Euler-Lagrange equations for variational problems on space curves

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We derive the Euler-Lagrange equations for a large class of variational problems on curves. Our result generalizes a recent result obtained in the literature. Moreover, it is simple and self-contained. It directly yields Euler-Lagrange equations in the form of equilibrium equations for the internal force and moment.

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I. INTRODUCTION

In the past years there has been growing interest in one-dimensional continuum models for rodlike objects arising in biology and engineering. Recently, one-dimensional variational problems have turned out to be relevant also for the description of pure bending deformations of thin elastic sheets (cf. [1–3]). Several of these one-dimensional models lead to variational problems on space curves which are invariant under rigid motions. The solutions to variational problems usually satisfy the so-called Euler-Lagrange (or equilibrium) equations associated with the given functional.

There exists a large body of literature in which variational derivations of equilibrium equations for curves are given (cf., e.g., [4,5]). The situation considered in this paper and, more generally, in rod theory is somewhat more general than that of a single curve γ. Namely, one considers a framed curve, i.e., a pair consisting of a space curve γ together with an adapted (right-handed) orthonormal frame r whose columns are called Cosserat directors (cf. [6,7] and Chap. 8 in [8]). For every space curve there is a natural adapted orthonormal frame: the so-called Frénet frame, consisting of the tangent to the curve, the normal, and the binormal. However, one can also choose other adapted frames. Since r takes values in SO(3), it automatically satisfies an ordinary differential equation (ODE) of the form

$$r'(t) = W(t)r(t),$$

where $W(t)$ is a skew-symmetric 3×3 matrix for each parameter value t. In other words, there exist functions κ, μ, and τ satisfying Eq. (3). They are determined by Eq. (3), and they are the curvatures and torsion of the framed curve (γ, r).

There is also a large body of literature concerned with the variational derivation of equilibrium equations for framed curves; cf., e.g., [9–11], Sec. 2.10 in Antman's book [8], Sec. 6.2 in [12], as well as Chap. XI, Sec. 3 of Salencon's book [5]. The classical derivation as, e.g., in [11] usually amounts to

$$\gamma_\varepsilon = \gamma + \varepsilon\dot{\gamma} + o(\varepsilon), \quad r_\varepsilon = r + \varepsilon\dot{r} + o(\varepsilon) \tag{1}$$

in our notation. Here, the correction $o(\varepsilon)$ in the second formula is necessary because of the condition $r_\varepsilon \in \mathrm{SO}(3)$, and for the same reason the virtual displacement $\dot{r}$ must be (up to a rotation) a skew-symmetric matrix.

In [13,14] the authors used variations of the curve to obtain the Euler-Lagrange equations associated with functionals depending on the torsion and on the curvature of the curve. Recently, certain formulas derived by Anderson in [15] have been used to derive Euler-Lagrange equations for a whole class of one-dimensional variational problems (cf. [2,16]). The particular formulas from [15] that have been used are slightly complicated, and their range of applicability is limited. In addition, their derivation as given in [15] relies upon a rather abstract and complicated mathematical machinery, called the variational bicomplex.

As observed in [2] it is, in principle, possible to derive the Euler-Lagrange equations for energy functionals like Eq. (2) making variations of the curve. However, it was observed in [2] that such an approach can lead to rather complicated expressions.

If the energy functional is of the form (2) then it seems more natural to directly perform variations $\kappa_\varepsilon = \kappa + \varepsilon\dot{\kappa}$ and so on and then to define the resulting varied curve and frame by integration. This is the idea exploited in the present paper. Calculating the variational derivative of an energy functional like Eq. (2) under these variations is straightforward. One useful difference to direct variations of the curve (i.e., adding a virtual displacement to the curve) is that γ_ε is automatically parametrized by arclength. The idea of performing variations of the curvatures is also at the heart of the example from [15] which was used as a starting point in [16]. It has been suggested that the formulas obtained using the machinery developed in [15] might be the only approach to derive equilibrium equations for some more complicated Lagrangians among those considered in [16].

The derivation given in the present paper shows that the results of [16] can be derived without using this machinery. In fact, our main result is more general than that in [16]. Moreover, the derivation presented is simple and self-contained. Instead of using the variational bicomplex formalism, it is entirely based on easy calculations and on the Lagrange multiplier rule. Thus it is, in spirit, quite close to the classical ideas described above, and it might not be new to specialists. Being self-contained, the derivation given below is rather flexible. It can easily be adapted to a range of related problems which do not fall into the framework of [16]. For example, one can consider several kinds of boundary conditions. At the end of this paper, we give some simple examples for such extensions.

