Integrable nonlinear Schrödinger equation on simple networks: Connection formula at vertices

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We study the case in which the nonlinear Schrödinger equation (NLSE) on simple networks consisting of vertices and bonds has an infinite number of constants of motion and becomes completely integrable just as in the case of a simple one-dimensional (1D) chain. Here the strength of cubic nonlinearity is different from bond to bond, and networks are assumed to have at least two semi-infinite bonds with one of them working as an incoming bond. The connection formula at vertices obtained from norm and energy conservation rules shows (1) the solution on each bond is a part of the universal (bond-independent) soliton solution of the completely integrable NLSE on the 1D chain, but is multiplied by the inverse of square root of bond-dependent nonlinearity; (2) nonlinearities at individual bonds around each vertex must satisfy a sum rule. Under these conditions, we also showed an infinite number of constants of motion. The argument on a branched chain or a primary star graph is generalized to other graphs, i.e., general star graphs, tree graphs, loop graphs and their combinations. As a relevant issue, with use of reflectionless propagation of Zakharov-Shabat's soliton through networks we have obtained the transmission probabilities on the outgoing bonds, which are inversely proportional to the bond-dependent strength of nonlinearity. Numerical evidence is also given to verify the prediction.

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I. INTRODUCTION

Transport in networks with vertices and bonds [1] received growing attention recently. The practical importance of this problem is caused by the fact that those networks mimic networks of nonlinear waveguides and optical fibers [2], Bose-Einstein condensates in optical lattices [3], superconducting ladders of Josephson junctions [4,5], double helix of DNA [6], vein networks in leaves [7,8], etc.

Most studies so far, however, are restricted to solving the linear Schrödinger equation to obtain the energy spectra in closed networks and transmission probabilities for open networks with semi-infinite leads. In fact, one can mention noteworthy contributions in the context of quantum graphs [9–16].

On the other hand, with introduction of the nonlinearity to the time-dependent Schrödinger equation, the network provides a nice playground where one can see interesting soliton propagations and nonlinear dynamics in general. There already exist some numerical studies of the soliton propagation through the discrete chain attached with small graphs [17–21], where the discrete nonlinear Schrödinger equation (DNLSE) plays a role. However, we see little exact analytical treatment of soliton propagation through networks, namely, through an assembly of continuum line segments connected at vertices, within a framework of nonlinear Schrödinger equation (NLSE) [22]. While there exist important analytical studies on initial value problems on the semi-infinite chain [23,24] and the finite chain [25,26], no corresponding ones in networks or graphs have appeared up to now. From a growing practical viewpoint, it is highly desirable to see a possibility of absolutely-stable soliton propagation through networks.

The subject is difficult due to the presence of vertices where the underlying one-dimensional (1D) chain should bifurcate or multifurcate in general. Here, not branching angles but the topology of bifurcations is essential. Under the ordinary continuity and smoothness conditions at each vertex as used for the linear Schrödinger equation on graphs [9–12,14,16], a soliton coming into the vertex along one of the bonds shows a complicated motion around the vertex such as reflection and emergence of the radiation there. Therefore NLSE on networks is far from being completely integrable, contrary to NLSE on a simple 1D chain [27].

In this paper, we shall investigate whether or not the NLSE on simple networks can have an infinite number of constants of motion and be completely integrable just as in the case of a simple 1D chain. Here the strength of cubic nonlinearity is different from bond to bond, and networks are assumed to have at least two semi-infinite bonds with one of them working as an incoming bond. We shall elucidate: if NLSE on networks would have an infinite number of constants of motion, what kind of connection formula should be required at each vertices and what kind of constraint should be imposed on strengths of nonlinearity around each vertex. We shall reveal how solutions of NLSE on networks will be mapped to those of the integrable NLSE on a 1D chain. Once this mapping will be found, the integrability properties like Lax pair, Backlund transformation, etc. are automatically maintained, and will not be addressed in this paper. Network models we shall choose are star graphs, tree graphs, loop graphs, and their combinations. As a relevant issue, with use of reflectionless propagation of Zakharov-Shabat's soliton [27] through networks we shall obtain the transmission probabilities on the outgoing bonds.

In Sec. II, using the simplest network [i.e., a primary star graph (PSG)], we shall analyze the norm and energy conservation rules. In Secs. II A and II B we show how these rules lead to connection formulas at a vertex. In Sec. II C we shall address the boundary condition to guarantee these connection formulas, finding the sum rule for strengths of nonlin-

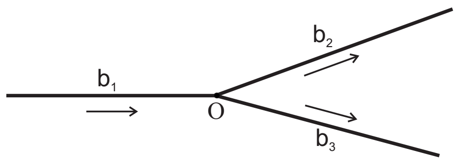

FIG. 1. Primary star graph consisting of three semi-infinite bonds connected at a vertex O.

earity at each bond. Here we also establish the relationship of the general solutions between NLSE on networks and NLSE on the 1D chain. In Sec. III an infinite number of constants of motion will be given for NLSE on PSG. In Sec. IV more general cases like tree graphs, loop graphs and their combinations are investigated. Section V presents relevant information on the analytic expression for transmission probabilities in the case of reflectionless propagation of Zakharov-Shabat's soliton through networks. Numerical verification of the result will also be carried out, by solving the discrete version of NLSE on PSG. Summary and discussion are given in Sec. VI. The Appendix is devoted to the way of numerically solving the corresponding DNLSE on PSG.

II. CONSERVATION RULES, CONNECTION FORMULA AND SUM RULES ON PRIMARY STAR GRAPH

A. Norm conservation rule

We consider an elementary branched chain or a primary star graph (PSG) in Fig. 1, where the vertex site is now taken as origin O. Space coordinate x_1 in bond b_1 is defined from $-\infty$ to 0 and coordinates x_2 in bond b_2 and x_3 in bond b_3 are commonly defined from 0 to $+\infty$. On each bond we have the nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial\Psi_k}{\partial t}+\frac{\partial^2\Psi_k}{\partial x_k^2}+\beta_k|\Psi_k|^2\Psi_k=0,\quad k=1,2,3, \tag{1}$$

with x_k defined on $-\infty<x_1<0,\ 0<x_2,x_3<\infty$. It should be noted that the strength of nonlinearity $\beta_k(>0)$ may be different among bonds. We shall explore the solution of NLSE on PSG which satisfies the following conditions at infinity: $\Psi_1(x_1)\to 0$ at $x_1\to-\infty$, $\Psi_k(x_k)\to 0$ at $x_k\to\infty$ for $k=2,3$. One of the physically important conditions for the solution in PSG is the norm conservation. The norm is defined as

$$N=\|\Psi\|^2=\int_{-\infty}^{0}|\Psi_1(x,t)|^2dx+\int_0^\infty|\Psi_2(x,t)|^2dx+\int_0^\infty|\Psi_3(x,t)|^2dx. \tag{2}$$

Let us find conditions for which the norm is conservative. For this purpose we calculate its time derivative,

$$\frac{d}{dt}N=\int_{-\infty}^{0}\frac{\partial|\Psi_1(x,t)|^2}{\partial t}dx+\int_0^\infty\frac{\partial|\Psi_2(x,t)|^2}{\partial t}dx+\int_0^\infty\frac{\partial|\Psi_3(x,t)|^2}{\partial t}dx. \tag{3}$$

From Eq. (1) we have the continuity equation,

$$\frac{\partial|\Psi_k(x,t)|^2}{\partial t}=-\frac{\partial}{\partial x}j_k(x,t)\equiv-2\frac{\partial}{\partial x}\mathrm{Im}\left[\Psi_k^*(x,t)\frac{\partial\Psi_k(x,t)}{\partial x}\right], \tag{4}$$

where $j_k(x,t)$ means the current density.

