

Asymptotic expansion for the resistance between two maximally separated nodes on an M by N resistor network

N. Sh. Izmailian^{1,2,3,4} and Ming-Chang Huang¹

¹*Department of Physics, Chung-Yuan Christian University, Chungli 320, Taiwan*

²*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan*

³*Yerevan Physics Institute, Alikhanian Br. 2, 375036 Yerevan, Armenia*

⁴*International Center for Advanced Study, Yerevan State University, 1 Alex Manoogian Street, Yerevan 375025, Armenia*

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We analyze the exact formulas for the resistance between two arbitrary nodes in a rectangular network of resistors under free, periodic and cylindrical boundary conditions obtained by Wu [J. Phys. A **37**, 6653 (2004)]. Based on such expression, we then apply the algorithm of Ivashkevich, Izmailian, and Hu [J. Phys. A **35**, 5543 (2002)] to derive the exact asymptotic expansions of the resistance between two maximally separated nodes on an $M \times N$ rectangular network of resistors with resistors r and s in the two spatial directions. Our results is $\frac{1}{s}R_{M \times N}(r, s) = c(\rho) \ln S + c_0(\rho, \xi) + \sum_{p=1}^{\infty} \frac{c_{2p}(\rho, \xi)}{S^p}$ with $S = MN$, $\rho = r/s$ and $\xi = M/N$. The all coefficients in this expansion are expressed through analytical functions. We have introduced the effective aspect ratio $\xi_{eff} = \sqrt{\rho\xi}$ for free and periodic boundary conditions and $\xi_{eff} = \sqrt{\rho\xi}/2$ for cylindrical boundary condition and show that all finite-size correction terms are invariant under transformation $\xi_{eff} \rightarrow 1/\xi_{eff}$.

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I. INTRODUCTION

The calculation of the resistance between arbitrary node of infinite networks of resistors is a well studied subject [1–3]. Resistor networks have been widely studied as models for conductivity problems and classical transport in disordered media [4–6].

Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [2,7], and discussions below), first-passage processes [8], to lattice Green's functions [9]. Past studies have been focused mainly on infinite lattices [10,11]. Little attention has been paid to finite network, even though the latter are those occurring in real life. Recently, Wu [12] has revisited the two-point resistance problem and deduced a closed-form expression for the resistance between arbitrary two nodes for finite networks with resistors r and s in the two spatial directions. Later, Jafarizadeh, *et al.* [13] proposed an algorithm for the calculation of the resistance between two arbitrary nodes in an arbitrary distance-regular networks. However, the exact expression obtained in [12] is in the form of a double summation whose mathematical and physical contents are not immediately apparent. Quite recently Essam and Wu based on the exact expression for the resistance between arbitrary two nodes for finite rectangular network obtained in [12] has derived the asymptotic expansion for the corner-to-corner resistance $[R_{M \times N}(r, s)]$ on an $M \times N$ rectangular resistor network under free boundary conditions. For the case $M=N$ and $r=s=1$ they computed the finite-size corrections to the corner-to-corner resistance up to order N^{-4} ,

$$R_{N \times N}(1, 1) = \frac{4}{\pi} \ln N + 0.077\,318 + \frac{0.266\,070}{N^2} - \frac{0.534\,779}{N^4} + O\left(\frac{1}{N^6}\right).$$

The computation of the asymptotic expansion of the corner-to-corner resistance (in other word the resistance between two maximally separated nodes) of a rectangular resistor network has been of interest for some time, as its value provides a lower bound to the resistance of compact percolation clusters in the Domany-Kinzel model of a directed percolation [15].

In experiments and in numerical studies of model systems, it is essential to take into account finite-size effects in order to extract correct infinite-volume predictions from the data. As soon as one has a finite system one must consider the question of boundary conditions on the outer surfaces or “walls” of the system. The systems under various boundary conditions have the same per-site free energy, internal energy, specific heat, etc, in the bulk limit, whereas the finite-size corrections are different. To understand the effects of boundary conditions on finite-size scaling and finite-size corrections, it is valuable to study model systems. Therefore, in recent decades there have many investigations on finite-size scaling, finite-size corrections, and boundary effects for model systems [14,16–22]. Of particular importance in such studies are exact results where the analysis can be carried out without numerical errors.

In this paper we will derive the exact asymptotic expansions for resistance between two maximally separated nodes on the rectangular network under free, periodic and cylindrical boundary conditions. We will show that the exact asymptotic expansion of the resistance between nodes of the network for all boundary conditions can be written as

$$\frac{1}{s}R_{M \times N}(r, s) = c(\rho) \ln S + c_0(\rho, \xi) + \sum_{p=1}^{\infty} \frac{c_{2p}(\rho, \xi)}{S^p}, \quad (1)$$

where $\rho = r/s$, $S = MN$ is the area of the lattice and $\xi = M/N$ is the aspect ratio. The all coefficients in this expansion are expressed through analytical functions. We will show that all finite-size correction terms are invariant under transforma-

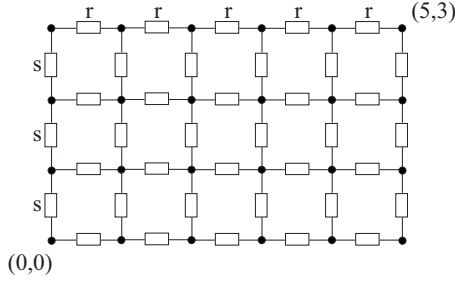


FIG. 1. A 6×4 rectangular network with free boundary conditions.

tion $\xi \rightarrow 1/(\rho\xi)$ for free and periodic boundary conditions and under transformation $\xi \rightarrow 4/(\rho\xi)$ for cylindrical boundary condition, which actually means that ξ_{eff}

$$\xi_{eff} = \xi\sqrt{\rho} \quad \text{for free and periodic b.c.,} \quad (2)$$

$$\xi_{eff} = \xi\sqrt{\rho}/2 \quad \text{for cylindrical b.c.} \quad (3)$$

can be regarded as the effective aspect ratio.

The organization of this paper is as follows: Based on the exact expression for the resistance between arbitrary two nodes for finite rectangular network under free, periodical and cylindrical boundary conditions obtained in [12] we express the resistance between two most separated nodes in terms of $G_{\alpha,\beta}(\rho, \xi)$ with $(\alpha, \beta) = (1/2, 0)$ and $(0, 1/2)$ (Sec. II). We then extend Ivashkevich, Izmailian, and Hu algorithm [16] to derive the exact asymptotic expansions of the resistance between two maximally separated nodes on the rectangular network for all boundary conditions and write down the expansion coefficients up to the second order (Sec. III). We also discuss our results in Sec. IV.

II. TWO-DIMENSIONAL RESISTOR NETWORKS

An electrical network can be regarded as a graph in which the resistance R_{ij} is associated to the edge between pair of connected nodes i and j . Denote the electric potential at the i th vertex by V_i and the net current flowing into the network at the i th vertex by I_i . When the potential difference occurs between points i and j , the current is given by the Ohm's law $I_{ij} = (V_i - V_j)C_{ij}$, where $C_{ij} = 1/R_{ij}$ is the conductance of the respective link. By the Kirchhoff's current law total current outflow from any point in the interior is zero, $\sum_j I_{ij} = 0$, we then find for the voltage

$$V_i = \sum_j V_j C_{ij} / C_i, \quad (4)$$

where $C_i = \sum_j C_{ij}$ and the sum is over all nodes j which are connected to i .

The two-point resistance has a probabilistic interpretation based on classical random walker walking on the network. The averaging property expressed by Eq. (4) implies that the voltage is a harmonic function on the interior points of the graph. This makes the basis for the probabilistic interpretation of the voltage [2,23–25]. The random walk determined by the electrical network is defined as finite state Markov

chain (for more details see [2]) with the transition probabilities P_{ij} that are weighted with the conductances as $P_{ij} = C_{ij}/C_i$. Then, when the constant voltage is applied to the graph such that $V_a = 1$ and $V_b = 0$, the voltage in an interior point x is determined as the hitting probability h_x that a walker starting at x reaches the point a before reaching b .