The Euler-Lagrange equations found here automatically arise in the form of equilibrium equations for the internal

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force and moment. Another advantage of the method presented here is that there are no hidden assumptions whereas, when using a large machinery as a black box, it can sometimes be difficult to trace back the hypotheses.

The Euler-Lagrange equations derived in this paper can be used as a general formula into which one can plug in the energy density (or Lagrangian) of a given variational problem in order to obtain the corresponding equilibrium equations. However, we also emphasize the method of derivation. It consists of few and simple steps, all carried out in detail below and each of which can be modified as need arises.

II. LAGRANGIAN

We start by formulating the variational problems to which we will apply our method. Let $T>0$ and let $\gamma:(0,T)\to\mathbb{R}^3$ be an arclength parametrized curve, i.e., $|\gamma'|=1$. Let $r:(0,T)\to\mathrm{SO}(3)$ be such that the first row agrees everywhere with γ'. Assume that the pair consisting of γ and r is an extremum of an energy functional of the form

$$E(\gamma,r)=\int_0^T \mathcal{L}([\kappa],[\mu],[\tau]). \tag{2}$$

Here, $\mathcal{L}$ is the Lagrangian, $[f]=(f,f',f'',\ldots,f^{(N)})$ (where N is some integer fixed from the outset), and $f^{(k)}$ denotes the kth derivative of the function f. The functions κ, μ, and τ are curvatures and torsions of the framed curve consisting of γ and r, in the sense that r solves the following ordinary differential equation:

$$r'=\begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & -\tau \\ -\mu & \tau & 0 \end{pmatrix} r. \tag{3}$$

For instance, if $\gamma''\neq 0$ and if we set $v=\gamma''/|\gamma''|$ and $n=\gamma'\times v$ then $r=(\gamma',v,n)$ is just the well-known Frenet frame of γ. It satisfies the Frenet equations, which are just Eq. (3) with $\mu\equiv 0$. Hence, the situation considered here is more general than that in [16]. One motivation to consider the more general form (3) is that the general equilibrium equations for inextensible elastic films derived in [1] arise from a variational problem on curves satisfying Eq. (3) with $\tau\equiv 0$, but $\kappa\neq 0$ and $\mu\neq 0$. Any orthonormal frame r along a curve γ satisfies a system of the form (3); cf. [13] for a more detailed discussion of this fact and of various kinds of adapted orthonormal frames.

III. VARIATIONS

By translating and rotating our coordinate system, we may assume with no loss of generality that $\gamma(0)=0$ and that $r(0)$ agrees with the 3×3 unit matrix. By definition, the first row of r agrees with γ'. We denote the second row of r by v and the third row by n, that is, $r=(\gamma',v,n)^T$. Then the initial data become $\gamma'(0)=(1,0,0)$, $v(0)=(0,1,0)$, and $n(0)=(0,0,1)$.

In order to derive the Euler-Lagrange equations satisfied by a critical point (γ,r) of E, we consider the natural variations $\kappa_\varepsilon=\kappa+\varepsilon\dot{\kappa}$, $\mu_\varepsilon=\mu+\varepsilon\dot{\mu}$, and $\tau_\varepsilon=\tau+\varepsilon\dot{\tau}$, where $\dot{\kappa}$, $\dot{\mu}$, and $\dot{\tau}$ are arbitrary smooth functions on $(0,T)$ with zero boundary values. The associated frame $r_\varepsilon=(\gamma'_\varepsilon,v_\varepsilon,n_\varepsilon)^T$ is obtained by solving Eq. (3) with $\kappa+\varepsilon\dot{\kappa}$ instead of κ, etc., and imposing the same initial values as the original frame, i.e., $(\gamma'_\varepsilon(0),v_\varepsilon(0),n_\varepsilon(0))=(\gamma'(0),v(0),n(0))$. The new (arclength parametrized) curve γ_ε is defined simply by integrating, i.e., $\gamma_\varepsilon(t)=\int_0^t\gamma'_\varepsilon$.