Using Eq. (4) in Eq. (3), we find that the norm is conservative only when the following connection formula at the vertex is satisfied:

$$\mathrm{Im}\left[\Psi_1^*\frac{\partial\Psi_1}{\partial x}\right]\Bigg|_{x=0}=\mathrm{Im}\left[\Psi_2^*\frac{\partial\Psi_2}{\partial x}\right]\Bigg|_{x=0}+\mathrm{Im}\left[\Psi_3^*\frac{\partial\Psi_3}{\partial x}\right]\Bigg|_{x=0}. \tag{5}$$

In Eq. (5) we prescribe $[\cdots]|_{x=0}$ to $\lim_{x_1\to0^-}[\cdots]$ for variables on bond b_1 and to $\lim_{x_{2,3}\to0^+}[\cdots]$ for variables on bond $b_{2,3}$. Hereafter the same prescription as above will be employed.

The equality in Eq. (5) implies the local current conservation condition at the vertex O,

$$j_1(0,t)=j_2(0,t)+j_3(0,t). \tag{6}$$

B. Energy conservation rule

The second important condition for the solution on PSG is the energy conservation. In PSG, the energy is defined as

$$E=E_1+E_2+E_3, \tag{7}$$

where

$$E_k=\int_{b_k}\left(\left|\frac{\partial\Psi_k}{\partial x}\right|^2-\frac{\beta_k}{2}|\Psi_k|^4\right)dx. \tag{8}$$

Let us take the time derivative

$$\frac{d}{dt}E_k=2\ \mathrm{Re}\left[\int_{b_k}\left(\frac{\partial\Psi_k^*}{\partial x}\frac{\partial^2\Psi_k}{\partial x\,\partial t}-\beta\Psi_k(\Psi_k^*)^2\frac{\partial\Psi_k}{\partial t}\right)dx\right]. \tag{9}$$

We then simplify each of integrands separately using NLSE for each bond in Eq. (1) as

$$\begin{aligned}I_1&\equiv\int_{b_k}\frac{\partial\Psi_k^*}{\partial x}\frac{\partial^2\Psi_k}{\partial x\,\partial t}dx\\&=i\int_{b_k}\frac{\partial\Psi_k^*}{\partial x}\frac{\partial^3\Psi_k}{\partial x^3}dx+i\beta_k\int_{b_k}\left[2|\Psi_k|^2\left|\frac{\partial\Psi_k^*}{\partial x}\right|^2+\left(\frac{\partial\Psi_k^*}{\partial x}\right)^2\Psi_k^2\right]dx\\&=\pm i\frac{\partial\Psi_k^*}{\partial x}\frac{\partial^2\Psi_k}{\partial x^2}\Bigg|_{x=0}-i\int_{b_k}\left|\frac{\partial^2\Psi_k}{\partial x^2}\right|^2dx\\&\quad+2i\beta_k\int_{b_k}|\Psi_k|^2\left|\frac{\partial\Psi_k^*}{\partial x}\right|^2dx+i\beta_k\int_{b_k}\left(\frac{\partial\Psi_k^*}{\partial x}\right)^2\Psi_k^2dx,\end{aligned} \tag{10}$$

$$
\begin{aligned}
I_2 &\equiv \beta_k \int_{b_k} \Psi_k (\Psi_k^*)^2 \frac{\partial \Psi_k}{\partial t} dx \\
&= i\beta_k \int_{b_k} \Psi_k (\Psi_k^*)^2 \frac{\partial^2 \Psi_k}{\partial x^2} dx + i\beta_k^2 \int_{b_k} |\Psi_k|^6 dx \\
&= \pm i\beta_k \Psi_k^* |\Psi_k|^2 \frac{\partial \Psi_k}{\partial x}\bigg|_{x=0} - i\beta_k \int_{b_k} \left(\frac{\partial \Psi_k}{\partial x}\right)^2 (\Psi_k^*)^2 dx \\
&\quad - 2i\beta_k \int_{b_k} |\Psi_k|^2 \left|\frac{\partial \Psi_k^*}{\partial x}\right|^2 dx + i\beta_k^2 \int_{b_k} |\Psi_k|^6 dx, \qquad (11)
\end{aligned}
$$

where "+" sign corresponds to the case of bond b_1 and "−" sign to the case of bonds $b_{2,3}$. This prescription will also be used in Eq. (12). In calculations in Eqs. (10) and (11), we supposed Ψ_k, Ψ_k^* and their derivatives tend to zero at $x = \pm\infty$. Applying the identities (10) and (11), we arrive at

$$
\begin{aligned}
\frac{d}{dt} E_k &= 2\,\mathrm{Re}(I_1 - I_2) \\
&\equiv \pm 2\,\mathrm{Re}\left(i \frac{\partial \Psi_k^*}{\partial x} \frac{\partial^2 \Psi_k}{\partial x^2} - i\beta \Psi_k^* |\Psi_k|^2 \frac{\partial \Psi_k}{\partial x} \right)\bigg|_{x=0} \\
&= \pm 2\,\mathrm{Re}\left[i \frac{\partial \Psi_k^*}{\partial x} \left(\frac{\partial^2 \Psi_k}{\partial x^2} + \beta \Psi_k |\Psi_k|^2 \right) \right]\bigg|_{x=0} \\
&= \pm 2\,\mathrm{Re}\left[\frac{\partial \Psi_k^*}{\partial x} \frac{\partial \Psi_k}{\partial t} \right]\bigg|_{x=0}. \qquad (12)
\end{aligned}
$$

Using Eq. (12) in the time derivative of Eq. (7), we find that the energy is conserved if the following rule is satisfied:

$$
\mathrm{Re}\left[\frac{\partial \Psi_1^*}{\partial x} \frac{\partial \Psi_1}{\partial t} \right]\bigg|_{x=0} = \mathrm{Re}\left[\frac{\partial \Psi_2^*}{\partial x} \frac{\partial \Psi_2}{\partial t} \right]\bigg|_{x=0} + \mathrm{Re}\left[\frac{\partial \Psi_3^*}{\partial x} \frac{\partial \Psi_3}{\partial t} \right]\bigg|_{x=0}, \qquad (13)
$$

which is another connection formula at the vertex O.

