PHYSICAL REVIEW E **82**, 011126 (2010)

Vicious walks with long-range interactions

Igor Goncharenko and Ajay Gopinathan*
School of Natural Sciences, University of California, Merced, California 95343, USA
(Received 27 April 2010; published 19 July 2010)

The asymptotic behavior of the survival or reunion probability of vicious walks with short-range interactions is generally well studied. In many realistic processes, however, walks interact with a long-ranged potential that decays in d dimensions with distance r as $r^{-d-\sigma}$. We employ methods of renormalized field theory to study the effect of such long-range interactions. We calculate the exponents describing the decay of the survival probability for all values of parameters σ and d to first order in the double expansion in $\varepsilon=2-d$ and $\delta=2-d-\sigma$. We show that there are several regions in the σ-d plane corresponding to different scalings for survival and reunion probabilities. Furthermore, we calculate the leading logarithmic corrections.

DOI: 10.1103/PhysRevE.82.011126 PACS number(s): 64.60.ae, 64.60.F−, 05.40.Jc, 64.60.Ht

I. INTRODUCTION

Systems consisting of diffusing particles or random walks interacting by means of a long-range potential are nonequilibrium systems, which describe different phenomena in physics, chemistry, and biology. From a physical perspective they are used to study metastable supercooled liquids [1,2], melting in type-II high-temperature superconductors [3], electron transport in quasi-one-dimensional conductors [4], and carbon nanotubes [5]. From a chemical viewpoint the interest in these systems lies in the fact that some diffusion-controlled reactions processes rely on the diffusion of long-range interacting particles which react after they are closer than an effective capture distance. Some examples include radiolysis in liquids [6], electronic energy transfer reactions [7], and a large variety of chemical reactions in amorphous media [8]. From a biological viewpoint, the investigation of these systems is helpful in understanding the dynamics of interacting populations in terms of predator-prey models [9,10] and membrane inclusions with curvature-mediated interactions [11,12].

Vicious walks (VWs) are a class of nonintersecting random walks, where the process is terminated upon the first encounter between walkers [13]. The fundamental physical quantity describing VW is the survival probability which is defined as the probability that no pair of particles has collided up to time t. Diffusing particles or walks that are not allowed to meet each other but otherwise remain free, we call pure VW. The behavior of pure VW is generally well known. The survival probability for such a system has been computed in the framework of renormalization-group theory in arbitrary spatial dimensions up to two-loop order [14–16]. These approximations have been confirmed by exact results available in one dimension from the solution of the boundary problem of the Fokker-Planck equation [9,10], using matrix model formalism [17] and Bethe ansatz technique [18]. On the other hand the effect of long-range interactions has been extensively investigated in many-body problems. It has been shown that the existence of long-range disorder leads to a rich phase diagram with interesting crossover effects [19–21]. If the potential is Coulomb-like ($\sim r^{-1-\sigma}$) then systems in one dimension behave similar to a one-dimensional version of a Wigner crystal [22] for $\sigma<0$ and similar to a Luttinger liquid for $\sigma\geq 0$ [23]. If the potential is logarithmic then in the long-time limit the dynamics of particles are described by nonintersecting paths [17,24]. The generalization of VW that includes the effect of long-range interactions has not attracted much attention in the literature. Up to our knowledge there was one attempt to study long-range VW [25]. Here, the authors considered the case of a long-range potential decaying as $gr^{-\sigma-d}$, where g is a coupling constant. It was shown by applying the Wilson momentum shell renormalization group that only one of the critical exponents characterizes long-range VW. For a specific value of σ ($\sigma=2-d$) they show that the exponent γ, which determines the decay of the asymptotic survival probability with time, is given by the expression

$$\gamma=\frac{p(p-1)}{4}u_1, \tag{1}$$

where p is the number of VW in the system, $u_1=\{\varepsilon/2+[(\varepsilon/2)^2+g]^{1/2}\}$, and $\varepsilon=2-d$. There are limitations to the above approach. First, it is restricted to a single form of the potential ($\sim r^{-2}$) and systems such as membrane inclusions and chemical reactions have different power-law potentials. Second, it considers identical walkers but one would like to have results if the diffusion constants of all walkers are different. Finally, it is not convenient to compute higher-loop corrections using the Wilson formalism.

In this paper we reconsider the problem of long-range VW using methods of Callan-Symanzik renormalized field theory in conjunction with an expansion in $\varepsilon=2-d$ and $\delta=2-d-\sigma$. We note that it is more convenient to compute logarithmic and higher-loop corrections by using this method. We derive the asymptotics of the survival and reunion probability for all values of the parameters (σ,d).

In this paper we will show that there are several regions in the σ-d plane in which we have different behaviors of the critical exponent. Our results are summarized in Table I. We note that results on the line $\sigma+d=2$ have been obtained before [25]. Regions I and IV correspond to Gaussian or mean-field behavior (see Fig. 3). In region II we found that the

*agopinathan@ucmerced.edu

TABLE I. Large-time asymptotics of the one-loop survival probability of p sets of particles with n_j particles in each set for different regions of the σ-d plane. We refer to Fig. 3 for specific value of σ and d in each region.

Region	Survival probability
I	$t^{-(d-2)/2}+t^{-(d+\sigma-2)/2}$
II	$t^{-(1/2)\Sigma_{ij}n_in_j\varepsilon}$
III	$t^{-(u_1/2)\Sigma_{ij}n_in_j(1+\delta/2\ \ln t)}$
IV	$t^{-(d-2)/2}$
V, $d=2$	$t^{-(\sqrt{g_0}/2)\Sigma_{ij}n_in_j(1+\delta/2\ \ln t)}$
VI, $\sigma=2-d$	$t^{-(u_1/2)\Sigma_{ij}n_in_j}$ [a]

[a] u_1 is defined by formula (1).

system reproduces pure VW. Logarithmic corrections in region III and at the short-range upper critical dimension $d=2$ have been obtained as series expansion in $\delta=2-\sigma-d$.

