Breathers in a nonautonomous Toda lattice with pulsating coupling

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We study a nonautonomous Toda lattice, with a periodically switched on-off coupling coefficient, describing a pulsating strength of neighbor particle interaction. It is shown that when the uncoupled oscillations are linear and under appropriate conditions for the duration of the time intervals where the coupling is switched off, breather solutions can be obtained analytically. Their dynamics and collisions are related to the soliton dynamics of the corresponding autonomous Toda lattice, while a "ratchet" effect is shown to result in breather deceleration, providing a mechanism for breather velocity and collision control.

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I. INTRODUCTION

Systems of coupled nonlinear oscillators forming finite or infinite lattices are widely utilized to model a large variety of phenomena in different areas of pure and applied science. The respective studies have been initiated as early as in 1953 by numerical experiments on the well-known FPU (Fermi-Pasta-Ulam) model [1]. A few years latter the continuum limit of the FPU model, namely the Korteweg–de Vries (KdV) equation, was shown to have self-localized soliton solutions that remain invariant under propagation and interact elastically [2]. Although integrability, required for soliton formation, is a very strict property which is very rarely met in physical models, there are many physical systems that can still have self-localized waves exhibiting robust propagation which are called solitary waves. The stability properties and the collisional characteristics of the latter are much more complex than those of their integrable counterparts.

Solitons and integrable systems of infinite degrees of freedom have been the subject of intense theoretical research and a set of integrable lattice equations have been studied and solved with utilization of the inverse scattering transform technique [3]. Such equations include for example the Toda lattice [4], the Ablowitz-Ladik equation [5], and the Calogero-Moser *N*-body problem [6]. From a more practical point of view, applications of soliton theory have been emerged in many different areas of physics in addition to mechanical systems [7,8], including solid state physics of polymers [9], biophysics [10], and Josephson junction arrays [11]. More recently, discrete solitons have been the subject of increasing interest in the field of nonlinear optics, where they can be experimentally observed in periodic photonic structures and waveguide arrays and provide potentiality for technological applications [12–16]. Furthermore, similar discretelike soliton formation and dynamics are encountered in Bose-Einstein condensates trapped in optical periodic potentials [17,18].

There are several cases of physical interest where the respective lattice models explicitly depend on time, i.e., the underlying dynamical system is nonautonomous. Such a dependence may result from the application of a driving force or an external field and appears either as an additional time-dependent term or as a time-dependent parameter in the lattice equations of motion [19–27]. The explicit time dependence of the lattice equations result in qualitatively different dynamics with respect to the time-independent cases, since it adds an additional (external) degree of freedom. When the time dependence can be considered as a small perturbation to a time-independent integrable Hamiltonian system the well-known Kolmogorov-Arnold-Moser (KAM) theorem [28] allows for the qualitative study of the perturbed system. According to KAM theorem, for small enough perturbations the invariants of the motion persist although slightly modified, while resonances between the degrees of internal degrees of freedom and the external time dependence modify drastically the local topology of the phase space. On the other hand, there exist no systematic methods for studying cases where the time dependence is strong enough to be considered as a perturbation. However, the strong explicit time dependence of certain characteristics of the lattice is related to interesting effects corresponding to parametrical driving and respective control capabilities of lattice dynamics.

II. NONAUTONOMOUS LATTICES WITH PULSATING COUPLING

A. Model

In this work we study a specific form of strong parametric driving of a nonlinear lattice, where the coupling between nearest neighbors depends explicitly on time. More specifically we study the case of a lattice with a periodically on-off switched coupling, describing a pulsating strength of neighbor particle interaction. It is shown that, when the uncoupled oscillations are linear and under appropriate conditions for the duration of the time intervals where the coupling is switched off, the existence of solitons and their dynamics under collision in the respective autonomous system with nonzero coupling, results in the existence of breathers with similar dynamics in the parametrically driven nonautonomous system. In general, the results apply in cases where the respective autonomous nonlinear lattice (when the coupling is switched on) is either integrable or nonintegrable.