C. Boundary condition at vertex and sum rule for strength of nonlinearity

The norm and energy are conserved if the (nonlinear) boundary conditions in Eqs. (5) and (13) at the vertex should hold. These boundary conditions are found to be satisfied by employing either one of the following linear connection formulas at the vertex O,

$$
\begin{gathered}
\alpha_1 \Psi_1|_{x=0} = \alpha_2 \Psi_2|_{x=0} = \alpha_3 \Psi_3|_{x=0}; \\
\frac{1}{\alpha_1} \frac{\partial \Psi_1}{\partial x}\bigg|_{x=0} = \frac{1}{\alpha_2} \frac{\partial \Psi_2}{\partial x}\bigg|_{x=0} + \frac{1}{\alpha_3} \frac{\partial \Psi_3}{\partial x}\bigg|_{x=0}, \qquad (14)
\end{gathered}
$$

or

$$
\begin{gathered}
\alpha_1 \frac{\partial \Psi_1}{\partial x}\bigg|_{x=0} = \alpha_2 \frac{\partial \Psi_2}{\partial x}\bigg|_{x=0} = \alpha_3 \frac{\partial \Psi_3}{\partial x}\bigg|_{x=0}; \\
\frac{1}{\alpha_1} \Psi_1|_{x=0} = \frac{1}{\alpha_2} \Psi_2|_{x=0} + \frac{1}{\alpha_3} \Psi_3|_{x=0}, \qquad (15)
\end{gathered}
$$

where α_1, α_2 and α_3 are arbitrary real constants.

Among many possible choices of α_1, α_2, and α_3, there is one special case in which an infinite number of constants of motion can be found and NLSE on PSG becomes completely integrable. We shall now consider this case by finding suitable values for α_1, α_2, and α_3.

Let us assume that there exists a bond-independent universal function $g(x,t)$ underlying PSG, which satisfies

$$
\begin{gathered}
\alpha_k \Psi_k|_{x=0} = g(0,t), \\
\alpha_k \frac{\partial \Psi_k}{\partial x}\bigg|_{x=0} = \frac{\partial g(x,t)}{\partial x}\bigg|_{x=0} \qquad (16)
\end{gathered}
$$

for $k=1$, 2 , and 3. The upper half of Eq. (14) is identical to the upper half of Eq. (16). With use of Eq. (16) the lower half of Eq. (14) can also be satisfied under the constraint,

$$
\frac{1}{\alpha_1^2} = \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}. \qquad (17)
$$

Similarly, using Eqs. (16) and (17), we find that Eq. (15) can be satisfied as well.

By the way, is there any bond-independent universal function which should satisfy Eq. (16)? The answer is yes. In fact, the general soliton solution of the integrable NLSE in 1D chain,

$$
i\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + \beta |\Psi|^2 \Psi = 0, \qquad (18)
$$

takes the form

$$
\Psi(x,t) = \sqrt{\frac{2}{\beta}}\, i q(x,t), \qquad (19)
$$

where $q(x,t)$ stands for the β-independent universal solution which satisfies the completely integrable NLSE with $\beta=2$,

$$
iq_t + q_{xx} + 2q|q|^2 = 0, \quad -\infty < x < +\infty. \qquad (20)
$$

Now we can introduce the solution of NLSE in Eq. (1) on PSG, which, on each bond, is composed of the universal soliton solution $q(x,t)$ on a simple 1D chain but multiplied by the inverse of square root of bond-dependent nonlinearity β_k,

$$
\Psi_k(x_k,t) = \sqrt{\frac{2}{\beta_k}}\, i q(x_k,t), \qquad (21)
$$

where the functions $q(x_1,t)$ and $q(x_{2,3},t)$ satisfy Eq. (20) and are defined on $(-\infty;0]$ and $[0,+\infty)$, respectively.

Then, noting the above fact and choosing

$$
\alpha_k = \sqrt{\beta_k} \quad (k=1,2,3), \qquad (22)
$$

Equation (16) at the vertex is reduced to

$$
\sqrt{\beta_k}\Psi_k|_{x=0} = \sqrt{2}\, i q(0,t);
$$

$$\sqrt{\beta_k}\frac{\partial \Psi_k}{\partial x}\bigg|_{x=0} = \sqrt{2}i\frac{\partial q(x_k,t)}{\partial x}\bigg|_{x=0} \tag{23}$$

with $k=1$, 2, and 3. Here $\sqrt{2}iq(x,t)$ corresponds to $g(x,t)$ in Eq. (16). At the same time, the constraint in Eq. (17) becomes

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3}, \tag{24}$$

which means the sum rule for strengths of nonlinearity around the vertex. Thus the general solution in Eq. (21) has proved to satisfy the boundary condition that guarantees the norm and energy conservation rules for PSG.

To summarize the results so far, the norm and energy conservation rules are satisfied by the connection formula in Eqs. (14) or (15) at the vertex. Among many possible choices of α_k, an interesting integrable case occurs when α_k takes the value in Eq. (22) and strengths of nonlinearity around the vertex satisfy the sum rule in Eq. (24). In this case the general soliton solution on PSG is given by Eq. (21). Equation (24) plays a crucial role: unless $\beta_1 \neq \beta_2, \beta_3$, no interesting bifurcation of a soliton propagation occurs at the vertex.

Equation (21) also guarantees the following relation at the vertex:

$$\lim_{x_1\to -0}\sqrt{\beta_1}\frac{\partial^n \Psi_1(x_1)}{\partial x_1^n} = \lim_{x_{2,3}\to +0}\sqrt{\beta_{2,3}}\frac{\partial^n \Psi_{2,3}(x_{2,3})}{\partial x_{2,3}^n} = \sqrt{2}i\frac{\partial^n q(x,t)}{\partial x^n}\bigg|_{x=0}, \tag{25}$$

for any $n=0,1,2,\ldots$, which means that, while the solution $\Psi(x,t)$ itself in Eq. (21) is neither continuous nor smooth at the vertex, its scaled version $q(x,t)$ shows no singularity there. The absence of singularity of any kind in the scaled function indicates that we see neither reflection nor emergence of radiation at the vertex, and Eq. (21) stands for the general soliton solution on PSG.

One might be skeptical about the ubiquity of the condition in Eqs. (14) and (15) or its explicit realization in Eqs. (23) and (24), when one will proceed to analysis of many other conservation rules. However, as will be proven in the next Section, the general solution on PSG given by Eq. (21) guarantees an infinite number of constants of motion on PSG, besides the norm and energy.