The remainder of this paper is organized as follows. Section II reviews the field theoretical formulation of long-range VW and describes Feynman rules and dimensionalities of various quantities. In Sec. III we derive the value of all fixed points and study their stability. Section IV presents results for the critical exponents and logarithmic corrections of various dynamical observables. Section V contains our concluding remarks. In the Appendix we give the details of the computation of some integrals that appear in Sec. III.

II. MODELING VW WITH LONG-RANGE INTERACTIONS

As the starting point of the description of our model we consider p sets of diffusing particles or random walks with n_i particles in each set $i=1,\ldots,p$, with a pairwise intraset interaction which includes a local or short-range part and a nonlocal or long-range tail. The local part determines the vicious nature of particles: if two walks belonging to the different sets are brought close to each other, both are annihilated. Particles belonging to the same set are supposed to be independent. At $t=0$ all particles start in the vicinity of the origin. We are interested in the survival and reunion probabilities of walks at time $t>0$.

A continuum description of a system of N Brownian particles X_i with two-body interactions is simplified by the coarse-graining procedure in which a large number of microscopic degrees of freedom are averaged out. Their influence is simply modeled as a Gaussian noise term in the Langevin equations. A convenient starting point for the description of the stochastic dynamics is the path-integral formalism. Then the system under consideration is modeled by the classical action

$$S=\int_0^{+\infty}dt\left(\sum_{i=1}^{N}\dot{X}_i^2/(2D_i)+\sum_{i<j}V(X_i-X_j)\right), \tag{2}$$

where t is the (imaginary) time and $X_i(t)$ is the d-dimensional vector denoting the position of ith particle at time t. D_i is an ith particle diffusion coefficient. The path-integral representation of the probability density function for the particle displacements from their original positions is given by the functional $\mathcal{Z}=\int\mathcal{D}X\exp[-S]$. The survival probability is defined as the expectation value

$$P(t)=\left\langle\prod_{i,j}[1-\delta(X_i(t)-X_j(t))]\right\rangle \tag{3}$$

with respect to the functional $\mathcal{Z}$. It is computed in the framework of usual perturbation theory and will be a sum of integrals over internal degrees of freedom. It is more convenient to perform these integrations in Fourier space. To do this we would need the Fourier transform of the interaction potential $V(r)$. We note that it is comprised of a short-range part of the form $V_0(r)=\lambda\delta(r)$ and a long-range part which decays with the distance r as a power law, $V_l(r)=gr^{-d-\sigma}$. The Fourier transform of the latter is divergent if $\sigma\geq 0$. We introduce the cutoff parameter a to regularize the singularity $V_l(r)=g(r^2+a^2)^{-(d+\sigma)/2}$. Fourier transformation of this function is given by the expression

$$V_l(q)=g\frac{\pi^{d/2}2^{\sigma}}{\Gamma\left(\frac{d+\sigma}{2}\right)}(q/a)^{\sigma/2}K_{\sigma/2}(aq), \tag{4}$$

where K_σ is the modified Bessel function with index σ. Small a expansion of Eq. (4) at leading order yields

$$V_l(q)\sim g\begin{cases}q^{\sigma}, & \text{if}\ \ \sigma\neq 0\\ \ln(aq), & \text{if}\ \ \sigma=0,\end{cases} \tag{5}$$

where we used the property $K_{-\sigma}(x)=K_\sigma(x)$ of the Bessel function. The nonuniversal coefficient coming from the Taylor expansion can be absorbed by the appropriate renormalization of the constant g. Special cases when σ is even give logarithmic behavior. Effectively it does not change our results. So we focus on the typical term q^σ.

The second quantized version of the action (2) can be constructed using standard methods [26,27]. The generalization of the action to the long-range interacting case is also known [28,29]. The result is

$$\begin{aligned}S(\phi_i,\phi_i^\dagger)&=\int dtd^dx\left\{\sum_i[\phi_i^\dagger\partial_t\phi_i+D_i\nabla\phi_i^\dagger\nabla\phi_i]\right\}\\&\quad+\int dtd^dxd^dy\sum_{i<j}\phi_i^\dagger(t,x)\phi_i(t,x)V_{ij}(x-y)\\&\quad\times\phi_j^\dagger(t,y)\phi_j(t,y).\end{aligned} \tag{6}$$

The first term describes the evolution of free random walks with diffusion constants D_i. The potential is

$$V_{ij}(x-y)=\lambda_{ij}\delta(x-y)+g_{ij}V(x-y), \tag{7}$$

and we refer to λ_{ij}, g_{ij} as short-range and long-range coupling constants, respectively.

A dynamic response functional associated with the action (6) is

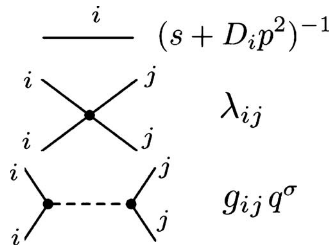

FIG. 1. Feynman rules for the theory (6). Notice that both λ and g vertices appear with different i and j indices and that g has momentum dependence.