The Hamiltonian describing particle dynamics in a lattice with an additional external on-site potential has the form

$$H = \sum_{n=1}^{N} \frac{1}{2} m \dot{y}_n^2 + \sum_{n=1}^{N} \Phi(y_n - y_{n-1}, t) + \sum_{n=1}^{N} \Phi_0(y_n, t), \quad (1)$$

where y_n denotes the displacement of the nth particle from the equilibrium position, Φ is the potential energy of the nearest-neighbor interaction and Φ_0 is the on-site potential. Note that without the explicit time dependence of the potentials Φ and Φ_0 the lattice is autonomous and the Hamiltonian is a constant of the motion. The equations of the motion are

$$\ddot{y}_n + \Phi'(y_n - y_{n-1}, t) - \Phi'(y_{n+1} - y_n, t) + \Phi_0'(y_n, t) = 0, \quad (2)$$

where the dot denotes time differentiation and the prime denotes differentiation with respect to y_n. The explicit time dependence of the interaction and the on-site potential can be due to various physical mechanisms parametrically driving the lattice. For the on-site potential the time dependence can be caused by any external field acting either uniformly $[\Phi_0''(y_n,t)=0]$ or nonuniformly $[\Phi_0''(y_n,t)\neq 0]$ on all sites. The latter can have the form of a wave standing or propagating along the lattice. Lattice dynamics are drastically affected by such a parametric driving and soliton motion depends crucially on the amplitude and the frequency of the drive [23,24]. On the other hand, the explicit time dependence of the interaction potential results in time-dependent (linear and/or nonlinear) diffraction properties of the lattice. Various physical mechanisms can be the source of such an explicit time dependence. For example, in a chain of motile elements the interactions may depend on time due to internal variables with autonomous dynamics which can modify the coupling strength or alternate the character of the interaction between attractive or repulsive. Such a interaction potential could have the form $\Phi(y_n - y_{n-1}, t) = V_n(\theta_{n,1}, \ldots, \theta_{n,M})\Psi(y_n - y_{n-1})$, where $\theta_{n,i}(i=1\ldots M)$ are internal variables representing the state of each motile element [25]. Another case of time-dependent interaction potential is a system consisting of moving particles connected by bonds which are sensitive to temperature variations or light incidence. In such lattices, the free equilibrium rest lengths are time dependent according to the varying external field, inducing an explicit time dependence on the interaction potential $\Phi(y_n - y_{n-1}, t) = \Psi(y_n - y_{n-1} + a_n(t))$. For example, this is the case of a Frenkel-Kontorova chain consisting of nanosize clusters (particles) and photochromic molecules (bonds) where a modulated incident light determines the time dependence of the interaction [26]. Such systems have interesting applications in concepts related to microscopic engines on the atomic scale where directed motion of the chain can perform useful functions [27]. In general, it can be shown, by appropriate change of variables, that an external driving term gives rise to time-dependent interaction potentials. Considering the system

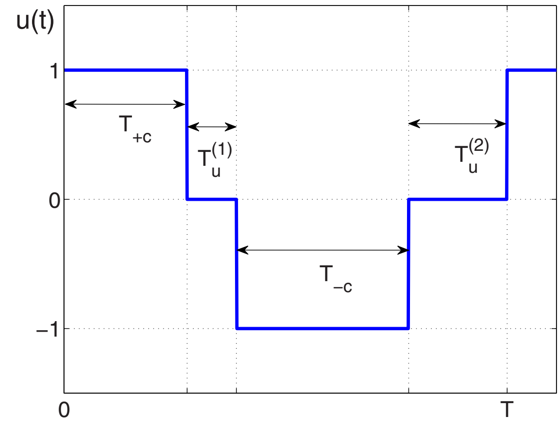

FIG. 1. (Color online) General form of the function $u(t)$.

$$\ddot{y}_n + \Phi'(y_n - y_{n-1}) - \Phi'(y_{n+1} - y_n) + \Phi_0'(y_n) = f_n(t) \quad (3)$$

and transforming to a new variable set $x_n = y_n + g_n(t)$ with $\ddot{g}_n(t) = -f_n(t)$ we have

$$\ddot{x}_n + \Phi'(x_n - x_{n-1} + \Delta g_n(t)) - \Phi'(x_{n+1} - x_n + \Delta g_{n+1}(t)) + \Phi_0'(x_n + g_n(t)) = 0, \quad (4)$$

where $\Delta g_n(t) = g_n(t) - g_{n-1}(t)$ and the interaction potential depends explicitly on time.