Consider a rectangular $M \times N$ network of resistors with resistances r and s on edges of the network in the respective horizontal and vertical directions. The closed-form expression for the resistance $R_{\{M \times N\}}(\mathbf{r}_1, \mathbf{r}_2)$ between arbitrary two nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ for free, periodic and cylindrical boundary conditions was obtained in [12].

In what follows, we will show that the resistance $R_{\{M \times N\}}(r, s)$ between two maximally separated nodes of the network for all above mentioned boundary conditions can be expressed in terms of $G_{\alpha,\beta}(\rho, M, N)$ only,

$$R_{M \times N}^{\text{free}}(r, s) = -r + \frac{\sqrt{rs}}{S} [G_{0,1/2}(\rho, M, N) + G_{1/2,0}(\rho, M, N)], \quad (5)$$

$$R_{M \times N}^{\text{per}}(r, s) = \frac{\sqrt{rs}}{S} [G_{0,1/2}(\rho, M/2, N/2) + G_{1/2,0}(\rho, M/2, N/2)], \quad (6)$$

$$R_{M \times N}^{\text{cyl}}(r, s) = \frac{\sqrt{rs}}{S} [G_{0,1/2}(\rho, M/2, N) + G_{1/2,0}(\rho, M/2, N)], \quad (7)$$

where $G_{\alpha,\beta}(\rho, \xi)$ is given by

$$G_{\alpha,\beta}(\rho, M, N) = M \operatorname{Re} \sum_{n=0}^{N-1} f\left(\pi \frac{n+\alpha}{N}\right) \times \coth \left[M \omega\left(\pi \frac{n+\alpha}{N}\right) + i\pi\beta \right] \quad (8)$$

for $(\alpha, \beta) \neq (0, 0)$. The function $\omega(x)$ is the same for all boundary conditions and given by

$$\omega(x) = \operatorname{arcsinh} \sqrt{\rho} \sin x \quad (9)$$

and function $f(x)$ is depend on boundary conditions and given by

$$f(x) = \frac{\cos^2 x \sqrt{1 + \rho \sin^2 x}}{\sin x} \quad \text{for free BCs,} \quad (10)$$

$$f(x) = \frac{1}{\sin x \sqrt{1 + \rho \sin^2 x}} \quad \text{for periodic BCs,} \quad (11)$$

$$f(x) = \frac{\cos^2 x}{\sin x \sqrt{1 + \rho \sin^2 x}} \quad \text{for cylindrical BCs.} \quad (12)$$

A. Two-dimensional network: Free boundary conditions

Consider a rectangular $M \times N$ network of resistors with free boundary conditions and with resistances r and s

on edges of the network in the respective horizontal and vertical directions. The example of a rectangular network with $M=6$, $N=4$ is shown in Fig. 1. The resistance between the two maximally separated nodes

on the network of resistors with free boundary conditions is the resistance between opposite corner nodes $(0,0)$ and $(M-1, N-1)$ of the network which is given by [14]

$$R_{M \times N}^{\text{free}}(r,s) = \frac{r(M-1)}{N} + \frac{s(N-1)}{M} + \frac{2}{MN} \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \frac{\left[\cos(\theta_m/2) \cos(\phi_n/2) - \cos\left(M - \frac{1}{2}\right) \theta_m \cos\left(N - \frac{1}{2}\right) \phi_n \right]^2}{r^{-1}(1 - \cos \theta_m) + s^{-1}(1 - \cos \phi_n)}, \quad (13)$$

where $\theta_m = \pi m/M$, $\phi_n = \pi n/N$. With the help of the identity

$$\sum_{n=0}^{N/2-1} \cot^2 \frac{\pi(n+1/2)}{N} = \frac{N(N-1)}{2}, \quad (14)$$

Eq. (13) can be rewritten in the following form:

$$R_{M \times N}^{\text{free}}(r,s) = -\frac{r(M-1)}{N} - \frac{s(N-1)}{M} + \frac{2}{MN} \sum_{m=0}^{M-1} \sum_{\substack{n=0 \\ (m,n) \neq (0,0)}}^{N-1} \frac{\left[\cos(\theta_m/2) \cos(\phi_n/2) - \cos\left(M - \frac{1}{2}\right) \theta_m \cos\left(N - \frac{1}{2}\right) \phi_n \right]^2}{r^{-1}(1 - \cos \theta_m) + s^{-1}(1 - \cos \phi_n)}. \quad (15)$$

Using the fact that $\cos(M - \frac{1}{2}) \theta_m = (-1)^m \cos(\theta_m/2)$ and $\cos(N - \frac{1}{2}) \phi_n = (-1)^n \cos(\phi_n/2)$ the Eq. (15) can be rewritten in the following form:

$$R_{M \times N}^{\text{free}}(r,s) = -\frac{r(M-1)}{N} - \frac{s(N-1)}{M} + \frac{4r}{MN} \sum_{m=0}^{M-1} \sum_{\substack{n=0 \\ m+n=\text{odd}}}^{N-1} \frac{\cos^2(\phi_n/2)(1 + \rho \sin^2(\phi_n/2))}{\sin^2(\theta_m/2) + \rho \sin^2(\phi_n/2)} - \frac{4r}{MN} \sum_{m=0}^{M-1} \sum_{\substack{n=0 \\ m+n=\text{odd}}}^{N-1} \cos^2(\phi_n/2). \quad (16)$$

There are two possibilities for the restriction $m+n=\text{odd}$ to hold, namely, m -odd, n -even and m -even, n -odd. Splitting the sum into two parts accordingly we obtain

$$R_{M \times N}^{\text{free}}(r,s) = -\frac{r(M-1)}{N} - \frac{s(N-1)}{M} + \frac{4r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} [f(m+1/2, n) + f(m, n+1/2)] - \frac{4r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} \cos^2 \frac{\pi n}{N} - \frac{4r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} \cos^2 \frac{\pi(n+1/2)}{N}, \quad (17)$$

where

$$f(m, n) = \frac{\cos^2 \frac{\pi n}{N} \left(1 + \rho \sin^2 \frac{\pi n}{N} \right)}{\sin^2 \frac{\pi m}{M} + \rho \sin^2 \frac{\pi n}{N}}. \quad (18)$$

Sums of the term $\cos^2(x)$ can be carried out using the identities

$$\sum_{n=0}^{N/2-1} \cos^2 \frac{\pi n}{N} = \frac{N}{4} + \frac{1}{2}, \quad \sum_{n=0}^{N/2-1} \cos^2 \frac{\pi(n+1/2)}{N} = \frac{N}{4}. \quad (19)$$

This yields

$$R_{M \times N}^{\text{free}}(r,s) = -\frac{r(M+N)}{N} - \frac{s(N-1)}{M} + \frac{4r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} [f(m+1/2, n) + f(m, n+1/2)]. \quad (20)$$

Now we first express double sums $\sum_{n=0}^{N/2-1} \sum_{m=0}^{M/2-1} f(m, n)$ in terms of $\sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(m, n)$, It is easy to show that $f(m, N-n) = f(M-m, n) = f(m, n)$ and thus

$$\begin{aligned}
 2 \sum_{n=0}^{N/2-1} \sum_{m=0}^{M/2-1} [f(m+1/2, n) + f(m, n+1/2)] &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [f(m+1/2, n) + f(m, n+1/2)] - \sum_{n=0}^{N/2-1} [f(M/2, n+1/2) - f(0, n+1/2)] \\
 &\quad - \sum_{m=0}^{M/2-1} [f(m+1/2, N/2) - f(m+1/2, 0)].
 \end{aligned} \tag{21}$$

With the help of the identities given by Eq. (14) and

$$\sum_{m=0}^{M/2-1} \frac{1}{\sin^2 \frac{\pi(m+1/2)}{M}} = \frac{M^2}{2}. \tag{22}$$