Writing $\dot{\gamma}'=:(d/d\varepsilon)|_{\varepsilon=0}\gamma'_\varepsilon$ and so on, and taking derivatives with respect to ε in the analog of Eq. (3) satisfied by $(\gamma'_\varepsilon,v_\varepsilon,n_\varepsilon)$, we see that $(\dot{\gamma}',\dot{v},\dot{n})$ solve the following system of ODEs:

$$\begin{pmatrix} \dot{\gamma}' \\ \dot{v} \\ \dot{n} \end{pmatrix}'=\begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & -\tau \\ -\mu & \tau & 0 \end{pmatrix}\begin{pmatrix} \dot{\gamma}' \\ \dot{v} \\ \dot{n} \end{pmatrix}+\begin{pmatrix} 0 & \dot{\kappa} & \dot{\mu} \\ -\dot{\kappa} & 0 & -\dot{\tau} \\ -\dot{\mu} & \dot{\tau} & 0 \end{pmatrix}\begin{pmatrix} \gamma' \\ v \\ n \end{pmatrix}. \tag{4}$$

One can use the variation of constants formula to deduce from this the expression for $(\dot{\gamma}',\dot{v},\dot{n})$. Defining

$$\xi=-\dot{\tau}\gamma'-\dot{\mu}v-\dot{\kappa}n \tag{5}$$

and $\Xi(t)=\int_0^t\xi$, the result can be concisely written as follows:

$$\begin{aligned} \dot{\gamma}' &= \Xi\times\gamma', \\ \dot{v} &= \Xi\times v, \\ \dot{n} &= \Xi\times n. \end{aligned} \tag{6}$$

Let us check the correctness of these formulas: using Eqs. (3) and (6) we have

$$\begin{aligned} (\Xi\times v)' &= \Xi\times(-\kappa\gamma'-\tau n)+\xi\times v \\ &= -\kappa\dot{\gamma}'-\tau\dot{n}-\dot{\kappa}\gamma'-\dot{\tau}n. \end{aligned}$$

Performing similar calculations with n and γ' instead of v, we find that $(\Xi\times\gamma',\Xi\times v,\Xi\times n)$ solves the ODE system (4). Since $\dot{\gamma}'(0)=\dot{v}(0)=\dot{n}(0)=0$ and since $\Xi(0)=0$, the uniqueness of solutions to ODEs (cf., e.g., [17]) proves that Eqs. (6) are correct.

IV. CONSTRAINTS ON THE VARIATIONS

The constraints on the variations that were used in [16] are

$$\gamma_\varepsilon(T)=\gamma(T), \tag{7}$$

$$(\gamma'_\varepsilon(T),v_\varepsilon(T),n_\varepsilon(T))=(\gamma'(T),v(T),n(T)). \tag{8}$$

These constraints are implicitly imposed in [16] because they are imposed in [15]. They are quite natural, because they arise automatically if one considers variations of γ that vanish near the end points, but they are not the only possible ones; see Sec. 3 in [13] and the examples below.

It is useful to strip off redundancies from constraint (8): obviously, since the frames are in SO(3), we do not need nine equations to ensure Eq. (8). Thus, we replace this constraint with the equivalent one,

$$R_\varepsilon(T) = 0, \tag{9}$$

where we have introduced

$$R_\varepsilon = (v_\varepsilon \cdot n)\gamma' - (\gamma'_\varepsilon \cdot v)v + (\gamma'_\varepsilon \cdot n)n.$$

Clearly, Eq. (8) implies Eq. (9) because γ', v, and n are orthogonal to each other. Conversely, if Eq. (9) holds, then $v_\varepsilon(T)\cdot n(T) = \gamma'_\varepsilon(T)\cdot v(T) = \gamma'_\varepsilon(T)\cdot n(T) = 0$. It is easy to prove that this implies Eq. (8), provided that ε is small. Of course there are other equivalent ways of expressing Eq. (9), but this one is particularly convenient, as will be seen below.

A key observation is that constraints (7) and (9) are just some integral constraints on κ_ε, τ_ε, and μ_ε: setting $\mathcal{G}_1(\kappa_\varepsilon,\tau_\varepsilon,\mu_\varepsilon) = \gamma_\varepsilon(T)$ and $\mathcal{G}_2(\kappa_\varepsilon,\tau_\varepsilon,\mu_\varepsilon) = R_\varepsilon(T)$, they have the general form

$$\mathcal{G}_1(\kappa_\varepsilon,\tau_\varepsilon,\mu_\varepsilon) = 0, \quad \mathcal{G}_2(\kappa_\varepsilon,\tau_\varepsilon,\mu_\varepsilon) = 0. \tag{10}$$

Notice that $\mathcal{G}_1$ and $\mathcal{G}_2$ are just some smooth nonlinear functionals defined on the function space in which the triple (κ,μ,τ) lives [typically $L^2((0,T),\mathbb{R}^3)$, thus allowing for very irregular κ, μ, and τ, if needed].