Since the explicit time dependence of the interaction potential is in general due to external fields with parameters that can be selected at will, it is interesting to study the relation between the dynamics of an autonomous lattice and a corresponding nonautonomous lattice and investigate the potentiality of controlling soliton formation and propagation properties by appropriate external fields. In the following we consider a case where the interaction potential in Eq. (1) has the following form:

$$\Phi(y_n - y_{n-1}, t) = u(t)\Psi(y_n - y_{n-1}) \quad (5)$$

with u representing the effect of the time-varying external fields to the interaction potential. This form of time dependence corresponds to flashing ratchet potentials [21]. More specifically, we focus on the case where the coupling (interaction potential) is periodically on-off switched, so that $u(t)$ is a periodic piecewise constant function taking the values 0 and ± 1, with the sign change corresponding to alternation between an attractive and repulsive character of the interaction. Therefore, $u(t)$ is defined as

$$u(t) = \begin{cases} +1, & kT < t \leq kT + T_{+c} \\ 0, & kT + T_{+c} < t \leq kT + T_{+c} + T_u^{(1)} \\ -1, & kT + T_{+c} + T_u^{(1)} < t \leq kT + T_{+c} + T_u^{(1)} + T_{-c} \\ 0, & kT + T_{+c} + T_u^{(1)} + T_{-c} < t \leq kT + T_{+c} + T_u^{(1)} + T_{-c} + T_u^{(2)} \end{cases} \quad (6)$$

where $k=0,\pm 1,\pm 2,\ldots$ The durations of the time intervals where the coupling is on and off are $T_{\pm c}$, $T_u^{(1,2)}$, respectively, and $T=T_{+c}+T_{-c}+T_u^{(1)}+T_u^{(2)}$ is the period of $u(t)$ (Fig. 1). Under such a time dependence, the evolution of the system is determined in two alternating phases: (a) in the coupled phase where the coupling with the nearest neighbors is switched on, the system evolves as in the case of the respective autonomous lattice; (b) in the uncoupled phase where the coupling is switched off each particle (lattice site) oscillates independently in the external on-site potential. In the following, we show an example of the critical dependence of lattice soliton formation and dynamics on the durations of time intervals $T_{\pm c}$, $T_u^{(1,2)}$ and investigate their capabilities as control parameters.

B. Analytical breather solutions

In the following we focus on the case where during the uncoupled phases, the external potential corresponds to a linear restoring force, i.e.,

$$\Phi_0(y_n,t)=v(t)\frac{\omega_0^2}{2}y_n^2+u(t)\Psi_0^{(nl)}(y_n), \tag{7}$$

where $\Psi_0^{(nl)}(y_n)$ is a nonlinear on-site potential acting only at the coupled phase and $v(t)$ is any function which is piecewise constant at the coupled and uncoupled phases. Note that a constant linear restoring force, corresponding to $v(t)=\text{const}$ is included as a special case.

In such cases, during the uncoupled phases, every particle oscillates linearly, with the same frequency ω_0 independently of the initial conditions (isochronicity property). For the case where the durations of the uncoupled linear phases $T_u^{(1,2)}$ are integer multiples of the period of the oscillations, that is when

$$T_u^{(1,2)}=m^{(1,2)}2\pi/\omega_0,\quad m=1,2,\ldots \tag{8}$$

the system, after evolving in the linear phase, returns at exactly the same state that it was at the end of the previous coupled phase. In an alternative description, in a $2N$-dimensional Poincare surface of section of the $2N+1$-dimensional extended phase space of the nonautonomous system, each point maps to itself after evolving in the linear uncoupled phase. Therefore, the existence of solutions of the autonomous coupled lattice directly implies the existence of related solutions of the nonautonomous lattice with pulsating coupling [29].