The sums $\sum_{n=0}^{N/2-1} [f(M/2, n+1/2) - f(0, n+1/2)]$, $\sum_{m=0}^{M/2-1} [f(m+1/2, N/2) + f(m+1/2, 0)]$ can be written as

$$\begin{aligned}
 \sum_{n=0}^{N/2-1} [f(M/2, n+1/2) - f(0, n+1/2)] &= -\frac{1}{\rho} \sum_{n=0}^{N/2-1} \cot^2 \frac{\pi(n+1/2)}{N} = -\frac{N(N-1)}{2\rho} \\
 \sum_{m=0}^{M/2-1} [f(m+1/2, N/2) - f(m+1/2, 0)] &= -\sum_{m=0}^{M/2-1} \frac{1}{\sin^2 \frac{\pi(m+1/2)}{M}} = -\frac{M^2}{2}.
 \end{aligned} \tag{23}$$

Plugging Eqs. (21) and (23) back in Eq. (20) we finally obtain

$$\begin{aligned}
 R_{M \times N}^{\text{free}}(r, s) &= -r + \frac{r}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} [f(m+1/2, n) + f(m, n+1/2)] \\
 &= -r + \frac{r}{MN} \sum_{n=0}^{N-1} \left(1 + \rho \sin^2 \frac{\pi n}{N} \right) \cos^2 \frac{\pi n}{N} \sum_{m=0}^{M-1} \left[\rho \sin^2 \frac{\pi n}{N} + \sin^2 \frac{\pi \left(m + \frac{1}{2} \right)}{M} \right]^{-1} \\
 &\quad + \frac{r}{MN} \sum_{n=0}^{N-1} \left(1 + \rho \sin^2 \frac{\pi(n+1/2)}{N} \right) \cos^2 \frac{\pi(n+1/2)}{N} \sum_{m=0}^{M-1} \left[\rho \sin^2 \frac{\pi(n+1/2)}{N} + \sin^2 \frac{\pi m}{M} \right]^{-1}.
 \end{aligned} \tag{24}$$

The sum over m in the Eq. (24) can be carried out using the identity [26]

$$\prod_{m=0}^{M-1} 4 \left[\sinh^2 \omega + \sin^2 \frac{\pi(m+\beta)}{M} \right] = 4 |\sinh(M\omega + i\pi\beta)|^2. \tag{25}$$

Taking the derivative over ω from the logarithm of the left and right side of the Eq. (25) we obtain

$$\sum_{m=0}^{M-1} \left[\sinh^2 \omega + \sin^2 \frac{\pi(m+\beta)}{M} \right]^{-1} = 2M \operatorname{Re} \frac{\coth[M\omega + i\pi\beta]}{\sinh 2\omega}. \tag{26}$$

Thus the sum over m in the Eq. (24) can be carried out as

$$\sum_{m=0}^{M-1} \left[\rho \sin^2 \frac{\pi n}{N} + \sin^2 \frac{\pi \left(m + \frac{1}{2} \right)}{M} \right]^{-1} = 2M \operatorname{Re} \frac{\coth \left[M\omega \left(\frac{\pi n}{N} \right) + i\pi/2 \right]}{\sinh 2\omega \left(\frac{\pi n}{N} \right)}, \tag{27}$$

$$\sum_{m=0}^{M-1} \left[\rho \sin^2 \frac{\pi(n+1/2)}{N} + \sin^2 \frac{\pi m}{M} \right]^{-1} = 2M \operatorname{Re} \frac{\coth M\omega \left(\frac{\pi(n+1/2)}{N} \right)}{\sinh 2\omega \left(\frac{\pi(n+1/2)}{N} \right)}, \tag{28}$$

where $\omega(x)$ is given by Eq. (A4). It is easy to see that

$$\sinh 2\omega(x) = 2\sqrt{\rho} \sin x \sqrt{1 + \rho \sin^2 x}. \quad (29)$$

Note that using more complicated approach the identity given by Eq. (27) has been obtained previously in [14,27].

Plugging Eqs. (27) and (28) back in Eq. (24) we obtain that $R_{M \times N}^{\text{free}}(r, s)$ can be written in the form given by Eq. (5).

B. Two-dimensional network: periodical boundary conditions

Consider a rectangular $M \times N$ resistor network with periodic boundary conditions. Using a closed-form expression for the resistance between arbitrary two nodes for finite network given by Eq. (43) of [12] we can obtain for the resistance between nodes $r_1=(0,0)$ and $r_2=(M/2, N/2)$ of the network the following expression:

$$R_{M \times N}^{\text{per}}(r, s) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{1 - \cos(M\theta_m + N\phi_n)}{r^{-1}(1 - \cos 2\theta_m) + s^{-1}(1 - \cos 2\phi_n)}, \quad (30)$$

$(m,n) \neq (0,0)$

Using the fact that $\cos(M\theta_m + N\phi_n) = (-1)^{m+n}$ the Eq. (30) can be rewritten in the following form:

$$R_{M \times N}^{\text{per}}(r, s) = \frac{r}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{1}{\sin^2 \theta_m + \rho \sin^2 \phi_n}. \quad (31)$$

$m+n=\text{odd}$

Splitting the sum into two parts accordingly we obtain

$$R_{M \times N}^{\text{per}}(r, s) = \frac{r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} \frac{1}{\sin^2\left(\frac{2\pi(m+1/2)}{M}\right) + \rho \sin^2\left(\frac{2\pi n}{N}\right)} + \frac{r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} \frac{1}{\sin^2\left(\frac{2\pi m}{M}\right) + \rho \sin^2\left(\frac{2\pi(n+1/2)}{N}\right)} \quad (32)$$

The sum over m in the Eq. (32) can be carried out using the identities given by Eq. (27) and (28). This yields

$$R_{M \times N}^{\text{per}}(r, s) = \frac{\sqrt{rs}^{N/2-1}}{N} \sum_{n=0}^{N/2-1} \frac{\coth\left[M\omega\left(\frac{2\pi n}{N}\right) + i\pi/2\right]}{\sin\frac{2\pi n}{N} \sqrt{1 + \rho \sin^2\frac{2\pi n}{N}}} + \frac{\sqrt{rs}^{N/2-1}}{N} \sum_{n=0}^{N/2-1} \frac{\coth M\omega\left(\frac{2\pi(n+1/2)}{N}\right)}{\sin\frac{2\pi(n+1/2)}{N} \sqrt{1 + \rho \sin^2\frac{2\pi(n+1/2)}{N}}}. \quad (33)$$

Introducing function $f(x)$ given by Eq. (11) we finally arrived to the Eq. (6).

C. Two-dimensional network: cylindrical boundary conditions

Consider a rectangular $M \times N$ resistor network embedded on a cylinder with periodic boundary in the direction of M and free boundaries in the direction of N . Using Eq. (46) of [12], the resistance between nodes $r_1=(0,0)$ and $r_2=(M/2, N-1)$ of the network is

$$R_{M \times N}^{\text{cyl}}(r, s) = -\frac{rM}{4N} + \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(\frac{\cos^2\left(\frac{1}{2}\phi_n\right) + \cos^2\left(N - \frac{1}{2}\right)\phi_n - 2 \cos\left(\frac{1}{2}\phi_n\right)\cos\left(N - \frac{1}{2}\right)\phi_n \cos M\theta_m}{r^{-1}(1 - \cos 2\theta_m) + s^{-1}(1 - \cos \phi_n)} \right). \quad (34)$$

$(m,n) \neq (0,0)$

Using the fact that $\cos(M\theta_m) = (-1)^m$ and $\cos(N - \frac{1}{2})\phi_n = (-1)^n \cos(\phi_n/2)$ the Eq. (34) can be rewritten in the following form:

$$R_{M \times N}^{\text{cyl}}(r, s) = -\frac{rM}{4N} + \frac{2r}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{\cos^2(\phi_n/2)}{\sin^2 \theta_m + \rho \sin^2(\phi_n/2)}. \quad (35)$$

