) can be casted in the form

$$B(t) = \int_0^\infty dr \frac{e^{-rt}}{r(\ln^2 r + \pi^2)} = B_a(t) + B_b(t) + B_c(t), \quad (70)$$

where

$$B_a(t) = \int_0^{1/t \ln t} dr \frac{e^{-rt}}{r(\ln^2 r + \pi^2)}, \quad (71)$$

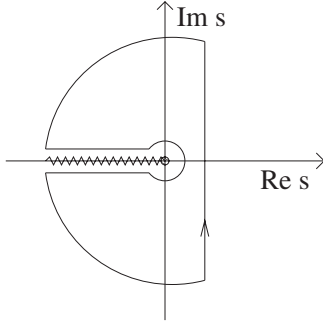


FIG. 2. Integration contour used in the evaluation of the function $B(t)$ given in Eq. (68).

$$B_b(t) = \int_{t \ln t / t}^{\ln t / t} dr \frac{e^{-rt}}{r(\ln^2 r + \pi^2)} \quad (72)$$

and

$$B_c(t) = \int_{\ln t / t}^{\infty} dr \frac{e^{-rt}}{r(\ln^2 r + \pi^2)}. \quad (73)$$

These three functions, $B_a(t)$, $B_b(t)$, and $B_c(t)$ will be evaluated separately.

a. Function $B_a(t)$

Since

$$B_a(t) = \int_0^{1/t \ln t} dr \frac{1}{r(\ln^2 r + \pi^2)} \left[1 + \sum_{m=1}^{\infty} \frac{(-rt)^m}{m!} \right], \quad (74)$$

and

$$\begin{aligned} & \left| \int_0^{1/t \ln t} dr \frac{1}{r(\ln^2 r + \pi^2)} \sum_{m=1}^{\infty} \frac{(-rt)^m}{m!} \right| \\ & \leq \int_0^{1/t \ln t} dr \frac{1}{r(\ln^2 r + \pi^2)} \sum_{m=1}^{\infty} (rt)^m \\ & \leq \int_0^{1/t \ln t} dr \frac{1}{r \ln^2 r} \sum_{m=1}^{\infty} \left(\frac{1}{t \ln t} t \right)^m \leq \frac{1}{\ln t} \int_0^{1/t \ln t} dr \frac{1}{r \ln^2 r} \\ & = \frac{1}{\ln t} \frac{1}{\ln(t \ln t)} = \mathcal{O}\left(\frac{1}{\ln^2 t}\right), \end{aligned} \quad (75)$$

then

$$\begin{aligned} B_a(t) &= \int_0^{1/t \ln t} dr \frac{1}{r(\ln^2 r + \pi^2)} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right) \\ &= \int_{-\infty}^{-1/\pi \ln(t \ln t)} \frac{dy}{\pi(y^2 + 1)} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right) \\ &= -\frac{1}{\pi} \left\{ \arctan \left[\frac{1}{\pi} \ln(t \ln t) \right] - \frac{\pi}{2} \right\} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right), \end{aligned} \quad (76)$$

where the change of variable $r \rightarrow e^{\pi y}$ was performed in the second line. For $t \gg 1$, one has

$$B_a(t) = \frac{1}{\ln(t \ln t)} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right) = \frac{1}{\ln t} + \mathcal{O}\left(\frac{\ln(\ln t)}{\ln^2 t}\right). \quad (77)$$

b. Function $B_b(t)$

The function $B_b(t)$ has the following upper bound:

$$\begin{aligned} |B_b(t)| &= \left| \int_{1/t \ln t}^{\ln t / t} dr \frac{e^{-rt}}{r(\ln^2 r + \pi^2)} \right| \leq \frac{1}{\ln^2 t} \int_{1/t \ln t}^{\ln t / t} dr \frac{e^{-rt}}{r} \\ &= \frac{1}{\ln^2 t} \int_{1/t \ln t}^{\ln t} du \frac{e^{-u}}{u} \leq \frac{e^{-1/\ln t}}{\ln^2 t} \int_{1/\ln t}^{\ln t} \frac{du}{u} \\ &= \frac{2e^{-1/\ln t} \ln(\ln t)}{\ln^2 t} = \mathcal{O}\left(\frac{\ln(\ln t)}{\ln^2 t}\right). \end{aligned} \quad (78)$$