These arguments hold for any kind of lattice including integrable and non integrable ones as well as any kind of solutions including periodic or localized solitary solutions. In the following, we focus on the Toda lattice which is known to be integrable and more specifically on its soliton solutions and show that these are directly related to breather solutions of the respective nonautonomous lattice. For the Toda lattice the interaction potential is exponential,

$$\Psi(y_n-y_{n-1})=e^{-(y_n-y_{n-1})} \tag{9}$$

and the function $v(t)$ is taken as

$$v(t)=1-u^2(t) \tag{10}$$

to ensure that the on-site potential is zero in the coupled phase as for a Toda lattice. Applying the canonical transformation from $(y_n,\dot{y}_n)$ to (r_n,s_n) where

$$r_n=y_n-y_{n-1}, \tag{11}$$

$$m\dot{y}_n=s_n-s_{n+1} \tag{12}$$

and considering $m=1$, Toda lattice solitons are given in the form

$$r_n^{sol}(t)=-\ln(\beta^2\,\text{sech}^2(\alpha n\mp\beta t)+1), \tag{13}$$

where $\beta=\sinh(\alpha)$. These solitons move with a velocity $v_{sol}=\beta/\alpha$ which increases with the height of the pulse.

Following the previous arguments, under the condition (8), the respective solutions of the nonautonomous lattice can be written as

$$r_n(t)=\begin{cases} r_n^{sol}(t-k(T_{+c}-T_{-c})), & kT<t\le kT+T_{+c}\\ A_n^{(1,k)}\sin(\omega_0 t+a_n^{(1,k)}), & kT+T_{+c}<t\le kT+T_{+c}+T_u^{(1)}\\ r_n^{sol}(-t+(1-k)(T_{+c}+kT_{-c})), & kT+T_{+c}+T_u^{(1)}<t\le kT+T_{+c}+T_u^{(1)}+T_{-c}\\ A_n^{(2,k)}\sin(\omega_0 t+a_n^{(2,k)}), & kT+T_{+c}+T_u^{(1)}+T_{-c}<t\le kT+T_{+c}+T_u^{(1)}+T_{-c}+T_u^{(2)} \end{cases} \tag{14}$$

with $(A_n^{(i,k)},a_n^{(i,k)})$, $i=1,2$ determined from matching conditions at the boundaries of the respective time intervals $t^{(i,k)}$ according to

$$A_n^{(i,k)}=\left([r_n^{sol}(t^{(i,k)})]^2+\left[\frac{\dot{r}_n^{sol}(t^{(i,k)})}{\omega_0}\right]^2\right)^{1/2}, \tag{15}$$

$$a_n^{(i,k)}=\tan^{-1}\left(\frac{\omega_0 r_n^{sol}(t^{(i,k)})}{\dot{r}_n^{sol}(t^{(i,sk)})}\right). \tag{16}$$

These solutions describe localized waves propagating as Toda solitons in the time intervals where the coupling is switched on, and oscillating periodically in the time intervals where the coupling is switched off. Note that in the time interval when $u(t)=-1$ the soliton travels in the opposite