$m+n=\text{odd}$

Splitting the sum into two parts accordingly we obtain

$$R_{M \times N}^{\text{cyl}}(r, s) = -\frac{rM}{4N} + \frac{2r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} \frac{\cos^2\frac{\pi n}{N}}{\sin^2\frac{2\pi(m+1/2)}{M} + \rho \sin^2\frac{\pi n}{N}} + \frac{2r}{MN} \sum_{m=0}^{M/2-1} \sum_{n=0}^{N/2-1} \frac{\cos^2\frac{\pi(n+1/2)}{N}}{\sin^2\frac{2\pi m}{M} + \rho \sin^2\frac{\pi(n+1/2)}{N}}. \quad (36)$$

Following along the same lines as in the case of free boundary conditions the Eq. (36) can be written as

$$R_{M \times N}^{\text{cyl}}(r, s) = \frac{r}{MN} \sum_{n=0}^{N-1} \cos^2 \frac{\pi n}{N} \sum_{m=0}^{M/2-1} \frac{1}{\sin^2 \frac{2\pi(m+1/2)}{M} + \rho \sin^2 \frac{\pi n}{N}} + \frac{r}{MN} \sum_{n=0}^{N-1} \cos^2 \frac{\pi(n+1/2)}{N} \sum_{m=0}^{M/2-1} \frac{1}{\sin^2 \frac{2\pi m}{M} + \rho \sin^2 \frac{\pi(n+1/2)}{N}}. \tag{37}$$

The sum over m in the Eq. (37) can be carried out using the identities given by Eq. (27) and (28). This yields

$$R_{M \times N}^{\text{cyl}}(r, s) = \frac{\sqrt{rs}}{2N} \sum_{n=0}^{N-1} \frac{\cos^2 \frac{\pi n}{N}}{\sin \frac{2\pi n}{N} \sqrt{1 + \rho \sin^2 \frac{2\pi n}{N}}} \coth \left[\frac{M}{2} \omega \left(\frac{2\pi n}{N} \right) + i\pi/2 \right] + \frac{\sqrt{rs}}{2N} \sum_{n=0}^{N-1} \frac{\cos^2 \frac{\pi(n+1/2)}{N}}{\sin \frac{2\pi(n+1/2)}{N} \sqrt{1 + \rho \sin^2 \frac{2\pi(n+1/2)}{N}}} \coth \frac{M}{2} \omega \left(\frac{2\pi(n+1/2)}{N} \right). \tag{38}$$

Introducing function $f(x)$ given by Eq. (12) we finally arrived to the Eq. (7).

III. ASYMPTOTIC EXPANSION OF $R_{M \times N}(r, s)$

In Sec. II we have shown that the resistance between two maximally separated nodes on an $M \times N$ rectangular network of resistors with various boundary conditions can be expressed, in terms of the function $G_{1/2,0}(\rho, M, N)$ and $G_{0,1/2}(\rho, M, N)$ only, [see Eqs. (5)–(7)].

Based on such results, one can use the method proposed by Ivashkevich, Izmailian, and Hu [16] to derive the asymptotic expansion of the $G_{\alpha,\beta}(\rho, M, N)$ in terms so-called Kronecker’s double series [28], which are directly related to elliptic θ functions (see Appendix A).

After reaching this point, one can easily write down all the terms of the exact asymptotic expansion for the resistance between two maximally separated nodes of the network $[R_{M \times N}(r, s)]$ using Eq. (A5). We have found that the exact asymptotic expansion of the $R_{M \times N}(r, s)$ for free, periodic and cylindrical boundary conditions can be written as Eq. (1).

(i) For the free boundary conditions we obtain

$$\frac{1}{s} R_{M \times N}^{\text{free}}(r, s) = \frac{2\sqrt{\rho}}{\pi} \left[2 \ln N + \int_0^\pi \varphi(x) dx + 2C_E + 4 \ln 2 - \frac{\pi\sqrt{\rho}}{2} - 2 \ln \theta_2(i\sqrt{\rho}\xi) \theta_4(i\sqrt{\rho}\xi) \right] - \frac{\sqrt{\rho}}{\pi} \sum_{p=1}^\infty \left(\frac{\pi^2 \xi}{S} \right)^p \frac{\Omega_{2p}}{p(2p)!} [K_{2p}^{0,1/2}(i\sqrt{\rho}\xi) + K_{2p}^{1/2,0}(i\sqrt{\rho}\xi)], \tag{39}$$

where integral $\int_0^\pi \varphi(x) dx$ is given by Eq. (A9).

Thus, the coefficients $c_{2p}(\rho, \xi)$ ($p=1, 2, \dots$) in the expansion Eq. (1) explicitly given by

$$c_{2p}(\rho, \xi) = \frac{\pi^{2p-1} \xi^p \sqrt{\rho}}{p(2p)!} \Omega_{2p} [K_{2p}^{0,1/2}(i\sqrt{\rho}\xi) + K_{2p}^{1/2,0}(i\sqrt{\rho}\xi)], \tag{40}$$

where the differential operators Ω_{2p} is given by Eq. (A7) and $K_{2p}^{0,1/2}(i\sqrt{\rho}\xi)$, $K_{2p}^{1/2,0}(i\sqrt{\rho}\xi)$ are the Kronecker’s double series which can all be expressed in terms of the elliptic $\theta_k(i\sqrt{\rho}\xi)$ ($k=2, 3, 4$) functions only (see Appendix D).

Here we list the first few coefficients in the expansion given by Eq. (1)

$$c(\rho) = \frac{2\sqrt{\rho}}{\pi}, \tag{41}$$

$$c_0(\rho, \xi) = \frac{2\sqrt{\rho}}{\pi} \left(2 \ln \frac{8}{\pi} + 2C_E - 1 - \ln \xi(1 + \rho) - \frac{\pi\sqrt{\rho}}{2} + \frac{\rho - 1}{\sqrt{\rho}} \arctan \sqrt{\rho} - 2 \ln \theta_2 \theta_4 \right), \tag{42}$$

$$c_2(\rho, \xi) = \frac{\pi\tau_0}{72} \left(5 - 3\rho + (1 + \rho)\tau_0 \frac{\partial}{\partial \tau_0} \right) (\theta_4^4 - \theta_2^4) = \frac{\pi\tau_0}{72} \left((5 - 3\rho)(\theta_4^4 - \theta_2^4) + \pi\tau_0(1 + \rho)\theta_3^4 \theta_4^4 + 4\tau_0(1 + \rho)(\theta_4^4 - \theta_2^4) \frac{\partial}{\partial \tau_0} \ln \theta_2 \right) \dots \tag{43}$$

To simplify the notation we have use the short hand

$$\theta_k = \theta_k(i\tau_0), \quad k = 2, 3, 4, \tag{44}$$

where $\tau_0 = \xi\sqrt{\rho}$.

We have also used the following relations between derivatives of the elliptic functions

$$\frac{\partial}{\partial \tau_0} \ln \theta_3 = \frac{\pi}{4} \theta_4^4 + \frac{\partial}{\partial \tau_0} \ln \theta_2 \quad \text{and}$$

$$\frac{\partial}{\partial \tau_0} \ln \theta_4 = \frac{\pi}{4} \theta_3^4 + \frac{\partial}{\partial \tau_0} \ln \theta_2.$$

Note that elliptic functions $\theta_2, \theta_3, \theta_4$ can be expressed through the complete elliptic integral of the first kind $K=K(k)$ and second kind $E=E(k)$ as

$$\theta_2 = \sqrt{\frac{2kK(k)}{\pi}}, \quad \theta_3 = \sqrt{\frac{2K(k)}{\pi}}, \quad \theta_4 = \sqrt{\frac{2k'K(k)}{\pi}}, \quad (45)$$

where

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}, \quad (46)$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 x} dx. \quad (47)$$

With the help of the identities

$$\frac{\partial}{\partial \tau_0} \ln \theta_2 = -\frac{1}{2} \theta_3^2 E, \quad \text{and} \quad \frac{\partial E}{\partial \tau_0} = \frac{\pi^2}{4} \theta_3^2 \theta_4^4 - \frac{\pi}{2} \theta_4^4 E$$

one can express all derivatives of the elliptic functions in terms of the elliptic functions $\theta_2, \theta_3, \theta_4$ and the complete elliptic integral of the second kind $E=E(k)$.