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}\phi^{\dagger} e^{-S(\phi_i,\phi_i^{\dagger})}, \quad (8)$$

where $\phi_i(x,t)$ is the complex scalar field. After the quantization we may treat $\phi_i^{\dagger}(x,t)$ as the creation operator which creates a particle of sort i at point x at time t. Having the dynamic response functional, correlation functions can be computed as functional averages (path integrals) of monomials of ϕ and $\phi^{\dagger}$ with the weight $\exp\{-S(\phi,\phi^{\dagger})\}$.

As a first step toward the renormalization-group analysis of this model, we discuss the dimensions of various quantities in Eq. (6) expressed in terms of momentum,

$$[t] = p^{-2}, \quad [\phi] = p^{d}, \quad [\lambda] = p^{2-d}, \quad [g] = p^{2-d-\sigma}. \quad (9)$$

The naive dimension of the coupling constant g allows us to identify the upper critical dimension $d_c(\sigma)=2-d-\sigma$. For $\sigma > 0$, the short-range term naively dominates the long-range term and we expect to have the behavior of the system similar to the case of pure VW. We will reserve the symbol ε ($\varepsilon=2-d$) to denote deviations from the short-range critical dimension $d_c=2$, and δ ($\delta=2-d-\sigma$) for the deviations from the long-range critical dimension $d_c(\sigma)$. If $\sigma=0$ then the critical dimension of the long-range part coincides with the short-range part and we have the nontrivial correction to the asymptotic behavior due to long-range interactions. This boundary separates mean-field or Gaussian behavior from long-range behavior. For $\sigma<0$ the long-range term dominates the short-range term and we expect to have nontrivial corrections to the behavior of the system.

Now we consider diagrammatic representation elements of model (6). In zero-loop approximation the vertex four-point function takes a simpler form after Laplace-Fourier transformation,

$$\Gamma_{ij}^{(2,2)}(s,p) = V_{ij}(p_1+p_2)\,\delta\left(\sum_k p_k\right). \quad (10)$$

The same transformation applied to the bare propagator yields

$$\Gamma_j^{(1,1)}(s,p) = (s + D_i p^2)^{-1}. \quad (11)$$

We note that there are no vertices in Eq. (6) that produce diagrams which dress the propagator, implying that there is no field renormalization. As a consequence the bare propagator (11) is the full propagator for the theory. Feynman rules are summarized in Fig. 1. There are two vertices in the theory: one is a short-range λ vertex and another is a long-range momentum-dependent g vertex. Each external line of the vertex corresponds to a functionally independent field. The propagator is formed by contracting appropriate lines from different vertices. We recall that the propagator is the correlation function of ϕ_i and $\phi_i^{\dagger}$ fields only.

Physical observables are computed with the help of correlation functions. The probability that p sets of particles with n_i particles in each set start at the proximity of the origin and finish at x_{i,α_i} (i index enumerates different sets and α_i index enumerates particles in set i) without intersecting each other can be obtained by generalizing Eq. (3). In the field theoretical formulation, this probability becomes the following correlation function:

$$G(t) = \int \prod_{i=1}^{p} \prod_{\alpha_i=1}^{n_i} d^d x_{i,\alpha_i} \langle \phi_i(t,x_{i,\alpha_i}) [\phi_i^{\dagger}(0,0)]^{n_i}\rangle. \quad (12)$$

In the Feynman representation it is the vertex with $2N$ ($N=\sum_j n_j$) external lines. In the first order of the perturbation theory one needs to contract these lines with corresponding lines of the vertices in Fig. 1. Since there are many independent fields in the correlation function (12) this operation can be done in many ways. It yields a combinatorial factor, $n_i n_j$, in front of each diagram, which is the number of ways of constructing a loop from the n_i lines of type i and n_j lines of type j on one hand and one line of type i and one line of type j on the other hand. In the next section we will see that the survival probability scales as $G(t) \sim t^{-\gamma}$, where γ is the critical exponent. If all walks are free, $\gamma=0$. In the presence of interactions we expect γ to be a universal quantity that does not depend on the intensity of the short-range interaction λ_{ij}. It is convenient to introduce the so-called truncated correlation function which is obtained from Eq. (12) by factoring out external lines,

$$\Gamma(t) = G(t)/(\Gamma^{(1,1)})^{2N}. \quad (13)$$

Another physical observable, the reunion probability, is defined as the probability that p sets of particles with n_i particles in each set start at the proximity of the origin and without colliding into each other finish at the proximity of some point at time t,

$$R(t) = \int d^d x \prod_{i=1}^{p} \langle \phi_i(t,x)^{n_i} [\phi_i^{\dagger}(0,0)]^{n_i}\rangle. \quad (14)$$

In the Feynman representation it is depicted as the watermelon diagram with $2N$ stripes. We note that if the theory is free this expression is the product of free propagators and at the large-time limit the return probability scales as $R_{\mathcal{O}}(t) \sim t^{-(N-1)d/2}$. If interactions are taken into account it becomes $R(t) \sim t^{-(N-1)d/2-2\gamma}$, where γ is the survival probability exponent. The reason why it enters with the factor of 2 is the following. If we cut a watermelon diagram of the reunion probability correlation function in the middle, then it produces two vertex diagrams with $2N$ external lines of the survival probability correlation function. As a result the reunion probability is the product of two survival probabilities. It remains true in all orders of perturbation theory. For a rigorous proof we refer to [16].