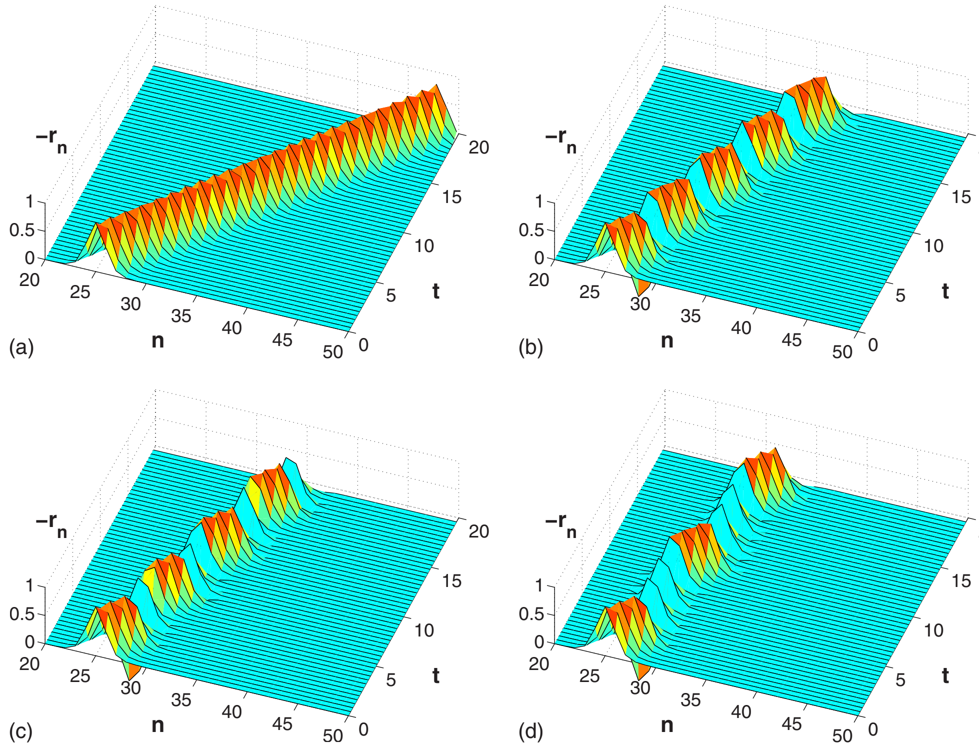

FIG. 2. (Color online) Evolution of a breather having an initial form of a Toda soliton with $\alpha=1$ with $T_{-c}=T_u^{(2)}=0$, $T_{+c}=2$, and $T_u^{(1)}=m2\pi/\omega_0$ with $m=0$ (a), $m=2$ (b), $m=3$ (c), $m=6$ (d).

direction in comparison to the time interval when $u(t)=+1$ while the velocity of the wave is zero during the uncoupled phases. Therefore, the mean velocity of the wave is

$$\langle v\rangle = \left(\frac{T_{+c}-T_{+c}}{T_{+c}+T_{-c}+T_u^{(1)}+T_u^{(2)}}\right)\frac{\beta}{\alpha} \tag{17}$$

with $T_u^{(1,2)}$ taking one of the discrete values shown in Eq. (8). Therefore, the periodically pulsating coupling results in a "ratchet" effect on soliton (breather) dynamics, since depending on the parameters of the function $u(t)$ it can decelerate the wave accordingly. Note that the ratio of the mean velocity in the nonautonomous lattice over the soliton velocity in the respective autonomous lattice is the same for all solitons having different heights and velocities and is determined by the form of $u(t)$.

First, we consider the case where $u(t)$ takes only the values 0,1 in the respective time intervals, that is $T_{-c}=T_u^{(2)}=0$. In Fig. 2, the evolution of a breather having an initial form of a Toda soliton given by Eq. (13) with $\alpha=1$, is shown for the case where $T_{+c}=2$. Figure 2(a) depicts the evolution of an autonomous Toda soliton (this is equivalent to considering $m=0$ in Eq. (8), and (b), (c), (d) a Toda lattice with periodically pulsating coupling with the duration of the uncoupled phase $T_u^{(1)}$ given by Eq. (8) for $m=2,3,6$, respectively. The frequency of the uncoupled oscillations ω_0 is taken equal to 2π, without loss of generality. In the coupled phase the solution of the nonautonomous system coincides with the soliton solution of the autonomous Toda lattice, while in the uncoupled phase the solution oscillates sinusoidally, with no transverse velocity. The intermittent transverse motion of the wave results in the reduction of the mean wave velocity given by Eq. (17). When the condition (8) is not met, the solitary wave deforms, splits and disperses under propagation as shown in Fig. 3 for the case corresponding to a Toda soliton with $\alpha=1$ and a $u(t)$ having $T_{-c}=T_u^{(2)}=0$, $T_{+c}=2$, $T_u^{(1)}=2\pi/\omega_0+0.01$. In Fig. 4, collisions between two breathers with velocities of opposite sign are shown for the case of two identical breathers corresponding to $\alpha=1$ and $m=1$ as well as two different breathers with $\alpha_1=1$, $\alpha_2=0.5$, and $m=1$. The breathers undergo completely elastic collisions and remain intact.