Thus we have obtained the explicit analytic formulas for all corrections—to scaling terms $c_{2p}(\rho, \xi)$ in the form of elliptic functions. Note, that quite recently, Essam and Wu [14] has derived the asymptotic expansion for the corner-to-corner resistance $[R_{M \times N}(r, s)]$ on an $M \times N$ rectangular resistor network under free boundary conditions in the form of Eq. (1) (see Eq. (40) in [14]), where the expansion coefficients $c_{2p}(\rho, \xi)$ (for $p=0, 1, 2$) are expressed through the fast convergent sums. For the case $M=N$ and $r=s$ ($\xi=1, \rho=1$) we reproduced the result of Essam and Wu [14],

$$\frac{1}{s} R_{N \times N}^{\text{free}}(s, s) = \frac{4}{\pi} \ln N + c_0 + \frac{c_2}{N^2} + \dots \quad (48)$$

with $c_0=0.077\ 318\ 893\ 909\ 458\ 76\dots$ and $c_2=0.266\ 070\ 441\ 638\ 478\ 37\dots$

(ii) For the periodic boundary conditions we obtain

$$\begin{aligned} \frac{1}{s} R_{M \times N}^{\text{tor}}(r, s) = & \frac{\sqrt{\rho}}{2\pi} \left[2 \ln \frac{N}{2} + \int_0^\pi \varphi(x) dx + 2C_E + 4 \ln 2 - 2 \ln \theta_2(i\sqrt{\rho}\xi) \theta_4(i\sqrt{\rho}\xi) \right] \\ & - \frac{\sqrt{\rho}}{4\pi} \sum_{p=1}^\infty \frac{\Omega_{2p}}{p(2p)!} \left(\frac{4\pi^2 \xi}{S} \right)^p [K_{2p}^{0,1/2}(i\sqrt{\rho}\xi) + K_{2p}^{1/2,0}(i\sqrt{\rho}\xi)], \end{aligned} \quad (49)$$

where integral $\int_0^\pi \varphi(x) dx$ is given by Eq. (A10).

The first few coefficients in the expansion are

$$c(\rho) = \frac{\sqrt{\rho}}{2\pi}, \quad (50)$$

$$c_0(\rho, \xi) = \frac{\sqrt{\rho}}{2\pi} \left(2 \ln \frac{4}{\pi} + 2C_E - \ln \xi(1+\rho) - 2 \ln \theta_2 \theta_4 \right), \quad (51)$$

$$\begin{aligned} c_2(\rho, \xi) = & \frac{\pi \tau_0}{72} \left(3\rho - 1 + (1+\rho) \tau_0 \frac{\partial}{\partial \tau_0} \right) (\theta_4^4 - \theta_2^4) \\ = & \frac{\pi \tau_0}{72} \left((3\rho - 1)(\theta_4^4 - \theta_2^4) + \pi \tau_0 (1+\rho) \theta_3^4 \theta_4^4 \right. \\ & \left. + 4\tau_0 (1+\rho) (\theta_4^4 - \theta_2^4) \frac{\partial}{\partial \tau_0} \ln \theta_2 \right) \dots \end{aligned} \quad (52)$$

For the case $M=N$ and $r=s$ ($\xi=1, \rho=1$) we obtain

$$\frac{1}{s} R_{N \times N}^{\text{tor}}(s, s) = \frac{1}{\pi} \ln N + c_0 + \frac{c_2}{N^2} + \dots \quad (53)$$

with $c_0=0.207\ 849\ 066\ 416\ 608\ 4\dots$ and $c_2=0.266\ 070\ 441\ 638\ 478\ 4\dots$

(iii) And for the cylindrical boundary conditions

$$\begin{aligned} \frac{1}{s} R_{M \times N}^{\text{cyl}}(r, s) = & \frac{\sqrt{\rho}}{\pi} \left[2 \ln N + \int_0^\pi \varphi(x) dx + 2C_E \right. \\ & \left. + 4 \ln 2 - 2 \ln \theta_2(i\sqrt{\rho}\xi/2) \theta_4(i\sqrt{\rho}\xi/2) \right] \\ & - \frac{\sqrt{\rho}}{2\pi} \sum_{p=2}^\infty \frac{\Omega_{2p}}{p(2p)!} \left(\frac{\pi^2 \xi}{S} \right)^p [K_{2p}^{0,1/2}(i\sqrt{\rho}\xi/2) \\ & + K_{2p}^{1/2,0}(i\sqrt{\rho}\xi/2)] \end{aligned} \quad (54)$$

where integral $\int_0^\pi \varphi(x) dx$ is given by Eq. (A11).

The first few coefficients in the exact asymptotic expansion are

$$c(\rho) = \frac{\sqrt{\rho}}{\pi}, \quad (55)$$

$$c_0(\rho, \xi) = \frac{\sqrt{\rho}}{\pi} \left(2 \ln \frac{8}{\pi} + 2C_E - \ln \xi(1 + \rho) - \frac{2}{\sqrt{\rho}} \arctan \sqrt{\rho} - 2 \ln \theta_2 \theta_4 \right), \quad (56)$$

$$\begin{aligned} c_2(\rho, \xi) &= \frac{\pi \tau_1}{72} \left(3\rho + 5 + (1 + \rho) \tau_1 \frac{\partial}{\partial \tau_1} \right) (\theta_4^4 - \theta_2^4) \\ &= \frac{\pi \tau_1}{72} \left((5 + 3\rho)(\theta_4^4 - \theta_2^4) + \pi \tau_1 (1 + \rho) \theta_3^4 \theta_4^4 + 4 \tau_1 (1 + \rho) (\theta_4^4 - \theta_2^4) \frac{\partial}{\partial \tau_1} \ln \theta_2 \right) \dots \end{aligned} \quad (57)$$

Here we have use the short hand

$$\theta_k = \theta_k(i\tau_1), \quad k = 2, 3, 4, \quad (58)$$

with $\tau_1 = \xi \sqrt{\rho} / 2$.

For the case $M=N$ and $r=s$ ($\xi=1, \rho=1$) we obtain

$$\frac{1}{s} R_{N \times N}^{cyl}(s, s) = \frac{2}{\pi} \ln N + c_0 + \frac{c_2}{N^2} + \dots \quad (59)$$

with $c_0 = 0.361\ 724\ 759\ 117\ 295\ 57\dots$ and $c_2 = -0.284\ 482\ 624\ 106\ 766\ 56\dots$

Let us now consider the behavior of the coefficients $c_{2k}(\rho, \xi)$ in the asymptotic expansion of the resistance between two maximally separated nodes on the rectangular network under the Jacobi transformation (see Appendix E). Using Eqs. (E4) and (E5) we can easily check that $c_{2k}(\rho, \xi)$ (for all k) are invariant under transformation $\tau_0 \rightarrow 1/\tau_0$, where

$$\tau_0 = \xi \sqrt{\rho} \quad \text{for free and periodic b.c.}, \quad (60)$$

$$\tau_0 = \xi \sqrt{\rho} / 2 \quad \text{for cylindrical b.c.} \quad (61)$$

Using the properties of the θ -functions and of the functions $K_{2p}^{\alpha, \beta}$ [see Eq. (E4) and (E5)] we can easily check from Eq. (A5) that the $\ln G_{\alpha, \beta}(\rho, M, N)$ have the following behavior under the transformation $\tau_0 \rightarrow 1/\tau_0$,

$$G_{1/2, 0}(\rho, M, N) \rightarrow G_{0, 1/2}(\rho, M, N). \quad (62)$$

Equations (5)–(7) and (62) imply that the resistance $R_{\{M \times N\}}(r, s)$ between two maximally separated nodes of the network for all above mentioned boundary conditions is invariant under transformation $\tau_0 \rightarrow 1/\tau_0$. This actually means that ξ_{eff} given by Eqs. (2) and (3) can be regarded as the effective aspect ratio.