In closing this section, we should give following two comments:

(i) In the community of the quantum graph, those who are engaged in the linear Schrödinger equation (LSE) use one parameter family of the standard continuity and smoothness conditions at a vertex. On the other hand, our connection formula in Eqs. (14) or (15) is more generic and includes this family in the special limit, $\alpha_1=\alpha_2=\alpha_3$. Our findings, however, are consistent with the general argument on mesoscopic quantum splitters with a branching point [28–30]. In fact, in the early days of quantum graphs, Exner and Šeba [11,12] obtained $3^2(=9)$ parameter family of the connection formula by assessing the self-adjointness of Schrödinger operator (that is equivalent to the norm conservation) on PSG, which certainly accommodates Eq. (14) and (15).

(ii) The norm and energy conservation rules in Eqs. (5) and (13) might be satisfied by the connection formula different from Eqs. (14) or (15) as

$$\breve{\alpha}_1\Psi_1|_{x=0} = \breve{\alpha}_2\Psi_2|_{x=0},$$

$$\frac{1}{\breve{\alpha}_1}\frac{\partial \Psi_1}{\partial x}\bigg|_{x=0} = \frac{1}{\breve{\alpha}_2}\frac{\partial \Psi_2}{\partial x}\bigg|_{x=0}$$

together with $\Psi_3|_{x=0}=0$ or $\frac{\partial \Psi_3}{\partial x}|_{x=0}=0$, or by another variant where the role of 2 and 3 are interchanged. However, this kind of connection formula will make the bond b_3 or b_2 disconnected from PSG, changing the topology of the underlying graph. Hence it should be discarded.

III. AN INFINITE NUMBER OF CONSERVATION RULES

In the previous section, we showed that the norm and energy conservation rules can be satisfied by the general solution in Eq. (21) which is composed of the solution of the integrable NLSE with $\beta=2$ in Eq. (20). Below we shall show that, so long as the general solution on PSG is described by parts of the universal scaled function $q(x,t)$ which is the soliton solution of Eq. (20), all the conservation laws for 1D chain should hold for PSG under the sum rule Eq. (24).

Exact analytical soliton solutions of NLSE on the infinite 1D chain are found by Zakharov and Shabat [27]. They also showed the theorem [27] that soliton solutions on the chain satisfy an infinite number of conservation laws given by $\int f_n(q(x,t))dx=(2i)^n C_n$, with C_n being constant, f_n is a polynomial of q and its derivatives with respect to x. Guided by their theorem, we now investigate the following quantity for PSG:

$$Q_n(t) \equiv \sum_{k=1}^{3}\beta_k^{-1}\int_{b_k} f_n[q(x_k,t)]dx_k, \tag{26}$$

where $q(x_k,t)$ is the solution of Eq. (20) in the bond b_k and $f_n[q(x,t)]$ obeys the recursion relation [see Eq. (35) of [27]],

$$f_{n+1} = q\frac{\partial}{\partial x}\left(\frac{1}{q}f_n\right) + \sum_{j+l=n} f_j f_l,$$

$$f_1 = |q|^2. \tag{27}$$

In fact, with use of Eq. (24), the rhs of Eq. (26) turns out

$$\beta_1^{-1}\int_{-\infty}^{0} f_n[q(x,t)]dx + (\beta_2^{-1}+\beta_3^{-1})\int_{0}^{+\infty} f_n[q(x,t)]dx = \beta_1^{-1}\int_{-\infty}^{+\infty} f_n[q(x,t)]dx = \beta_1^{-1}(2i)^n C_n, \tag{28}$$

where the second equality is due to the conservation rule for the 1D chain [27] and C_n is constant. Hence Q_n has proved to be a constant of motion.

It is easy to see that f_n is the $(2n)$th order polynomial of q and its derivatives with respect to x, written in the following form:

$$f_n = \sum_{s=1}^{n} b_s P_{n,2s}(q,q_x,q_{xx},\ldots), \tag{29}$$

where $P_{n,2s}=q^{k_1}(q^*)^{k_2}q_x^{k_3}(q_x^*)^{k_4}\cdots$ with $k_1+k_2+k_3+\cdots=2s$.

Noting Eq. (21), one can obtain an infinite number of conservation laws in PSG,

$$(2i)^n C_n \beta_1^{-1} = \frac{1}{2}\sum_{k=1}^{3}\int_{b_k}\sum_{s=1}^{n} b_s\left(\frac{\beta_k}{2}\right)^{s-1} P_{n,2s}(\Psi_k,\Psi_{k,x},\ldots)dx_k. \tag{30}$$

In Eq. (30), the cases $n=1$ and 3 give the norm and energy conservation rules in Eqs. (2) and (7) with Eq. (8), respectively. The current conservation rule is now given by

$$(2i)^2 C_2 \beta_1^{-1} = \frac{1}{2}\sum_{k}\int_{b_k}\left(\Psi_k^*\frac{\partial \Psi_k}{\partial x_k}\right)(x_k,t)dx_k. \tag{31}$$

Some higher-order conservation rules are as follows:

$$(2i)^4 C_4 \beta_1^{-1} = \frac{1}{2}\sum_{k}\int_{b_k}\left(\Psi_k\frac{\partial^3 \Psi_k^*}{\partial x_k^3} + \frac{3\beta_k}{2}\Psi_k\frac{\partial \Psi_k^*}{\partial x_k}|\Psi_k|^2\right) \times (x_k,t)dx_k, \tag{32}$$

$$(2i)^5 C_5 \beta_1^{-1} = \frac{1}{2}\sum_{k}\int_{b_k}\left[\left|\frac{\partial^2 \Psi_k}{\partial x_k^2}\right|^2 + \frac{\beta_k^2}{2}|\Psi_k|^6 - \frac{\beta_k}{2}\left(\frac{\partial}{\partial x_k}|\Psi_k|^2\right)^2 - 3\beta_k\left|\frac{\partial \Psi_k}{\partial x_k}\right|^2|\Psi_k|^2\right](x_k,t)dx_k. \tag{33}$$

The above results are also true for more general star graphs consisting of M semi-infinite bonds connected at a single vertex. In such cases, the initial soliton at an incoming bond splits into $M-1$ solitons in the remaining outgoing bonds. In this case, on the rhs of the lower halves of Eqs. (14) and (15), the summation is taken over all the outgoing $M-1$ bonds. Correspondingly, the extended version of Eqs. (24) is given by

$$\frac{1}{\beta_1} = \sum_{j=1}^{M-1}\frac{1}{\beta_j}. \tag{34}$$

IV. OTHER TYPES OF GRAPHS

Now we explore the propagation of soliton solutions of NLSE on other kind of graphs which include an incoming semi-infinite bond and at least one outgoing semi-infinite bonds. We shall see conservation rules to hold under the extended sum rule for strengths of nonlinearity around vertices.

An example of the graph for which the soliton solution of NLSE can be obtained analytically is a tree graph in Fig. 2.