In Fig. 5 we show the evolution of a Toda soliton with $\alpha=1$ for the case of a $u(t)$ taking the values $0,\pm1$ in the respective time intervals. The durations of the uncoupled

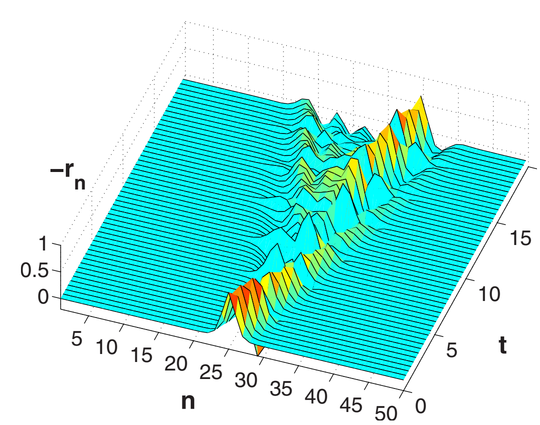

FIG. 3. (Color online) Evolution of an initial form as in Fig. 2(b), but with $T_u^{(1)}=2\pi/\omega_0+0.01$.

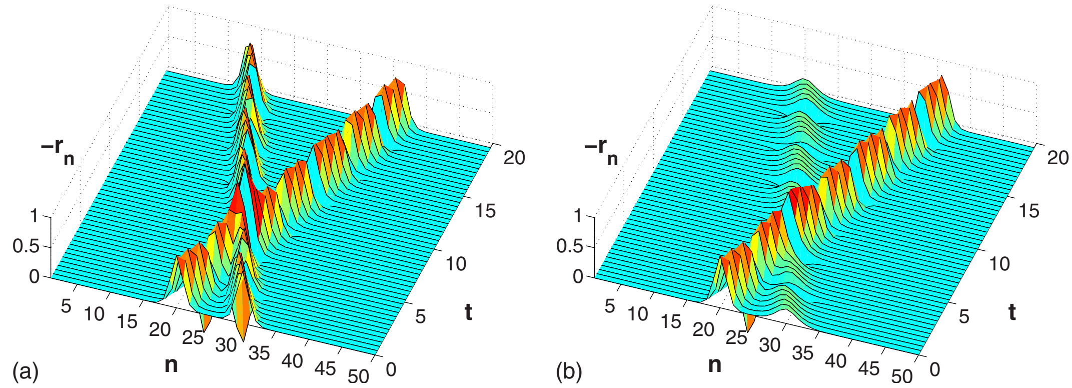

FIG. 4. (Color online) Collisions between two breathers with velocities of opposite sign corresponding to Toda solitons having $\alpha_1 = \alpha_2 = 1$ (a), $\alpha_1 = 1$, $\alpha_2 = 0.5$ (b). The parameters of the function $u(t)$ are $T_{-c} = T_u^{(2)} = 0$, $T_{+c} = 2$ and $T_u^{(1)} = 2\pi/\omega_0$.

phases are $T_u^{(1)} = T_u^{(2)} = 2\pi/\omega_0$. Depending on the values of T_{+c} and T_{-c} the breather can have a positive [Fig. 5(a)] or negative [Fig. 5(b)] mean velocity according to Eq. (17). When $T_{+c} = T_{-c}$ the mean velocity is zero and the breather undergoes a periodic "swinging" motion, the amplitude of which depends on the value of $T_{\pm c}$ [Fig. 5(c)]. Therefore, although the original Toda soliton of the respective autonomous lattices has always a definite nonzero velocity, depending on the form of the function $u(t)$ we can control the mean breather velocity of the breather of the nonautonomous system. Moreover, we can control breather collisions as shown in Fig. 6, where depending on T_{+c} and T_{-c} we can either prevent collisions [Fig. 6(a)] or make two breathers collide periodically [Fig. 6(b)].