III. RENORMALIZATION OF OBSERVABLES

While computing correlation functions like Eq. (12) perturbatively one faces divergent integrals when $d = d_c$. The convenient scheme developed for dealing with these divergences follows Callan-Symanzik renormalization-group analysis [30,31]. Within this scheme we start with the bare correlation function $G(t;\lambda,g)$, where $\lambda=\{\lambda_{ij}\}$, and $g=\{g_{ij}\}$ denote the set of bare short-range and long-range coupling constants. In the renormalized theory it becomes $G_R(t;\lambda_R,g_R,\mu)$. From dimensional analysis it follows that

$$G_R(t;\lambda_R,g_R,\mu) = G_R(t\mu;\lambda_R,g_R), \tag{15}$$

where μ is the renormalization scale. The scale invariance leads to the expression

$$G_R(t;\lambda_R,g_R,\mu) = Z(\lambda_R,g_R,\mu)G(t;\lambda,g). \tag{16}$$

Here, functions Z are chosen in such a way that $G_R(t,\lambda_R,g_R,l)$ remains finite when the cutoff is removed at each order in a series expansion of λ_R, g_R, ε, and δ. From the fact that $G(t,\lambda,g)$ does not depend on the renormalization scale μ, we get the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g} + \beta_u\frac{\partial}{\partial u} - \gamma\right)G_R = 0, \tag{17}$$

where the β functions are defined by

$$\beta_\lambda(\lambda_R,g_R) = \mu\frac{\partial}{\partial\mu}\lambda_R, \quad \beta_g(\lambda_R,g_R) = \mu\frac{\partial}{\partial\mu}g_R, \tag{18}$$

and the function γ is defined by

$$\gamma(\lambda_R,g_R) = \mu\frac{\partial}{\partial\mu}\ln Z. \tag{19}$$

The renormalization-group functions are understood as the expansion in double series of coupling constants λ and g and deviations from the critical dimension ε and δ. We take $\delta = O(\varepsilon)$. The coefficient $Z(\lambda_R,g_R,\mu)$ is fixed by the normalization conditions. It is more convenient to impose these conditions on the Laplace transform of the truncated correlation function (13). One sets the following condition then:

$$\Gamma_R(\mu) = 1, \tag{20}$$

when $s=\mu$. We note that the same multiplicative renormalization factor Z yields Γ finite. From this fact one can infer that

$$\Gamma(\mu;\lambda,g) = Z(\mu;\lambda,g)^{-1}. \tag{21}$$

If we express unrenormalized couplings in terms of renormalized ones (21), we will obtain the equation for finding Z explicitly.

Equation (17) can be solved by the method of characteristics. Within this method we let couplings depend on the scale which is parametrized by $\mu(x)=x\mu$. Here, x is introduced as a parametrization variable of the renormalization group (RG) flow and is not to be confused with position. Henceforth, x will refer to this parametrization variable. We introduce running couplings $\bar{\lambda}(x)$ and $\bar{g}(x)$. They satisfy the equations

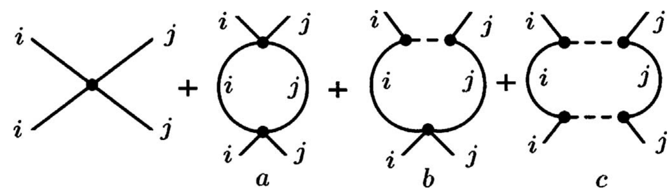

FIG. 2. One-loop Feynman diagrams contributing to λ_{Rij}.

$$x\frac{d}{dx}\bar{g}(x) = \beta_g(\bar{\lambda}(x),\bar{g}(x)), \quad x\frac{d}{dx}\bar{\lambda}(x) = \beta_\lambda(\bar{\lambda}(x),\bar{g}(x)). \tag{22}$$

The renormalized value should be defined by the initial conditions $\bar{\lambda}(1)=\lambda_R$ and $\bar{g}(1)=g_R$. The solution of the equation is then

$$G_R(t) = \exp\left[\int_1^{\mu t}\gamma(\bar{\lambda}(x),\bar{g}(x))dx/x\right]G_R(\mu^{-1};\bar{\lambda}(\mu t),\bar{g}(\mu t),\mu). \tag{23}$$

Next we calculate the first-order contribution to the renormalized vertices. The λ vertex is renormalized by the set of diagrams that are shown in Fig. 2. We notice that there are no diagrams producing the momentum-dependent g vertex in the theory (6). This statement is the corollary of the fact that only independent fields of power one enters into the expression of the vertex and there are no higher powers of fields. Also we keep in mind that the renormalized couplings are defined by the value of the vertex function taken at zero external momenta. It produces the following expression:

$$\lambda_{Rij} = \lambda_{ij} - \tfrac{1}{2}(\lambda_{ij}^2 I_1 + 2\lambda_{ij}g_{ij}I_2 + g_{ij}^2 I_3),$$

$$g_{Rij} = g_{ij}, \tag{24}$$

where $I_k = I_k(\sigma;D_i,D_j)$ are one-loop integrals corresponding to the diagrams a, b, c in Fig. 2, respectively. Using the Feynman rules we can explicitly write them down as follows:

$$I_k = \int\frac{d^dq}{(2\pi)^d}\frac{q^{(k-1)\sigma}}{2s + (D_i + D_j)q^2}, \quad k = 1,2,3. \tag{25}$$

We will use dimensional regularization procedure to compute these integrals. The details of the computation are summarized in the Appendix. We note that integrals will diverge logarithmically at different values of the spatial dimension d. For this reason it leads to different critical behaviors in different regions of the σ-d plane (see Fig. 3). These regions correspond to four possibilities for $\varepsilon = 2-d$ and $\delta = 2-d-\sigma$ to be positive or negative. Only if $\delta = O(\varepsilon)$ or, in other words, if both ε and δ are infinitesimally small but the ratio ε/δ is finite we expect nonzero fixed points of the renormalization-group flow. Similar approximation has been used before [19] but for different models with long-range disorder. It allows us to follow the standard procedure of deriving the β functions, which consists of two steps.