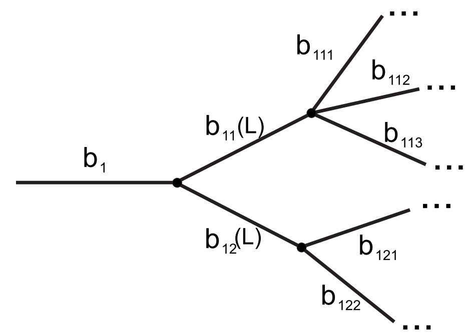

FIG. 2. Tree graph. $b_1\sim(-\infty,0)$, $b_{11},b_{12}\sim(0,L)$, and $b_{1ij}\sim(0,+\infty)$ with $i,j=1,2,\ldots$. Notations in parentheses, (L), indicate sizes of finite bonds. The similar notations are used in Figs. 3 and 4.

Hereafter, for an arbitrary one of bonds in the tree graph, we shall employ an abbreviation like $b_\Gamma\equiv b_{1ij\cdots}$. On each bond b_Γ we have NLSE given by Eq. (1) and for each vertex the following conditions is satisfied:

$$\frac{1}{\beta_\Gamma} = \sum_{k}\frac{1}{\beta_{\Gamma k}}, \tag{35}$$

which is again available from the norm and energy conservation rules. The soliton solution satisfying these conditions can be written as

$$\Psi_\Gamma(x_\Gamma,t) = \sqrt{\frac{2}{\beta_\Gamma}}iq(x_\Gamma+s_\Gamma,t;s_\Gamma), \quad x_\Gamma\in b_\Gamma. \tag{36}$$

Here parameter s_Γ is the length of the path that soliton passes from b_1 through b_Γ. For tree graphs this parameter is given as

$$s_1=s_{1i}=l, \quad s_{1ij}=l+L_{1i},$$

$$s_\Gamma\equiv s_{1ij\cdots lm}=l+L_{1i}+L_{1ij}+\cdots+L_{1ij\cdots l}, \tag{37}$$

where $-l$ stands for an initial location of the solution (arbitrary part) on the left-most semi-infinite bond, and $L_{1i},L_{1ij},\ldots,L_{1ij\cdots l}$ are finite lengths of the bonds prior to $b_\Gamma\equiv b_{1ij\cdots lm}$.

Below, applying the induction method, we give a proof of conservation rules for soliton solutions of NLSE on any tree graph. Let us denote the tree graph in Fig. 2 as G and assume the conservation rules to hold in G: $\sum_{b_\Gamma\in G}\beta_\Gamma^{-1}\int_{b_\Gamma}f_n[q(x_\Gamma+s_\Gamma,t)]dx_\Gamma=(2i)^n C_n\beta_1^{-1}$. Then we construct an enlarged tree graph in the following way: First, cut an arbitrary one of the right-most semi-infinite bond $b_\Lambda\sim(0,+\infty)$ at a point A located by distance L_Λ from the nearest vertex and then attach M semi-infinite bonds to the point A which now becomes a new vertex point. Namely, the bond b_Λ is now replaced by the finite bond $\hat{b}_\Lambda\sim(0,L_\Lambda)$ connected with M semi-infinite bonds $\hat{b}_{\Lambda m}\sim(0,+\infty)$ with $m=1,\ldots,M$. The enlarged tree graph thus obtained is denoted as G'. In the same way as in Eq. (26), the general conserved quantity for G' is given by

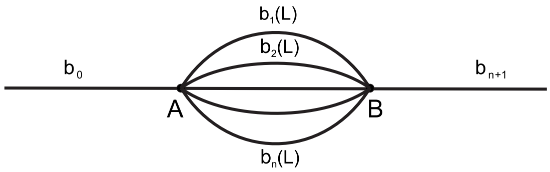

FIG. 3. Graph with loop. $b_0 \sim (-\infty,0), b_{n+1} \sim (0,+\infty), b_k \sim (0,L)$ with $k=1,2,\ldots,n$.

$$\sum_{b_\Gamma \in G-b_\Lambda} \beta_\Gamma^{-1} \int_{b_\Gamma} f_n(q(x_\Gamma + s_\Gamma,t))dx_\Gamma + \beta_\Lambda^{-1} \times \int_{\hat{b}_\Lambda} f_n(q(x_\Lambda + s_\Lambda,t))dx_\Lambda + \sum_{m=1}^{M} \beta_{\Lambda m}^{-1} \times \int_{\hat{b}_{\Lambda m}} f_n(q(x_{\Lambda m} + s_{\Lambda m} + L_{\Lambda m},t))dx_{\Lambda m}$$
$$= \sum_{b_\Gamma \in G} \beta_\Gamma^{-1} \int_{b_\Gamma} f_n(q(x_\Gamma + s_\Gamma,t))dx_\Gamma - \beta_\Lambda^{-1} \times \int_{L_\Lambda}^{+\infty} f_n(q(x + s_\Lambda,t))dx + \sum_{m=1}^{M} \beta_{\Lambda m}^{-1} \int_{L_\Lambda}^{+\infty} f_n(q(x + s_\Lambda,t))dx$$
$$= (2i)^n C_n \beta_1^{-1} - \left(\beta_\Lambda^{-1} - \sum_{m=1}^{M} \beta_{\Lambda m}^{-1}\right) \int_{L_\Lambda}^{+\infty} f_n(q(x + s_\Lambda,t))dx. \tag{38}$$

Here $\Sigma_{b_\Gamma \in G-b_\Lambda}$ and $\Sigma_{b_\Gamma \in G}$ imply summations over all bonds in G except for b_Λ and over all bonds of G, respectively. It is clear that the final expression becomes constant $(2i)^n C_n \beta_1^{-1}$ under the sum rule in Eq. (35). Thus, starting from PSG in Fig. 1 and repeating the above procedure, we can get the conservation rules for all possible tree graphs.

Another example for which soliton can be easily obtained is a graph with loops (see Fig. 3). These graphs consist of two semi-infinite bonds whose edges are connected with n bonds having finite lengths. Again, requiring the following conditions for the coefficients of NLSE:

$$\frac{1}{\beta_0} = \sum_{k=1}^{n} \frac{1}{\beta_k} = \frac{1}{\beta_{n+1}}$$

we can write the soliton solution by Eqs. (36).

Also, the exact soliton solution can be obtained for the graph in Fig. 4 where the corresponding condition for the parameters, β_k is required. This graph can be considered as a loop graph connected with three semi-infinite bonds.

In these different types of graphs, the soliton solution of NLSE can be constructed under the conditions given by Eqs. (35) and (36). It should be noted that throughout in our approach the graphs are supposed to have at least two semi-infinite bonds.