c. Function $B_c(t)$

The function $B_c(t)$ has the following upper bound:

$$\begin{aligned} |B_c(t)| &= \left| \int_{\ln t / t}^{\infty} dr \frac{e^{-rt}}{r(\ln^2 r + \pi^2)} \right| \leq \frac{1}{\pi^2} \int_{\ln t / t}^{\infty} dr \frac{e^{-rt}}{r} \\ &\leq \frac{1}{\pi^2} \frac{1}{\ln t} \int_{\ln t / t}^{\infty} dr e^{-rt} = \frac{1}{\pi^2 t \ln t} = \mathcal{O}\left(\frac{1}{t \ln t}\right). \end{aligned} \quad (79)$$

From Eqs. (70) and (77)–(79), one finally has

$$B(t) = \frac{1}{\ln t} + \mathcal{O}\left[\frac{\ln(\ln t)}{\ln^2 t}\right], \quad (80)$$

which can be inserted in (69) to yield

$$\begin{aligned} \chi(t) &\sim 2\pi \left(\frac{q-1}{q^2} \right) \int dt B(t) \\ &= 2\pi \left(\frac{q-1}{q^2} \right) \left[\int dt \left(\frac{1}{\ln t} - \frac{1}{\ln^2 t} \right) + \int dt \frac{1}{\ln^2 t} \right] \\ &\sim 2\pi \left(\frac{q-1}{q^2} \right) \frac{t}{\ln t} [1 + o(1)]. \end{aligned} \quad (81)$$

2. Spatial correlation function

In the thermodynamic limit ($N \rightarrow \infty$), the spatial correlation function can be casted as

$$\begin{aligned} q_r(t \rightarrow \infty) &= \frac{G_r(0, \mu)}{G_0(0, \mu)} \\ &= \frac{1}{G_0(0, \mu)} \int_{[-\pi, \pi]^d} \frac{d^d p}{(2\pi)^d} \frac{e^{i\vec{r} \cdot \vec{p}}}{\epsilon + 2\mu \left(1 - \frac{1}{d} \sum_{i=1}^d \cos p_i \right)}, \end{aligned} \quad (82)$$

as seen in Sec. II B, and $\epsilon = 2(1 - \mu)$. It is straightforward to see also that

$$\begin{aligned}
q_r(t \rightarrow \infty) &= \frac{1}{G_0(0, \mu)} \int_{[-\pi, \pi]^d} \frac{d^d p}{(2\pi)^d} \\
&\times \int_0^\infty dy e^{-y[\epsilon + 2\mu(1 - 1/d \sum_{i=1}^d \cos p_i)]} e^{i\vec{r} \cdot \vec{p}} \\
&= \frac{1}{G_0(0, \mu)} \int_0^\infty \frac{dy}{\epsilon} e^{-y - 2\mu y/\epsilon} \prod_{i=1}^d \int_{-\pi}^{\pi} I_{r_i} \left(\frac{2\mu y}{\epsilon d} \right),
\end{aligned} \tag{83}$$

where $I_\nu(z)$ is the modified Bessel function, which behaves asymptotically as ([36])

$$I_\nu(z \sim \infty) \sim \frac{e^{z - \nu^2/2z}}{\sqrt{2\pi z}}. \tag{84}$$

Therefore, when the system approaches the critical point ($\epsilon \sim 0^+$), one has

$$\begin{aligned}
q_r(t \rightarrow \infty) &= \frac{1}{G_0(0, \mu)} \left(\frac{d}{4\pi\mu} \right)^{d/2} \epsilon^{d-2/2} \int_0^\infty dy e^{-y - r^2 \epsilon d/4\mu y} \\
&= \left(\frac{d}{2\pi\mu} \right)^{d/2} \frac{1}{G_0(0, \mu)} \frac{[2(1-\mu)]^{d-2/2}}{(r/\xi)^{d-2/2}} K_{d-2/2}(r/\xi),
\end{aligned} \tag{85}$$

where $K_\nu(z)$ is the Macdonald's function (or modified Bessel function of the third kind)

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2} \right)^\nu \int_0^\infty dy e^{-y - z^2/4y} t^{-\nu-1}, \quad |\arg z| < \frac{\pi}{4}, \tag{86}$$

and

$$\xi = \sqrt{\frac{\mu}{2d}} (1-\mu)^{-1/2}. \tag{87}$$

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