First, we express the unrenormalized coupling constants in terms of the renormalized coupling constants. For the

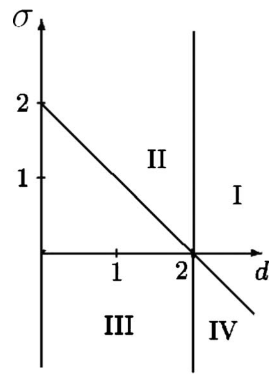

FIG. 3. The critical behavior of vicious walks with long-range interactions in the different regions of the (σ,d) plane. Regions I and IV correspond to the mean-field short-range behavior, region II is the critical short-range behavior, and region III is the long-range behavior. The lines $d=2$ and $\sigma+d=2$ represent regions V and VI, respectively.

short-range coupling constant λ it can be done by solving the quadratic equation in Eq. (24). Expanding the square root and keeping terms up to the second order we infer that

$$\lambda_{ij} = \lambda_{Rij} + \frac{1}{2}\left(\lambda_{Rij}^2\frac{a_d}{\varepsilon} + 2\lambda_{Rij}g_{Rij}\frac{b_d}{\delta} + g_{Rij}^2\frac{c_d}{2\delta-\varepsilon}\right),$$

$$g_{ij} = g_{Rij}, \tag{26}$$

where a_d, b_d, and c_d coefficients have been found explicitly in the Appendix. Now we introduce dimensionless renormalized couplings,

$$\bar{g}_{Rij} = a_d(2s)^{-\delta/2}, \quad \bar{\lambda}_{Rij} = b_d(2s)^{-\varepsilon/2}. \tag{27}$$

An important observation is that $c_d a_d = b_d^2$ which can be verified by explicit substitution (see the Appendix). Multiplying the first and second equations in Eqs. (26) by the factors a_d and b_d, respectively, and using redefinitions (27) we can condense all prefactors in the right-hand side of the equations into the dimensionless constants.

Second, we differentiate Eqs. (26) with respect to the scaling parameter μ. Using definitions (18) and the fact that bare couplings do not depend on the scale, we derive

$$\beta_{\lambda,ij} = -\varepsilon\bar{\lambda}_{Rij} + (\bar{\lambda}_{Rij} + \bar{g}_{Rij})^2,$$

$$\beta_{g,ij} = -\delta\bar{g}_{Rij}, \tag{28}$$

where the right-hand side is understood as the leading contribution to the β functions from the double expansions in λ, g and ε, δ. From Eqs. (28) we see that it is convenient to introduce new coupling constants $u_{Rij} = \bar{\lambda}_{Rij} + \bar{g}_{Rij}$. After this step the renormalization-group equations read

$$\beta_{u,ij} = -\varepsilon u_{Rij} + u_{Rij}^2 - g_{Rij},$$

$$\beta_{g,ij} = -\delta g_{Rij}. \tag{29}$$

We note that in the last equations the g coupling constant has been redefined, $\sigma\bar{g}_{Rij} \to g_{Rij}$.

TABLE II. Fixed points for flow equations (29) and the corresponding eigenvalues (λ_1,λ_2) of the stability matrix (32). We note that u_1 and u_2 are values given by Eq. (31). LR stands for long range.

Fixed point	(u_*, g_*)	(λ_1,λ_2)
Gaussian	(0, 0)	(ε,δ)
Pure VW	$(\varepsilon,0)$	$(-\varepsilon,\delta)$
LR stable	$(u_1,0)$	$(-\sqrt{\varepsilon^2-4g},0)$
LR unstable	$(u_2,0)$	$(\sqrt{\varepsilon^2-4g},0)$

Fixed points are zeros of the β functions. If $\delta \neq 0$ then the last equation in Eqs. (29) is zero only when $g_*=0$. Then the first equation has two solutions $u=0$ and $u=\varepsilon$. If $\delta=0$ then g plays the role of a parameter and the fixed points are determined by the roots of the quadratic equation

$$0 = -\varepsilon u + u^2 - g, \tag{30}$$

which are real if $g \geq -(\varepsilon/2)^2$, and we find

$$u_{1,2} = \varepsilon/2 \pm \sqrt{(\varepsilon/2)^2 + g}. \tag{31}$$

All fixed points are listed in Table II. The stability of these fixed points is determined by the matrix of partial derivatives,

$$\beta_* = -\begin{pmatrix} \partial\beta_u/\partial u & \partial\beta_u/\partial g \\ \partial\beta_g/\partial u & \partial\beta_g/\partial g \end{pmatrix}_{u=u_*,g=g_*}. \tag{32}$$

Eigenvalues are listed in Table II. The Gaussian fixed point is stable in all directions for $\varepsilon<0$ and $\delta<0$, which corresponds to region I in Fig. 3. In this region we find both short-range (pure VW) and long-range mean-field behaviors depending on the sign of σ. On the contrary, for $\varepsilon>0$ and $\delta>0$ we find that the Gaussian fixed point is unstable (irrelevant) in all directions and the short-range (pure VW) fixed point is stable (relevant) only in the u direction. It means that long-range interactions will play a leading role. This region corresponds to region III in Fig. 3. Next for $\varepsilon>0$ and $\delta<0$ we find that the short-range (pure VW) fixed point is stable in all directions. It means that the system is insensitive to the long-range tail. This region corresponds to region II in Fig. 3. Finally, for $\varepsilon<0$ and $\delta>0$ we find that the short-range (pure VW) fixed point is unstable in all directions and the system will be described by mean field at long time.