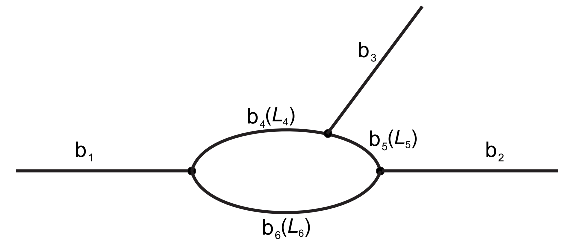

FIG. 4. Loop with semi-infinite bonds. $b_1 \sim (-\infty,0), b_2, b_3 \sim (0,+\infty), b_k \sim (0,L_k)$ with $k=4,5,6$. $L_6 = L_4 + L_5$.

V. INJECTION OF A SINGLE SOLITON AND TRANSMISSION PROBABILITIES AT $t \to +\infty$

Here we calculate transmission probabilities for a single soliton which is incoming through an semi-infinite bond b_0 and outgoing through the semi-infinite bonds $\beta_{\Gamma m}$.

A single (bright) soliton on a graph, which takes the general form in Eq. (21), is described with use of parts of Zakharov-Shabat's soliton with $\beta=2$ [27]: Ψ_γ lying on individual bonds b_γ is given by

$$\Psi_\gamma(x_\gamma,t) = \frac{a\sqrt{2}}{\sqrt{\beta_\gamma}} \frac{\exp\left[i\frac{v}{2}x_\gamma - i\left(\frac{v^2}{4} - a^2\right)t\right]}{\cosh[a(x_\gamma + l - vt)]}, \tag{39}$$

where $v,-l$ and a are bond-independent parameters characterizing velocity, initial center of mass and amplitude of a soliton, respectively. In the simplest graph (: PSG) in Fig. 5, the soliton at bond b_1 splits into two parts and appears in both of b_2 and b_3. This is a novel feature of the soliton propagation through a branched chain and networks in general.

As can be seen from Eq. (39), the center of mass of the soliton (CMS) on each bond b_γ is located at $x_\gamma = -l$ at $t=0$. However, coordinates on the individual semi-infinite bonds are defined on the limited interval. For example, on outgoing bonds b_2 and b_3, their coordinates x_2 and x_3 are defined in the interval $(0,+\infty)$. If $-l<0$, therefore, CMS on b_2 and b_3 is initially located outside of the real bonds. In such cases we call the soliton as a "ghost soliton." When CMS belongs to a bond we use a term "real soliton." In Fig. 5 ghost solitons are plotted by broken curve, while real ones are plotted by solid line. The soliton dynamics here is governed by a single characteristic time $\tau \equiv \frac{l}{v}$. While for $0 \le t \le \tau$ the soliton at b_1 is a real one and those at b_2 and b_3 are ghosts, for $\tau \le t$ the soliton at b_1 is a ghost and those at b_2 and b_3 are real. The incoming real soliton on b_1 and outgoing ghost solitons at b_2 and b_3 arrive at the vertex O. At $t=0$ with $l \gg 1$, the soliton

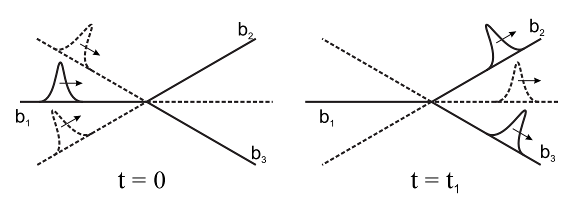

FIG. 5. Splitting of soliton. $t_1 > \tau$. Dashed lines represent ghost solitons.

lying on the bond b_1 is exclusively responsible for the norm N. On the other hand, at $t \gg 1$, the solitons running through the bonds b_2 and b_3 are exclusively responsible for the norm. Therefore we can naturally define transmission probabilities at $t \to +\infty$.

Let us consider the general graph with incoming semi-infinite bond $b_1(-\infty,0)$ and n outgoing semi-infinite bonds $b_{\Gamma m}(0,+\infty)$, $m=1,2,\ldots,n$. According to a combination of sum rules for nonlinearity coefficients we have

$$\frac{1}{\beta_1} = \sum_{l=1}^{n} \frac{1}{\beta_{\Gamma m}}. \tag{40}$$

From this rules it follows that the limit at $t \to +\infty$, transmission coefficients vanish on the part of graph of intermediate part (between incoming and outgoing bonds).

Transmission probability for arbitrary bond $b_{\Gamma m}$ are defined as

$$T_{\Gamma m} \equiv \frac{1}{N}\int_0^{+\infty} |\Psi_{\Gamma m}(x,t)|^2 dx. \tag{41}$$

In the case of a single soliton solution with $v>0$ we have

$$T_{\Gamma m} = \frac{1}{N}\frac{2a^2}{\beta_{\Gamma m}}\int_0^{+\infty} \frac{dx}{\cosh^2[a(x+l-s_{\Gamma m}-vt)]} = \frac{1}{N}\frac{2a^2}{\beta_{\Gamma m}}\int_{l-s_{\Gamma m}-vt}^{+\infty} \frac{dx}{\cosh^2(ax)}. \tag{42}$$

At the limit $t \to +\infty$, we have

$$T_{\Gamma m} \to \frac{1}{N}\frac{2a^2}{\beta_{\Gamma m}}\int_{-\infty}^{+\infty} \frac{dx}{\cosh^2(ax)} = \frac{\beta_1}{\beta_{\Gamma m}}. \tag{43}$$

We should recognize the reflection probability at the bond b_1 is vanishing,

$$R = \frac{1}{N}\frac{2a^2}{\beta_1}\int_{-\infty}^{0} \frac{dx}{\cosh^2[a(x+l-vt)]} = \frac{1}{N}\frac{2a^2}{\beta_1}\int_{-\infty}^{l-vt} \frac{dx}{\cosh^2(ax)} = 0. \tag{44}$$

The last equality is justified at the limit $t \to +\infty$.

According to the sum rule (40) one can see the unitarity to be satisfied.

$$\sum_m T_{\Gamma m} = 1. \tag{45}$$

The result in Eq. (43) provides new analytic expressions for the transmission probability for open networks with incoming and outgoing semi-infinite bonds.

We have checked the result in Eq. (43) using a numerical simulation of the discrete nonlinear Schrödinger equation (DNLSE) on the discrete version of PSG in Fig. 8. Figure 6 shows: the soliton starting at lattice point $x=50$ in the branch 1 enters the vertex at $x=200$ and is smoothly split into a pair of smaller solitons in the branches 2 and 3, with neither reflection nor emergence of radiation at the vertex. In Fig. 7 transmission probabilities T_2 and T_3 are plotted as a function of $\frac{\beta_1}{\beta_2}$ keeping the sum rule $\frac{\beta_1}{\beta_2}+\frac{\beta_1}{\beta_3}=1$. We can confirm the linear law predicted in Eq. (43).