IV. DISCUSSION

In Fig. 2 we plot the conventional aspect-ratio (ξ) dependence of the finite-size correction term $c_0(\xi)$ for the resistance between two maximally separated nodes on a $M \times N$ resistor network with the free (solid lines), toroidal (dashed lines), and cylindrical (dot-dashed lines) boundary conditions for several values of the factor ρ : (a) for $\rho=1$ and (b) for $\rho=4$. We use the logarithmic scales in the horizontal axis. The finite-size correction term $c_0(\xi)$ at first decrease until

$\xi = \xi_{min}$: $\xi_{min}=1$ for $\rho=1$ and $\xi_{min}=1/2$ for $\rho=4$. Note that $\xi_{min}=1/\sqrt{\rho}$ for arbitrary value of ρ . With further increase of ξ it reverses directions, increasing monotonically to infinity. For large enough ξ ($\xi \gg 1$), the finite-size properties of the resistor network with cylindrical boundary condition and those of the torus become the same, which means that the boundaries along the shorter direction determine the finite-size properties of the system; for both cylindrical boundary condition and the torus, the boundary condition along the shorter direction is the periodic one. For small enough ξ ($\xi \ll 1$), the finite-size properties of the resistor network with free boundary condition and those of the cylinder become the same because the boundaries along the shorter directions for these two are the same, that is, the free boundary condition.

In Fig. 3 we plot the effective aspect-ratio (ξ_{eff}) dependence of the finite-size correction term c_0 for the resistance between two maximally separated nodes on a $M \times N$ resistor network with the free (solid lines), toroidal (dashed lines), and cylindrical (dot-dashed lines) boundary conditions for several values of the factor ρ : (a) for $\rho=1$ and (b) for $\rho=4$. We use the logarithmic scales in the horizontal axis. We can see that finite-size correction terms c_0 are invariant under transformation $\xi_{eff} \rightarrow 1/\xi_{eff}$. The finite-size correction term $c_0(\xi)$ at first decrease until $\xi_{eff}=1$ for all boundary conditions and for arbitrary value of ρ . With further increase of ξ_{eff} it reverses directions, increasing monotonically to infinity. In Fig. 4 we plot the ρ dependence of the finite-size correction term $c_0(\rho)$ for the resistance between two maximally separated nodes on a $M \times N$ resistor network with the free (solid lines), toroidal (dashed lines) and cylindrical (dot-dashed lines) boundary conditions for several values of the aspect-ratio ξ : (a) for $\xi=1$ and (b) for $\xi=4$. The finite-size correction term $c_0(\rho)$ at first decrease until $\rho = \rho_{min}$. Note that value of ρ_{min} depends on the boundary conditions as well on the value of the aspect ratio ξ . With further increase of ρ it reverses directions, increasing monotonically to infinity.

In the present paper, we study the two-point resistor problem on planar $M \times N$ rectangular lattices with free, periodic, and cylindrical boundary conditions. Using the exact expression for the resistance between arbitrary two nodes for finite rectangular network obtained in [12] and the ITHs algorithm [16], we derive the exact asymptotic expansion of the corner-to-corner resistance on the rectangular network for all above mentioned boundary conditions. All corrections to scaling are analytic.

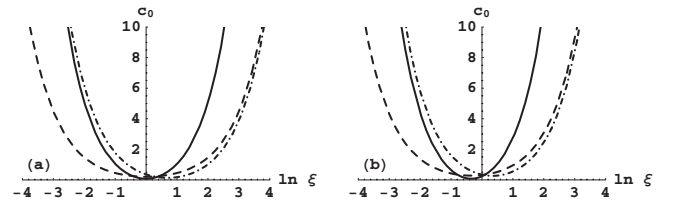


FIG. 2. Conventional aspect-ratio (ξ) dependence of finite-size correction term c_0 for the resistance between two maximally separated nodes on a $M \times N$ resistor network with the free (solid lines), toroidal (dashed lines) and cylindrical (dot-dashed lines) boundary conditions: (a) for $\rho=1$ and (b) for $\rho=4$. We use the natural logarithmic scales for the horizontal axis.

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 APPENDIX A: ASYMPTOTIC EXPANSION OF $G_{\alpha,\beta}(\rho, \mathcal{M}, \mathcal{N})$

Using the expansion of the coth x

$$\coth x = 1 + 2 \sum_{m=1}^{\infty} e^{-2mx}$$

we can transform the Eq. (8) in the following form:

$$\begin{aligned} G_{\alpha,\beta}(\rho, \mathcal{M}, \mathcal{N}) &= \text{Re} \mathcal{M} \sum_{n=0}^{\mathcal{N}-1} f\left(\frac{\pi(n+\alpha)}{\mathcal{N}}\right) \coth \left[\mathcal{M} f\left(\frac{\pi(n+\alpha)}{\mathcal{N}}\right) + i\pi\beta \right] \\ &= \mathcal{M} \sum_{n=0}^{\mathcal{N}-1} f\left(\frac{\pi(n+\alpha)}{\mathcal{N}}\right) + 2 \text{Re} \mathcal{M} \sum_{n=0}^{\mathcal{N}-1} \sum_{m=1}^{\infty} f\left(\frac{\pi(n+\alpha)}{\mathcal{N}}\right) e^{-2m[\mathcal{M}\omega(\pi(n+\alpha)/\mathcal{N}) + i\pi\beta]}, \end{aligned} \quad (\text{A1})$$

where $f(x)$ is given by Eqs. (10)–(12) for free, periodic and cylindrical b.c., respectively.

Using Taylor's theorem, the asymptotic expansion of the $f(x)$ for all boundary conditions can be written in the following form:

$$f(x) = \frac{1}{x} \left[1 + \sum_{p=1}^{\infty} \frac{\kappa_{2p}}{(2p)!} x^{2p} \right], \quad (\text{A2})$$

where $\kappa_2 = \rho - 5/3$, $\kappa_4 = -3\rho^2 - 14\rho + 67/15$, etc. for free boundary conditions $\kappa_2 = -\rho + 1/3$, $\kappa_4 = 9\rho^2 + 2\rho + 7/15$, etc. for periodic boundary conditions and $\kappa_2 = -\rho - 5/3$, $\kappa_4 = 9\rho^2 + 14\rho + 67/15$, etc. for cylindrical boundary conditions.

Note that function $f(x)$ can be represent as

$$f(x) = \frac{1}{x} \exp \left\{ \sum_{p=1}^{\infty} \frac{\varepsilon_{2p}}{(2p)!} x^{2p} \right\}, \quad (\text{A3})$$

where coefficients ε_{2p} and κ_{2p} are related to each other through relation between moments and cumulants (Appendix B)

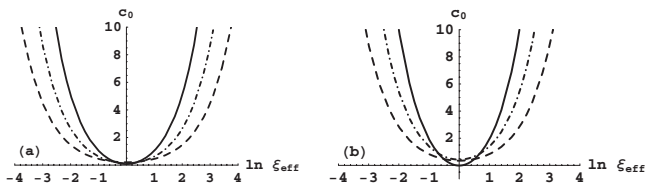


FIG. 3. Effective aspect-ratio (ξ_{eff}) dependence of finite-size correction term c_0 for the resistance between two maximally separated nodes on a $M \times N$ resistor network with the free (solid lines), toroidal (dashed lines) and cylindrical (dot-dashed lines) boundary conditions: (a) for $\rho=1$ and (b) for $\rho=4$. We use the natural logarithmic scales for the horizontal axis.

$$\kappa_2 = \varepsilon_2,$$

$$\kappa_4 = \varepsilon_4 + 3\varepsilon_2^2;$$

We will need also the Taylor expansion of the $\omega(x)$ given by Eq. (9)

$$\omega(x) = x \left(\lambda + \sum_{p=1}^{\infty} \frac{\lambda_{2p}}{(2p)!} x^{2p} \right), \quad (\text{A4})$$

where $\lambda = \sqrt{\rho}$, $\lambda_2 = -\sqrt{\rho}(1+\rho)/3$, $\lambda_4 = \sqrt{\rho}(1+10\rho+9\rho^2)/5$, etc. for all boundary conditions.