IV. CALCULATION OF CRITICAL EXPONENTS AND DISCUSSION

Here, we describe our method of computing critical exponents. It is based on formula (21) from the previous section. First, we obtain the leading divergent part of the correlation function. The renormalized correlation function depends on the scale μ but it appears in all formulas in combination with time: μt. Second, since we have found the bare coupling constant as a function of renormalized (dressed) couplings we express correlation function in terms of dressed couplings. Finally, using the normalization condition (20) and the definition (19) we differentiate Z with respect to $\mu\partial/\partial\mu$ to obtain the exponent γ. The poles should cancel after this operation.

In Sec. II it was explained that the truncated correlation function in the one-loop approximation is given by the formula

$$\Gamma(t;\lambda,g) = 1 - \sum_{i,j} n_i n_j(\lambda_{ij}I_1 + g_{ij}I_2). \tag{33}$$

Here, integrals are the same as in Eq. (25).

We start our analysis with region I. Notice that truncated correlation function $\Gamma(t)$ and survival probability $G(t)$ have similar large-time behavior. We use large momentum cutoff to compute integrals I_1 and I_2 as in formula (A9) in the Appendix. The renormalization of coupling constants is trivial in this case. Therefore, the leading contribution to the survival probability is given by

$$G(t) \sim t^{(2-d)/2} + g_0 t^{(2-d-\sigma)/2}, \tag{34}$$

where g_0 is nonuniversal coefficient and we will not need its exact value. We notice that if $\sigma>0$ the second term will decay faster than the first term and in the long-time limit it will produce the same behavior as mean-field pure VW. On the other hand if $\sigma<0$ the first term will decay faster and long-range interactions will play a leading role. Many authors observed similar behavior in various systems with long-range defects [19–21]. Intuitively, if the potential falls off rapidly with distance, then the system effectively has a short-range potential implying that particles interact only when they are close to each other.

Region IV exhibits similar behavior. Now the integral I_2 is computed with the help of the dimensional regularization (A3) and the integral I_1 remains the same. From the fact (15) one can infer that the survival probability scales as

$$G(t) \sim t^{(2-d)/2}. \tag{35}$$

Short-range behavior dominates because the running coupling constant will flow toward the Gaussian fixed point at long-time limit which is the only stable fixed point in this region. This result is exact regardless of the number of loops one takes into account.

In region II the computation is as follows:

$$\ln Z = \sum n_i n_j\left(\lambda_{ij}\frac{a_d}{\varepsilon} + g_{ij}t^{(2-d-\sigma)/2}\right), \tag{36}$$

so plugging the result from Eq. (A4) to Eq. (36) we obtain at the fixed point $(\lambda_* = \varepsilon,\ g=0)$

$$\gamma = -\frac{1}{2}\sum n_i n_j \varepsilon, \tag{37}$$

and we reproduce the pure VW behavior. This result is the reflection of the fact that the renormalization-group trajectories run away to stable pure VW fixed point. It is with agreement with the results obtained by Katori and Tanemura in [17] for $d=1$, and the logarithmic intraset particle interactions. The irrelevance of the long-range interaction in lower dimensions is a typical phenomenon observed in various out of equilibrium interacting particle systems.

We now consider regions III, V, and VI. Integrals in Eq. (33) are computed via dimensional regularization. Taking the inverse of Eq. (33) and then logarithm one can obtain at the leading order

$$\ln Z = \sum n_i n_j\left(\lambda_{ij}\frac{a_d}{\varepsilon} + g_{ij}\frac{b_d}{\delta}\right), \tag{38}$$

where a_d and b_d are defined in the Appendix in Eqs. (A4) and (A5). We note that after taking the derivative the poles in Eq. (38) will cancel in the limit of $\delta = O(\varepsilon)$. Also one recalls the expansion (26) and the redefinitions in Eq. (27). Using Eq. (19) we show that the expression for the function γ which determines critical exponent takes the form

$$\gamma = -\frac{1}{2}\sum_{ij} n_i n_j u_R. \tag{39}$$

Evaluated at the stable fixed point $[u_1 = \varepsilon/2 + \sqrt{(\varepsilon/2)^2 + g}]$ it gives the following result:

$$\gamma = -\frac{1}{2}\sum_{ij} n_i n_j u_1, \tag{40}$$

and the survival probability scales as $G(t) \sim t^{\gamma}$.

We will now find the logarithmic corrections to this scaling law. The running coupling constant can be found from the flow equations (29): $\bar{g}(x) = e^{-\delta x}g$. In the case $\delta, \varepsilon = 0$ (the intersection of regions V and VI) the flow equation for $\bar{u}(x)$ is

$$x\frac{d\bar{u}(x)}{dx} = -\bar{u}^2(x) + g, \tag{41}$$

and the solution is

$$\bar{u}(x) = \sqrt{g}\tanh(\sqrt{g}\ln x + \phi_0) \sim \sqrt{g}\tanh(\sqrt{g}\ln x), \tag{42}$$

where ϕ_0 is the initial condition and we do not need its exact form. After plugging this expression into Eq. (23) we infer

$$\int_1^{\mu t}\gamma(\bar{u},\bar{g})\frac{dx}{x} \sim \ln[\cosh(\sqrt{g}\ln \mu t)]. \tag{43}$$