V. EULER-LAGRANGE EQUATIONS

Define the energy functional $\mathcal{E}$ on the space of curvatures by

$$\mathcal{E}(\kappa,\mu,\tau) = \int_0^T \mathcal{L}([\kappa],[\mu],[\tau]).$$

Since (γ,r) is an extremum for E under constraints (7) and (9), the triple (κ,μ,τ) is an extremum for $\mathcal{E}$ under these constraints—now interpreted in form (10). Thus, the Lagrange multiplier rule (cf., e.g., [18]) tells us that there exist $\lambda_0 \in \mathbb{R}$ and $\lambda_1,\lambda_2 \in \mathbb{R}^3$, with at least one among λ_0, λ_1, and λ_2 being nonzero, such that the following equation holds:

$$\lambda_0 \int_0^T \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\kappa_\varepsilon,\mu_\varepsilon,\tau_\varepsilon) = \lambda_1 \cdot \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \gamma_\varepsilon(T) + \lambda_2 \cdot \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} R_\varepsilon(T). \tag{11}$$

The case $\lambda_0 = 0$ cannot be ruled out *a priori*. It arises if the boundary conditions contain redundancies. Ours do not contain any, unless the solution curve is degenerate. We will prove this later. Now we consider the generic case when $\lambda_0 \neq 0$, so after dividing Eq. (11) by λ_0 we may assume without loss of generality that $\lambda_0 = 1$. After integrating by parts, the left-hand side of Eq. (11) is simply

$$\int_0^T \left[\sum_{k=0}^N (-1)^k (\partial_{\kappa^{(k)}}\mathcal{L})^{(k)}\right]\dot{\kappa} + \int_0^T \left[\sum_{k=0}^N (-1)^k (\partial_{\tau^{(k)}}\mathcal{L})^{(k)}\right]\dot{\tau} + \int_0^T \left[\sum_{k=0}^N (-1)^k (\partial_{\mu^{(k)}}\mathcal{L})^{(k)}\right]\dot{\mu}. \tag{12}$$

As before, $f^{(k)}$ denotes the kth derivative of f.

To calculate the right-hand side of Eq. (11), we use Eq. (6) to find that simply

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} R_\varepsilon(T) = \Xi(T),$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \gamma_\varepsilon(T) = \int_0^T \dot{\gamma}' = \int_0^T \Xi \times \gamma'.$$

By integration by parts, the right-hand side of Eq. (11) therefore equals

$$\int_0^T \lambda_1 \cdot \Xi \times \gamma' + \lambda_2 \cdot \xi = \int_0^T \xi \cdot \tilde{\Lambda}, \tag{13}$$

where we have introduced

$$\tilde{\Lambda} = \lambda_2 - \lambda_1 \times \int_t^T \gamma'.$$

The functions $\dot{\tau}$, $\dot{\kappa}$, and $\dot{\mu}$ are arbitrary and mutually independent. Thus, inserting Eqs. (12) and (13) into Eq. (11) and recalling definition (5) of ξ, we obtain the "constitutive" Euler-Lagrange equations,

$$\sum_{k=0}^N (-1)^k (\partial_{\kappa^{(k)}}\mathcal{L})^{(k)} = -\Lambda_3,$$

$$\sum_{k=0}^N (-1)^k (\partial_{\mu^{(k)}}\mathcal{L})^{(k)} = -\Lambda_2,$$

$$\sum_{k=0}^N (-1)^k (\partial_{\tau^{(k)}}\mathcal{L})^{(k)} = -\Lambda_1, \tag{14}$$

where $\Lambda_1 = \tilde{\Lambda}\cdot\gamma'$ and $\Lambda_2 = \tilde{\Lambda}\cdot v$ and $\Lambda_3 = \tilde{\Lambda}\cdot n$. Set $\Lambda = (\Lambda_1,\Lambda_2,\Lambda_3)^T$ and $\eta = (\eta_1,\eta_2,\eta_3)^T$, where $\eta_1 = \lambda_1\cdot\gamma'$, $\eta_2 = \lambda_1\cdot v$, and $\eta_3 = \lambda_1\cdot n$. Then using $\tilde{\Lambda}' = \lambda_1 \times \gamma'$ and Eq. (3), we find that Λ and η satisfy the "structural" equations