In what follows, we shall not use the special values of these coefficients assuming the possibility for generalizations.

The asymptotic expansion of $G_{\alpha,\beta}(\rho, \mathcal{M}, \mathcal{N})$ can be derived in the similar way as it has done in Ref. [16] for the second derivative of the partition function with twisted boundary condition (see Eq. (18) in Ref. [16]). Following along the same line as in Ref. [16] we can obtain for the asymptotic expansion of $G_{\alpha,\beta}(\rho, \mathcal{M}, \mathcal{N})$ the following expression:

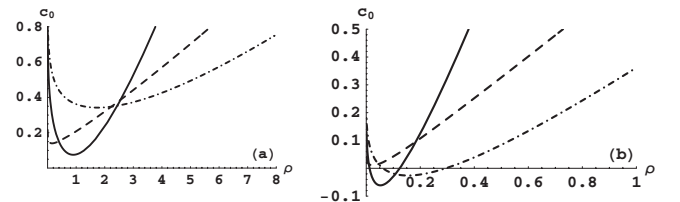


FIG. 4. The (ρ) dependence of finite-size correction term c_0 for the resistance between two maximally separated nodes on a $M \times N$ resistor network with the free (solid lines), toroidal (dashed lines), and cylindrical (dot-dashed lines) boundary conditions: (a) for $\xi=1$ and (b) for $\xi=4$.

$$G_{\alpha,\beta}(\rho, \mathcal{M}, \mathcal{N}) = \frac{2S}{\pi} \left[\ln \mathcal{N} + \frac{1}{2} \int_0^\pi \varphi(x) dx + C_E + 2 \ln 2 - 2 \ln |\theta_{\alpha,\beta}(i\lambda\xi)| \right] + \frac{1}{2} \left(\kappa_2 \frac{\partial}{\partial \lambda} + \lambda_2 \frac{\partial^2}{\partial \lambda^2} \right) \ln \left| \frac{\theta_{\alpha,\beta}(i\lambda\xi)}{\eta(i\lambda\xi)} \right| - \pi \xi \sum_{p=2}^\infty \left(\frac{\pi^2 \xi}{S} \right)^{p-1} \frac{\Omega_{2p}}{p(2p)!} \operatorname{Re} K_{2p}^{\alpha,\beta}(i\lambda\xi), \quad (\text{A5})$$

where $S = \mathcal{M}\mathcal{N}$, $\xi = \mathcal{M}/\mathcal{N}$, C_E is the Euler constant, $\eta(\tau)$ is the Dedekind- η function

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^\infty [1 - e^{2\pi i n \tau}], \quad (\text{A6})$$

$K_{2p+2}^{\alpha,\beta}(\tau)$ is Kronecker's double series (Appendix D) and $\theta_{\alpha,\beta}(\tau)$ is elliptic theta function (Appendix C).

The differential operators Ω_{2p} that have appeared here can be expressed via coefficients $\omega_{2p} = \varepsilon_{2p} + \lambda_{2p} \frac{\partial}{\partial \lambda}$ as

$$\begin{aligned} \Omega_2 &= \omega_2, \\ \Omega_4 &= \omega_4 + 3\omega_2^2 \vdots \end{aligned} \quad (\text{A7})$$

The function $\varphi(x)$ is defined as

$$\varphi(x) = f(x) - \frac{1}{x} - \frac{1}{\pi - x}. \quad (\text{A8})$$

Thus, the $\int_0^\pi \varphi(x) dx$ depend on the boundary conditions and given by

$$\begin{aligned} \int_0^\pi \varphi(x) dx &= -1 + 2 \ln \frac{2}{\pi} - \ln(1 + \rho) \\ &+ \frac{\rho - 1}{\sqrt{\rho}} \arctan \sqrt{\rho} \quad \text{for free b.c.}, \end{aligned} \quad (\text{A9})$$

$$\int_0^\pi \varphi(x) dx = 2 \ln \frac{2}{\pi} - \ln(1 + \rho) \quad \text{for periodic b.c.}, \quad (\text{A10})$$

$$\begin{aligned} \int_0^\pi \varphi(x) dx &= 2 \ln \frac{2}{\pi} - \ln(1 + \rho) - \frac{2}{\sqrt{\rho}} \arctan \sqrt{\rho} \\ &\text{for cylindrical b.c.} \end{aligned} \quad (\text{A11})$$

We are interested in the asymptotic expansion of $G_{\alpha,\beta}(\rho, \mathcal{M}, \mathcal{N})$ with $(\alpha, \beta) = (0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$.

APPENDIX B: RELATION BETWEEN MOMENTS AND CUMULANTS

Moments Z_k and cumulants F_k which enters the expansion of exponent

$$\exp \left\{ \sum_{k=1}^\infty \frac{x^k}{k!} F_k \right\} = 1 + \sum_{k=1}^\infty \frac{x^k}{k!} Z_k$$

are related to each other as [29]

$$Z_1 = F_1,$$

$$Z_2 = F_2 + F_1^2,$$

$$Z_3 = F_3 + 3F_1F_2 + F_1^3,$$

$$Z_4 = F_4 + 4F_1F_3 + 3F_2^2 + 6F_1^2F_2 + F_1^4 \vdots,$$

$$Z_k = \sum_{r=1}^k \sum \binom{F_{k_1}}{k_1!}^{i_1} \cdots \binom{F_{k_r}}{k_r!}^{i_r} \frac{k!}{i_1! \dots i_r!},$$

where summation is over all positive numbers $\{i_1 \dots i_r\}$ and different positive numbers $\{k_1, \dots, k_r\}$ such that $k_1 i_1 + \dots + k_r i_r = k$.

APPENDIX C: ELLIPTIC THETA FUNCTIONS

In this appendix we gather all the definitions and properties of the Jacobi's θ -functions and Kronecker's double series needed in this paper. We adopt the following definition of the elliptic θ -functions:

$$\begin{aligned} \theta_{\alpha,\beta}(z, \tau) &= \sum_{n \in \mathbb{Z}} \exp \left\{ \pi i \tau \left(n + \frac{1}{2} - \alpha \right)^2 + 2\pi i \left(n + \frac{1}{2} - \alpha \right) \left(z - \frac{1}{2} + \beta \right) \right\} \\ &= \eta(\tau) \exp \left\{ \pi i \tau B_2^\alpha + 2\pi i \left(\frac{1}{2} - \alpha \right) \left(z - \frac{1}{2} + \beta \right) \right\} \prod_{n=0}^\infty [1 - e^{2\pi i \tau(n+\alpha) - 2\pi i(z+\beta)}] [1 - e^{2\pi i \tau(n+1-\alpha) + 2\pi i(z+\beta)}] \end{aligned} \quad (\text{C1})$$

where B_2^α is the Bernoulli polynomial $B_2^\alpha = \alpha^2 - \alpha + \frac{1}{6}$ and $\eta(\tau)$ is Dedekind η -function:

$$\eta(\tau) = e^{\pi i \tau B_2/2} \prod_{n=1}^{\infty} [1 - e^{2\pi i n \tau}]. \quad (\text{C2})$$

where $B_2 \equiv B_2^0 = 1/6$.