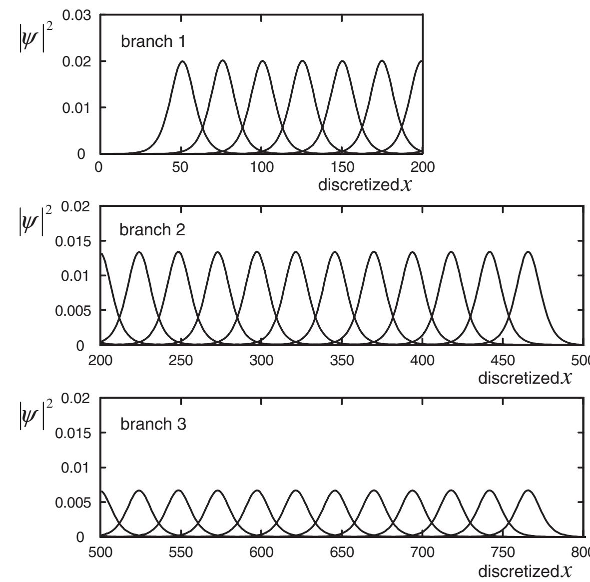

FIG. 6. Time evolution of a soliton propagation through a vertex (numerical result): an example. Space distribution of wave function probability is depicted in every time interval $T=50.0$ with time used commonly in branches 2 and 3. Abscissa represents discrete lattice coordinates defined in Fig. 8. Strength of nonlinearity at each bond are $\beta_1=1, \beta_2=1.5, \beta_3=3$. Initial profile is Zakharov-Shabat soliton in Eq. (39) at $t=0$ with parameters $a=0.1, v=0.1$. Time difference in numerical iteration is $\Delta t=0.1$. For the numerical method to solve NLSE on PSG, see the Appendix.

VI. SUMMARY AND DISCUSSIONS

We have derived (nonlinear) conditions under which the solution of the nonlinear Schrödinger equation on simple networks satisfies the norm and energy conservation rules,

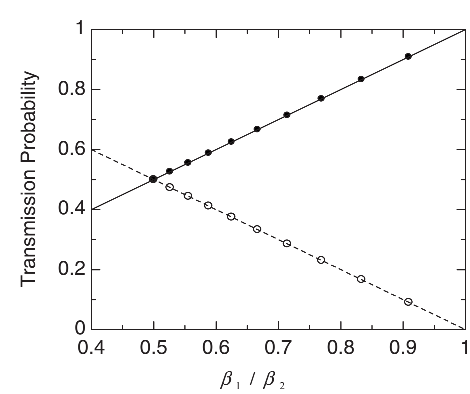

FIG. 7. Transmission probabilities (TPs) as a function of $\frac{\beta_1}{\beta_2}$ in PSG in Figs. 1 and 8. Symbols and lines denote numerical and theoretical results, respectively. A solid line with ● and a broken line with ○ correspond to T_2 and T_3, respectively. Numerical method is the same as used in Fig. 6 and the theoretical result is given by Eq. (43).

and obtained generic connection formulas at vertices. Then we have elucidated the special case for which exact soliton solutions of NLSE on networks have infinite number of conservation rules. Here the strength of cubic nonlinearity is different from bond to bond, and networks are assumed to have at least two semi-infinite bonds with one of them used as an incoming bond. We find (1) the solution on each bond is a part of the universal (bond-independent) soliton solution of the completely integrable NLSE on the 1D chain, but is multiplied by the inverse of square root of bond-dependent nonlinearity; (2) the inverse nonlinearity at an incoming bond should be equal to the sum of inverse nonlinearities at the remaining outgoing bonds. If this sum rule will be broken, no interesting bifurcation of a soliton propagation occurs at vertices. Using the above two findings, we also showed an infinite number of constants of motion. The argument on a branched chain or a primary star graph (PSG) is generalized to other graphs, i.e., general star graphs, tree graphs, loop graphs and their combinations. To see all conservation rules to hold there, a set of strengths of nonlinearity should satisfy the generalized sum rule around each vertex, which we proved by the induction method. As a relevant issue, with use of reflectionless propagation of Zakharov-Shabat's soliton through networks we have obtained the transmission probabilities on the outgoing bonds, which are inversely proportional to the bond-dependent strength of nonlinearity. Extension of the work to networks with plural number of incoming bonds and to closed networks would be important, which will be done in due course.

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APPENDIX: NUMERICAL METHOD TO SOLVE NONLINEAR SCHRÖDINGER EQUATION ON PRIMARY STAR GRAPH

With use of space discretization ($\partial^2\Psi/\partial x^2 \Rightarrow \Psi_{i+1}-2\Psi_i+\Psi_{i-1}$), the nonlinear Schrödinger equation in the 1D continuum with neither branches nor vertex

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} - \beta|\Psi|^2\Psi$$

can be reduced to

$$i\frac{\partial\Psi_i}{\partial t} = -\Psi_{i-1} + (2-\beta|\Psi_i|^2)\Psi_i - \Psi_{i+1}, \tag{A1}$$

which is rewritten in a matrix form as

$$i\frac{\partial}{\partial t}\begin{pmatrix}\vdots\\ \Psi_{i-1}\\ \Psi_i\\ \Psi_{i+1}\\ \vdots\end{pmatrix} = H(t)\begin{pmatrix}\vdots\\ \Psi_{i-1}\\ \Psi_i\\ \Psi_{i+1}\\ \vdots\end{pmatrix} \tag{A2}$$

with

$$H(t) \equiv \begin{pmatrix} & & (i-1) & (i) & (i+1) & & \\ & & \vdots & \vdots & \vdots & & \\ \cdots & -1 & 2-\beta|\Psi_{i-1}|^2 & -1 & 0 & 0 & \cdots \\ \cdots & 0 & -1 & 2-\beta|\Psi_i|^2 & -1 & 0 & \cdots \\ \cdots & 0 & 0 & -1 & 2-\beta|\Psi_{i+1}|^2 & -1 & \cdots \\ & & \vdots & \vdots & \vdots & & \end{pmatrix}. \tag{A3}$$

Then, by carrying out the time discretization with time difference Δt, Eq. (A2) reduces to

$$\begin{pmatrix}\vdots\\ \Psi_{i-1}(t+\Delta t)\\ \Psi_i(t+\Delta t)\\ \Psi_{i+1}(t+\Delta t)\\ \vdots\end{pmatrix} = \exp[-iH(t)\Delta t]\begin{pmatrix}\vdots\\ \Psi_{i-1}(t)\\ \Psi_i(t)\\ \Psi_{i+1}(t)\\ \vdots\end{pmatrix}. \tag{A4}$$