$$\Lambda' = \begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & -\tau \\ -\mu & \tau & 0 \end{pmatrix}\Lambda + \begin{pmatrix} 0 \\ \eta_3 \\ -\eta_2 \end{pmatrix}, \tag{15}$$

$$\eta' = \begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & -\tau \\ -\mu & \tau & 0 \end{pmatrix}\eta. \tag{16}$$

Coupling system (14)–(16), we obtain the closed Euler-Lagrange system.

VI. DEGENERATE CASE

To complete the proof, we finally consider the degenerate case when $\lambda_0 = 0$ in Eq. (11). This amounts to having zero on the left-hand sides of system (14), i.e.,

$$\Lambda = 0 \tag{17}$$

identically. Hence, $\Lambda' = 0$, so Eq. (15) implies that $\eta_3 = \eta_2 = 0$. Hence, their derivatives vanish as well, so Eq. (16)

implies that $\eta_1'=0$, that $\kappa\eta_1=0$, and that $\mu\eta_1=0$. Hence, $\eta_1=0$ unless both κ and μ vanish identically. But if κ and μ vanish identically, then Eq. (3) implies that $\gamma''=0$, i.e., that γ is a straight line, and one can explicitly solve Eq. (3). Hence, in this case the solution is trivial. It remains to rule out the case that $\eta_1=\eta_2=\eta_3=0$. By definition of these functions, this implies that $\lambda_1=0$. But then Eq. (17) implies that also $\lambda_2=0$. This contradicts the fact that not all multipliers λ_0, λ_1, and λ_2 are zero. Thus, the only possibility is the trivial one considered above when μ and κ vanish identically and γ is a straight line.

VII. RELATION TO A RECENT RESULT IN THE LITERATURE

The system (14)–(16) generalizes the system obtained in [16]. To see this, replace the symbols F_t, F_n, and F_b in [16] with $-\eta_1$, $-\eta_2$, and $-\eta_3$, and the symbols M_t, M_n, and M_b in [16] with $-\Lambda_1$, $-\Lambda_2$, and $-\Lambda_3$. Moreover, recall that multiplication of a skew-symmetric matrix is equivalent to taking the vector product with an appropriate vector: we define $\omega=(-\tau,-\mu,\kappa)^T$. (Up an irrelevant sign convention regarding τ, this is the same ω as in [16], except that in [16] one has $\mu\equiv 0$.) Then Eq. (16) is equivalent to

$$\eta' + \omega \times \eta = 0, \tag{18}$$

and Eq. (15) is equivalent to

$$\Lambda' + \omega \times \Lambda + (1,0,0)^T \times \eta = 0. \tag{19}$$

Observe that the symbol t in formula (3) of [16] is just the vector $(1,0,0)^T$, according to their notational convention. Now evidently Eqs. (18) and (19) are equivalent to Eq. (3) in [16] and the first and third equations in Eq. (14) are equivalent to Eq. (4) in [16]. The second equation in Eq. (14) appears because we allowed μ to be nonzero. Finally, notice that if, as in [16], the Lagrangian $\mathcal{L}$ also depends on some further functions $\chi_1,\ldots,\chi_m$ and their derivatives, then one trivially obtains further Euler-Lagrange equations of the form

$$\partial_{\chi_k}\mathcal{L} - (\partial_{\dot{\chi}_k}\mathcal{L})' + (\partial_{\chi_k}\mathcal{L})'' + \cdots = 0$$

simply from the variations $(\chi_k)_\varepsilon=\chi_k+\varepsilon\dot{\chi}_k$. This is Eq. (5) in [16].

VIII. MODIFICATIONS AND EXAMPLES

As the results obtained above generalize those in [16], all examples considered in that paper fall into the present framework. We have proven that the Euler-Lagrange system (14)–(16) holds for them.