The elliptic θ -functions satisfies the heat equation

$$\frac{\partial}{\partial \tau} \theta_{\alpha,\beta}(z, \tau) = \frac{1}{4\pi i} \frac{\partial^2}{\partial z^2} \theta_{\alpha,\beta}(z, \tau). \quad (\text{C3})$$

The relation of the functions $\theta_{\alpha,\beta}(z, \tau)$ with the usual θ -functions $\theta_i(z, \tau)$ $i=1, \dots, 4$ is the following:

$$\theta_{0,0}(z, \tau) = \theta_1(z, \tau),$$

$$\theta_{0,1/2}(z, \tau) = \theta_2(z, \tau),$$

$$\theta_{1/2,0}(z, \tau) = \theta_4(z, \tau),$$

$$\theta_{1/2,1/2}(z, \tau) = \theta_3(z, \tau).$$

In this paper we will only need these functions evaluated at $z=0$ and $\tau=i\tau_0$ is a pure imaginary aspect ratio. To simplify the notation we will use the shorthand

$$\theta_{\alpha,\beta}(i\tau_0) = \theta_{\alpha,\beta}(0, i\tau_0)$$

$$\theta_i(i\tau_0) = \theta_i(0, i\tau_0)$$

APPENDIX D: KRONECKER'S DOUBLE SERIES

Kronecker's double series can be defined as [28]

$$K_p^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{e^{-2\pi i(n\alpha+m\beta)}}{(n+\tau m)^p}.$$

In this form, however, they cannot be directly applied to our analysis. In [16] it was shown that Kronecker's double series can be written as

$$K_p^{\alpha,\beta}(\tau) = B_p^\alpha - p \sum_{m \neq 0} \sum_{n=0}^{\infty} (n+\alpha)^{p-1} e^{2\pi i m[\tau(n+\alpha)-\beta]} \quad (\text{D1})$$

The Kronecker's double series $K_{2p}^{\alpha,\beta}(\tau)$ for the cases $(\alpha, \beta) = (0, 0), (0, 1/2), (1/2, 0), (1/2, 1/2)$ can all be expressed in terms of the elliptic $\theta(\tau)$ functions only.

Equations for $K_{2p}^{\alpha,\beta}(\tau)$ with $p=2, 3, 4, 5$ and other useful relations for elliptic θ -functions and Kronecker's double series can be found in Refs. [16–19].

Here we write down the Kronecker's double series $K_{2p}^{0,1/2}(i\tau_0)$ and $K_{2p}^{1/2,0}(i\tau_0)$ that have appeared in our asymptotic expansions

$$K_2^{0,1/2}(i\tau_0) = -\frac{1}{\pi} \frac{\partial}{\partial \tau_0} \ln \frac{\theta_2(i\tau_0)}{\eta(i\tau_0)} = \frac{\theta_3^4(i\tau_0) + \theta_4^4(i\tau_0)}{12},$$

$$K_2^{1/2,0}(i\tau_0) = -\frac{1}{\pi} \frac{\partial}{\partial \tau_0} \ln \frac{\theta_4(i\tau_0)}{\eta(i\tau_0)} = -\frac{\theta_3^4(i\tau_0) + \theta_2^4(i\tau_0)}{12},$$

$$K_4^{0,1/2}(i\tau_0) = \frac{1}{30} \left[\frac{7}{8} \theta_2^8(i\tau_0) - \theta_3^4(i\tau_0) \theta_4^4(i\tau_0) \right],$$

$$K_4^{1/2,0}(i\tau_0) = \frac{1}{30} \left[\frac{7}{8} \theta_4^8(i\tau_0) - \theta_2^4(i\tau_0) \theta_3^4(i\tau_0) \right],$$

$$K_6^{0,1/2}(i\tau_0) = \frac{1}{84} \left[\theta_3^4(i\tau_0) + \theta_4^4(i\tau_0) \right] \left[\frac{31}{16} \theta_2^8(i\tau_0) + \theta_3^4(i\tau_0) \theta_4^4(i\tau_0) \right],$$

$$K_6^{1/2,0}(i\tau_0) = -\frac{1}{84} \left[\theta_3^4(i\tau_0) + \theta_2^4(i\tau_0) \right] \left[\frac{31}{16} \theta_4^8(i\tau_0) + \theta_3^4(i\tau_0) \theta_2^4(i\tau_0) \right].$$

$$\vdots \quad (\text{D2})$$

Note that when $\tau_0 \rightarrow \infty$ we have limits $\theta_4(i\tau_0) \rightarrow 1$, $\theta_3(i\tau_0) \rightarrow 1$ and $\theta_2(i\tau_0) \rightarrow 2 \exp(-\frac{\pi\tau_0}{4}) \rightarrow 0$, and Kronecker's double series reduce to the Bernoulli polynomials:

$$\lim_{\tau_0 \rightarrow \infty} K_{2p}^{\alpha,\beta}(i\tau_0) = B_{2p}^\alpha.$$

The case $\tau_0 \rightarrow 0$ can be obtained by using Jacobi's imaginary transformation of the $\theta(i\tau_0)$ -functions. In this case $\theta_2(i\tau_0) \rightarrow \frac{1}{\sqrt{\tau_0}}$, $\theta_3(i\tau_0) \rightarrow \frac{1}{\sqrt{\tau_0}}$ and $\theta_4(i\tau_0) \rightarrow \frac{2}{\sqrt{\tau_0}} \exp(-\frac{\pi}{4\tau_0}) \rightarrow 0$ and the Kronecker's function can again be reduce to the Bernoulli polynomials.

APPENDIX E: JACOBI TRANSFORMATION

We also need the behavior of the θ functions, Dedekind's η -function and the Kronecker functions $K_{2p}^{0,1/2}$ under the Jacobi transformation

$$\tau \rightarrow \tau' = -1/\tau. \quad (\text{E1})$$

The result for the θ functions and Dedekind's η -function when $z=0$ is given in Ref. [30].

$$\theta_3(\tau') = (-i\tau)^{1/2} \theta_3(\tau),$$

$$\theta_2(\tau') = (-i\tau)^{1/2} \theta_4(\tau),$$

$$\theta_4(\tau') = (-i\tau)^{1/2} \theta_2(\tau),$$

$$\eta(\tau') = (-i\tau)^{1/2} \eta(\tau). \quad (\text{E2})$$

The result for the Kronecker functions $K_{2p}^{0,1/2}$ and $K_{2p}^{1/2,0}$ can be obtain from the relation between coefficients in Laurent expansion of the Weierstrass function and Kronecker functions (see Appendix F in [16]) and is given by

$$K_{2p}^{0,1/2}(\tau') = \tau^{2p} K_{2p}^{1/2,0}(\tau),$$

$$K_{2p}^{1/2,0}(\tau') = \tau'^{2p} K_{2p}^{0,1/2}(\tau). \quad (\text{E3})$$

In particular, in the case of pure imaginary aspect ratio $\tau = i\tau_0$, the θ -functions and $K_{2p}^{0,1/2}$, $K_{2p}^{1/2,0}$ -functions transforms under Eq. (E1) as follows:

$$\theta_3(i/\tau_0) = \tau_0^{1/2} \theta_3(i\tau_0),$$

$$\theta_2(i/\tau_0) = \tau_0^{1/2} \theta_4(i\tau_0),$$

$$\theta_4(i/\tau_0) = \tau_0^{1/2} \theta_2(i\tau_0),$$

$$\eta(i/\tau_0) = \tau_0^{1/2} \eta(i\tau_0), \quad (\text{E4})$$

$$K_{2p}^{0,1/2}(i/\tau_0) = (i\tau_0)^{2p} K_{2p}^{1/2,0}(i\tau_0),$$

$$K_{2p}^{1/2,0}(i/\tau_0) = (i\tau_0)^{2p} K_{2p}^{0,1/2}(i\tau_0), \quad (\text{E5})$$

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