It is obvious that Eq. (A4) conserves the norm because of the unitarity of $\exp[-iH(t)\Delta t]$. It is straightforward that the diagonalization

$$P^{-1}H(t)P = \begin{pmatrix}\epsilon_1 & 0 & \cdots & \cdots\\ 0 & \epsilon_2 & 0 & \cdots\\ \vdots & \vdots & & \end{pmatrix}$$

gives rise to

$$\exp[-iH(t)\Delta t] = P\begin{pmatrix}\exp(-i\epsilon_1\Delta t) & 0 & \cdots & \cdots\\ 0 & \exp(-i\epsilon_2\Delta t) & 0 & \cdots\\ \vdots & \vdots & & \end{pmatrix}P^{-1}.$$

In case of a branched chain, i.e., a primary star graph (PSG),

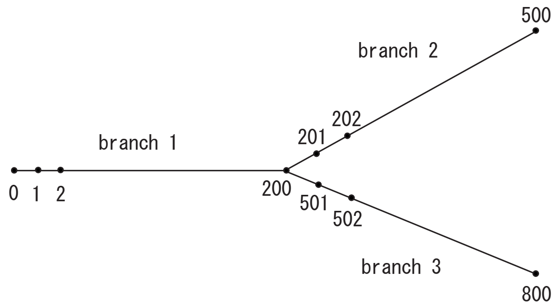

FIG. 8. Space-discrete version of primary star graph used for numerical simulations.

we consider its discretized counterpart and introduce the numbering as in Fig. 8. Equations to generalize Eq. (A1) are given by

$$i\frac{\partial\Psi_i^{(1)}}{\partial t} = -\Psi_{i-1}^{(1)} + (2-\beta_1|\Psi_i^{(1)}|^2)\Psi_i^{(1)} - \Psi_{i+1}^{(1)}, \quad \text{(A5)}$$

$$i\frac{\partial\Psi_j^{(2)}}{\partial t} = -\Psi_{j-1}^{(2)} + (2-\beta_2|\Psi_j^{(2)}|^2)\Psi_j^{(2)} - \Psi_{j+1}^{(2)}, \quad \text{(A6)}$$

$$i\frac{\partial\Psi_k^{(3)}}{\partial t} = -\Psi_{k-1}^{(3)} + (2-\beta_3|\Psi_k^{(3)}|^2)\Psi_k^{(3)} - \Psi_{k+1}^{(3)}, \quad \text{(A7)}$$

which correspond to bonds b_1, b_2 and b_3, respectively.

The important problem is to search for the connection formula at the vertex, which will be resolved as follows. Let call the end of b_1 as K site. Similarly the starts of b_2 and b_3 are taken as L and M sites, respectively. Introducing virtual wave functions $\Psi_{L-1}^{(2)}$ and $\Psi_{M-1}^{(3)}$ and establish for their relationship with $\Psi_K^{(1)}$. As a manifold of the global solution on PSG, we assume a discretized version of Eq. (21):

$$\Psi_j^{(k)}(t) = \frac{\sqrt{2}}{\sqrt{\beta_k}} iq(x_j,t), \quad \text{(A8)}$$

where k (=1,2,3) denotes individual bonds and the discrete lattice variable i runs over PSG in Fig. 8. Because of the continuity of $g(x_j,t)$ at the vertex, we obtain a connection formula:

$$\sqrt{\beta_1}\Psi_K^{(1)} = \sqrt{\beta_2}\Psi_{L-1}^{(2)} = \sqrt{\beta_3}\Psi_{M-1}^{(3)}. \quad \text{(A9)}$$

On the other hand, with use of suitable parameters s_2 and s_3, a virtual wave function $\Psi_{K+1}^{(1)}$ should be

$$\Psi_{K+1}^{(1)} = \frac{1}{s_2+s_3}\left(s_2\sqrt{\frac{\beta_2}{\beta_1}}\Psi_L^{(2)} + s_3\sqrt{\frac{\beta_3}{\beta_1}}\Psi_M^{(3)}\right). \quad \text{(A10)}$$

Then Eqs. (A5)–(A7) at the vertex can be explicitly rewritten as

$$i\frac{\partial\Psi_K^{(1)}}{\partial t} = -\Psi_{K-1}^{(1)} + (2-\beta_1|\Psi_K^{(1)}|^2)\Psi_K^{(1)} - \frac{1}{s_2+s_3}\left(s_2\sqrt{\frac{\beta_2}{\beta_1}}\Psi_L^{(2)} + s_3\sqrt{\frac{\beta_3}{\beta_1}}\Psi_M^{(3)}\right), \quad \text{(A11)}$$

$$i\frac{\partial\Psi_M^{(2)}}{\partial t} = -\sqrt{\frac{\beta_1}{\beta_2}}\Psi_K^{(1)} + (2-\beta_2|\Psi_M^{(2)}|^2)\Psi_M^{(2)} - \Psi_{M+1}^{(2)}, \quad \text{(A12)}$$

$$i\frac{\partial\Psi_L^{(3)}}{\partial t} = -\sqrt{\frac{\beta_1}{\beta_3}}\Psi_K^{(1)} + (2-\beta_3|\Psi_L^{(3)}|^2)\Psi_L^{(3)} - \Psi_{L+1}^{(3)}. \quad \text{(A13)}$$

Lining up $\Psi_i^{(1)}$, $\Psi_j^{(2)}$ and $\Psi_k^{(3)}$ vertically and rewriting Eqs. (A5)–(A7) with Eqs. (A11)–(A13) in a matrix form, we obtain the equation like Eq. (A2) with Eq. (A3), but with a modified real matrix $\hat{H}(t)$. To conserve the norm, $\hat{H}(t)$ should be symmetric, which imposes the following relationship:

$$\frac{s_2}{s_2+s_3}\sqrt{\frac{\beta_2}{\beta_1}} = \sqrt{\frac{\beta_1}{\beta_2}}, \quad \frac{s_3}{s_2+s_3}\sqrt{\frac{\beta_3}{\beta_1}} = \sqrt{\frac{\beta_1}{\beta_3}}. \quad \text{(A14)}$$

As a result, we have the sum rule for three kind of strength of nonlinearity:

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3}, \quad \text{(A15)}$$

which agrees with Eq. (24) obtained from the norm and energy conservation rules for the PSG in the text. Using Eq. (A14), Eq. (A11) can be replaced by

$$i\frac{\partial\Psi_K^{(1)}}{\partial t} = -\Psi_{K-1}^{(1)} + (2-\beta_1|\Psi_K^{(1)}|^2)\Psi_K^{(1)} - \left(\sqrt{\frac{\beta_1}{\beta_2}}\Psi_L^{(2)} + \sqrt{\frac{\beta_1}{\beta_3}}\Psi_M^{(3)}\right). \quad \text{(A16)}$$

By numerically solving Eqs. (A5)–(A7) with Eqs. (A16), (A12), and (A13) under any initial condition, one obtains nonlinear dynamics of solitons without reflection at the vertex. Figure 6 is obtained under the initial profile in Eq. (39) with Eqs. (A15) and (24). Figure 7 is calculated with use of Eq. (41) for a soliton on each outgoing bond at large enough time.

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