However, the method outlined above allows one to consider modifications of these examples as well. For instance, it allows more general constraints than just Eqs. (7) and (8). In other words, it allows one to study not only extrema among all curves with a fixed end point and running into this end point by a fixed angle, but also extrema among a class of curves with other kinds of conditions at the end points. For instance:

(i) One very natural and straightforward modification is to drop constraint (8). If we interpret γ as the centerline of a rod, then this amounts to fixing the end point of the centerline while allowing the directors to be twisted and letting γ run into its end point from arbitrary angles. The corresponding Euler-Lagrange equations are obtained from the above by setting $\lambda_2=0$. This can be checked by going through the above derivation, but without including condition (8).

What does $\lambda_2=0$ imply for system (14)–(16)? To see this, we check how that Lagrange multiplier appears in that system: looking at the definition of the Λ_i, we see that $\lambda_2=0$ if and only if $\Lambda_1(T)=\Lambda_2(T)=\Lambda_3(T)=0$. Thus, we have proven that if γ,r is a nontrivial (i.e., γ is not a straight line) extremum of Eq. (2) under the boundary condition (7) alone, then there exist solutions Λ,η of system (15) and (16) such that Eqs. (14) are satisfied and such that $\Lambda_1(T)=\Lambda_2(T)=\Lambda_3(T)=0$.

(ii) Another simple modification is to allow the curve to slip within a certain plane rather than fixing its end point. A motivation for this comes from the Euler-Lagrange equations for developable surfaces as derived in [1]. It corresponds to imposing constraint (8) together with the constraint

$$[\gamma_\varepsilon(T) - \gamma(T)] \cdot w = 0, \tag{20}$$

for some fixed vector $w\in\mathbb{R}^3$. Thus, Eq. (20) replaces the stronger end point fixing constraint (7). In order to include this constraint we now define

$$\mathcal{G}_1(\kappa_\varepsilon,\tau_\varepsilon,\mu_\varepsilon) = [\gamma_\varepsilon(T) - \gamma(T)] \cdot w.$$

Then Eq. (11) becomes

$$\lambda_0 \int_0^T \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{L}(\kappa_\varepsilon,\mu_\varepsilon,\tau_\varepsilon) = \hat{\lambda}_1 \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \gamma_\varepsilon(T)\cdot w + \lambda_2 \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} R_\varepsilon(T).$$

We could now repeat the whole procedure outlined above. However, we can also observe that this equation becomes Eq. (11) if we set

$$\lambda_1 = \hat{\lambda}_1 w.$$

Therefore, analogous to the vanishing of λ_2 in the previous example, here we have the extra bit of information that λ_1 is parallel to w. [One has more information because Eq. (20) is only one constraint, whereas Eq. (7) is three.] How does this knowledge appear in system (14)–(16)? Looking at the definition of η we see that it means that $\eta(0)$ is parallel to w.

We conclude that if γ,r is a nontrivial (i.e., γ is not a straight line) extremum of Eq. (2) under the boundary conditions (8) and (20), then there exist solutions Λ,η of system (15) and (16) such that Eqs. (14) are satisfied and such that $\eta(0)$ is parallel to w.

(iii) One can also impose an extra constraint like $\int_0^T\tau=\text{const}$ as in [13], simply by including $\mathcal{G}_3(\tau)=\int_0^T\tau$ as a third constraint functional. Summarizing, imposing other constraints than Eqs. (7) and (8) above will simply lead to different constraint functionals $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$ to which one can still apply the same scheme as the one carried out above for cases (7) and (8).

(iv) A natural example which falls into our framework arises in a variant (obtained in [1]) of the equilibrium equations from [2]. It involves lines of curvature (instead of geodesics as in [2]) on a surface. A line of curvature can be regarded as a curve together with an orthonormal frame r, called the Darboux frame, which satisfies Eq. (3) with $\tau \equiv 0$, but neither κ nor μ vanishes identically.

Observe that none of the above modifications (i)–(iii) satisfies the conditions required in [16]. Also example (iv) does not fall into the framework of [16].

IX. CONCLUSION

It is possible to generalize the results from [16], and to do so by a different method than the one used in [16]. This different method has several features: it is simple and self-contained, and it makes no hidden assumptions. It also allows the Lagrangian to depend on three curvatures rather than just two. This allows one to study framed curves, which arise, e.g., in problems for curves lying on a nonflat surface. In fact, it applies to any situation where one has some orthonormal frame along the curve. Finally, it allows one to consider more general boundary conditions (cf. Sec. VIII). From a formal viewpoint, an interesting feature of the method described above is that it is basic and self-contained (up to the use of the Lagrange multiplier rule).

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