Thus, the survival probability is

$$G(t) \sim \cosh(\sqrt{g}\ln t)^{-(1/2)\Sigma n_i n_j}. \tag{44}$$

In the limit of large time $\cosh(\sqrt{g}\ln t) \sim t^{\sqrt{g}}$, implying $\gamma = -\frac{1}{2}\Sigma_{ij} n_i n_j\sqrt{g}$, which is consistent with Eq. (40). For negative coupling constant $g<0$ the solution in Eq. (42) becomes

$$\bar{u}(x) \sim -\sqrt{|g|}\tan(\sqrt{|g|}\ln x). \tag{45}$$

The integral (43) is divergent if $t > \exp(\pi/2\sqrt{|g|})$ which leads to the result that the survival probability is zero beyond this time. For smaller times one has $G(t) \sim \cos(\sqrt{|g|}\ln t)^{-(1/2)\Sigma n_i n_j}$. Thus, up to one-loop order approximation, it implies that if walks are attracted to each other then all of them will annihilate at some finite time. This might be a signature of faster than power-law decay and we expect to have corrections to this behavior at higher-loop approximation.

Next we consider the case when $\varepsilon=0$ and $\delta\neq 0$ but δ remains small, i.e., region V. The flow equation for $\bar{u}(x)$ is

$$x\frac{d\bar{u}(x)}{dx} = -\bar{u}^2(x) + gx^{-\delta}, \tag{46}$$

and the solution can be found by the method of perturbation. Up to the first order,

$$\bar{u}(x) = \sqrt{g}\tanh(\sqrt{g}\ln x) + \delta\sqrt{g}\ln(x)\tanh(\sqrt{g}\ln x). \tag{47}$$

After plugging this expression into Eq. (23) we infer

$$\int_1^{\mu t}\gamma(\bar{u},\bar{g})\frac{dx}{x} \sim -\frac{1}{2}\sum n_i n_j\left(\ln(t^{\sqrt{g}}) + \frac{1}{2}\delta\sqrt{g}\ln^2(t)\right). \tag{48}$$

Therefore, we have the correction to the survival probability in the form

$$G \sim t^{-(1/2)\Sigma n_i n_j\sqrt{g}[1+\delta/2(\ln t)]}. \tag{49}$$

Now we extend our analysis to the case when $\varepsilon > 0$, corresponding to regions III and VI. The evolution of the coupling constant is

$$x\frac{d}{dx}\bar{u}(x) = \varepsilon\bar{u} - \bar{u}^2 + gx^{\delta}. \tag{50}$$

We choose the ansatz in the form $\bar{u}(x)=u_0(x)+\delta v(x)$. For $\delta=0$ (i.e., region VI) the equation for $u_0(x)$ reads

$$x\frac{d}{dx}u_0(x) = \varepsilon u_0 - u_0^2 + g, \tag{51}$$

and we reproduce the result (40). We now extend to the case where $\varepsilon,\delta>0$ (region III). Here, we will need the exact solution to Eq. (51) to find the corrections,

$$u_0(x) = \frac{Cx^{u_1-u_2}u_1 + u_2}{1 + Cx^{u_1-u_2}}, \tag{52}$$

where $C=(u_R-u_2)/(u_1-u_R)$. The logarithmic correction follows from the form of the perturbation. The equation for $v(x)$ is

$$x\frac{d}{dx}v(x) = \varepsilon v - 2u_0 v - g\ln x. \tag{53}$$

The solution can be found explicitly as a combination of hypergeometric functions. In the most interesting case, $\varepsilon=1$ ($d=1$) the hypergeometric functions are degenerate and become linear functions. Corrections to the integral then read

$$\int_1^{\mu t}\gamma dx/x \sim \frac{1}{2}\delta u_1\ln^2(t) + \ln(t)(t)^{u_1-u_2}. \tag{54}$$

In the limit of large time only the first term contributes to the exponent and the survival probability scales as

$$G \sim t^{-(1/2)\Sigma n_i n_j u_1(1+\delta/2\ln t)}. \tag{55}$$

V. CONCLUSION

In summary, we studied long-range vicious walks using the methods of Callan-Symanzik renormalized field theory. Our work confirms the previously known RG fixed-point structure including their stability regions. We calculated the critical exponents for all values of σ and d to first order in ε expansion and to all orders in δ expansion, which have hitherto been known only for $d+\sigma=2$. Our results indicate that, depending on the exact values of d and σ, the system can be dominated by either short-range (pure VW) or long-range behaviors. In addition, we calculated the leading logarithmic corrections for several dynamical observables that are typically measured in simulations.

We hope that our work stimulates further interest in long-range vicious walks. It would be interesting to see further simulation results for the critical exponents for $d>1$ and for logarithmic corrections. Also, it would be interesting to have analytical and numerical results for other universal quantities such as scaling functions and amplitudes.

ACKNOWLEDGMENTS

The authors would like to acknowledge UC Merced start-up funds and a James S. McDonnell Foundation Award for Studying Complex Systems.

APPENDIX

Effective four-point function (one-particle irreducible) that appeared in Eq. (24) is composed of usual short-range and new momentum-dependent vertices. This gives rise to integrals (25). The first integral $\mu=1$ has been evaluated in [14] by using alpha representation $1/(q^2+s)=\int_0^{+\infty}d\alpha e^{i(q^2+s)\alpha}$ and the result is

$$I_1 = K_d(2s)^{-\varepsilon/2}\Gamma(\varepsilon/2). \tag{A1}$$

We notice that since there is no angular dependence one can perform $d-1$ integrations and one will be left with one-dimensional integral. To compute this integral we use the formula [32]

$$\int_0^{+\infty}dx\frac{x^{\nu-1}}{P+Qx^2} = \frac{1}{2P}\left(\frac{P}{Q}\right)^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right). \tag{A2}$$

We see that in our case $P=s$, $Q=(D_i+D_j)$, and $\nu=d+(\mu-1)\sigma$. This immediately gives the result

$$I_\mu = \frac{K_d}{2}\left(\frac{1}{(D_i+D_j)}\right)^{[d+(\mu-1)\sigma]/2}s^{[d+(\mu-1)\sigma]/2-1}\times\Gamma\left(\frac{d+(\mu-1)\sigma}{2}\right)\Gamma\left(1-\frac{d+(\mu-1)\sigma}{2}\right), \tag{A3}$$

where $K_d=2^{d-1}\pi^{-d/2}\Gamma^{-1}(d/2)$ is the surface area of d-dimensional unit sphere.

It is convenient to define

$$a_d = \frac{K_d}{2}\left(\frac{2}{(D_i+D_j)}\right)^{d/2}(2s)^{-\varepsilon/2}, \tag{A4}$$

$$b_d = \frac{K_d}{2}\left(\frac{2}{(D_i+D_j)}\right)^{(d+\sigma)/2}(2s)^{-\delta/2}, \tag{A5}$$

$$c_d = \frac{K_d}{2}\left(\frac{2}{(D_i + D_j)}\right)^{(d+2\sigma)/2}(2s)^{-(2\delta-\varepsilon)/2}. \tag{A6}$$

So integral I_μ in the limit of $\delta = O(\varepsilon)$ can be written as

$$I_1 = \frac{a_d}{\varepsilon}, \quad I_2 = \frac{b_d}{\delta}, \quad I_3 = \frac{c_d}{2\delta - \varepsilon}. \tag{A7}$$

We used an expansion $\Gamma(\varepsilon/2) \sim 2/\varepsilon$ for small ε. An important property of coefficients (A4)–(A6) is that

$$c_d a_d = b_d^2, \tag{A8}$$

which can be verified by direct substitution.

Now we compute mean-field integrals,

$$I_\mu = \int d^d q dt q^{d+\sigma} \exp[-t(D_i + D_j)q^2] \sim t^{-(d+\sigma-2)/2}, \tag{A9}$$

where we assumed that the large momentum cutoff is imposed and corresponding coupling constants have been renormalized. The nonuniversal coefficient is not important.

[1] C. A. Angell, Science **267**, 1924 (1995).
[2] D. S. Dean and A. Lefevre, Phys. Rev. E **69**, 061111 (2004).
[3] D. R. Nelson and P. Le Doussal, Phys. Rev. B **42**, 10113 (1990).
[4] M. Nakamura, A. Sekiyama, H. Namatame, A. Fujimori, H. Yoshihara, T. Ohtani, A. Misu, and M. Takano, Phys. Rev. B **49**, 16191 (1994).
[5] C. Kane, L. Balents, and M. P. A. Fisher, Phys. Rev. Lett. **79**, 5086 (1997).
[6] J. M. Park and M. W. Deem, Eur. Phys. J. B **10**, 35 (1999).
[7] J. Klafter and J. Jortner, J. Chem. Phys. **73**, 1004 (1980).
[8] V. Kuzovkov and E. Kotomin, Rep. Prog. Phys. **51**, 1479 (1988).
[9] S. Redner and P. L. Krapivsky, Am. J. Phys. **67**, 1277 (1999); P. L. Krapivsky and S. Redner, J. Phys. A **29**, 5347 (1996).
[10] A. J. Bray and R. A. Blythe, Phys. Rev. Lett. **89**, 150601 (2002).
[11] B. J. Reynwar, Nature (London) **447**, 461 (2007).
[12] B. J. Reynwar, BioInterphases **3**, FA117 (2008).
[13] M. E. Fisher, J. Stat. Phys. **34**, 667 (1984).
[14] J. Cardy and M. Katori, J. Phys. A **36**, 609 (2003).
[15] S. Mukherji and S. M. Bhattacharjee, Phys. Rev. E **48**, 3427 (1993); **52**, 3301(E) (1995).
[16] S. Mukherji and S. M. Bhattacharjee, J. Phys. A **26**, L1139 (1993).
[17] M. Katori and H. Tanemura, Phys. Rev. E **66**, 011105 (2002).
[18] B. Derrida and M. R. Evans, J. Phys. A **32**, 4833 (1999).
[19] A. Weinrib and B. I. Halperin, Phys. Rev. B **27**, 413 (1983).
[20] V. Blavats'ka, C. von Ferber, and Y. Holovatch, Phys. Rev. E **64**, 041102 (2001).
[21] V. V. Prudnikov, P. V. Prudnikov, and A. A. Fedorenko, Phys. Rev. B **62**, 8777 (2000).
[22] H. J. Schulz, Phys. Rev. Lett. **71**, 1864 (1993).
[23] A. D. Mironov and A. V. Zabrodin, Phys. Rev. Lett. **66**, 534 (1991).
[24] F. Linder, J. Trangia, S. Dahmen, and H. Hinrichsen, J. Phys. A: Math. Theor. **41**, 185005 (2008).
[25] S. Mukherji and S. M. Bhattacharjee, Phys. Rev. E **63**, 051103 (2001); **64**, 059902(E) (2001).
[26] M. Doi, J. Phys. A **9**, 1465 (1976).
[27] M. Peliti, J. Phys. (France) **46**, 1469 (1985).
[28] G. D. Mahan, *Many-Particle Physics* (Plenum, New York, 1993).
[29] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 2003).
[30] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Clarendon, Oxford, 2002).
[31] D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984).
[32] I. S. Gradshtein and I. M. Ryjik, *Tables of Integrals* (Academic, New York, 1965).