III. CONCLUSIONS

In conclusion, we have obtained analytical breather solutions of a nonautonomous Toda lattice with periodically pulsating coupling under certain conditions for the duration of the uncoupled phases of the system. These breathers are directly related to solitons of the respective autonomous Toda lattice and have the property of undergoing purely elastic collisions. Moreover, it is shown that depending on the duration of the uncoupled phase these breathers undergo a "ratchet" effect. The latter provides capabilities of breather

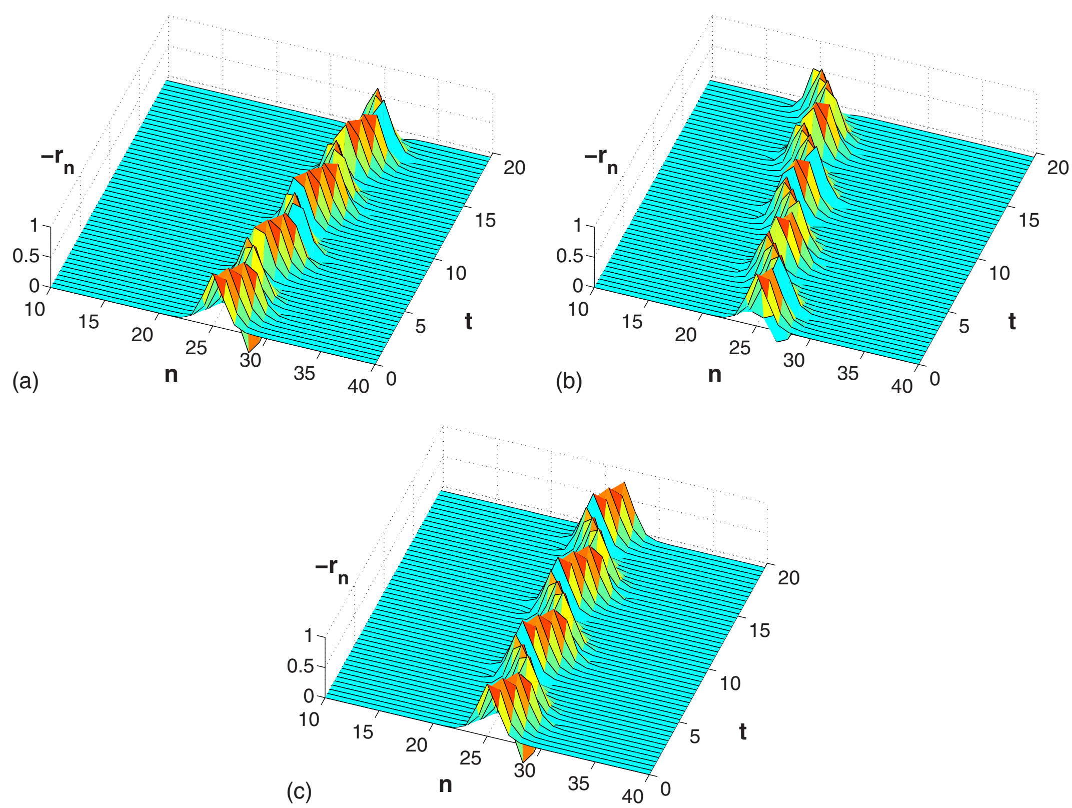

FIG. 5. (Color online) Evolution of a breather having an initial form of a Toda soliton with $\alpha = 1$ with $T_u^{(1)} = T_u^{(2)} = 2\pi/\omega_0$ and $T_{+c} = 2$, $T_{-c} = 1$ (a), $T_{+c} = 1$, $T_{-c} = 2$ (b), $T_{+c} = 2$, $T_{-c} = 2$ (c).

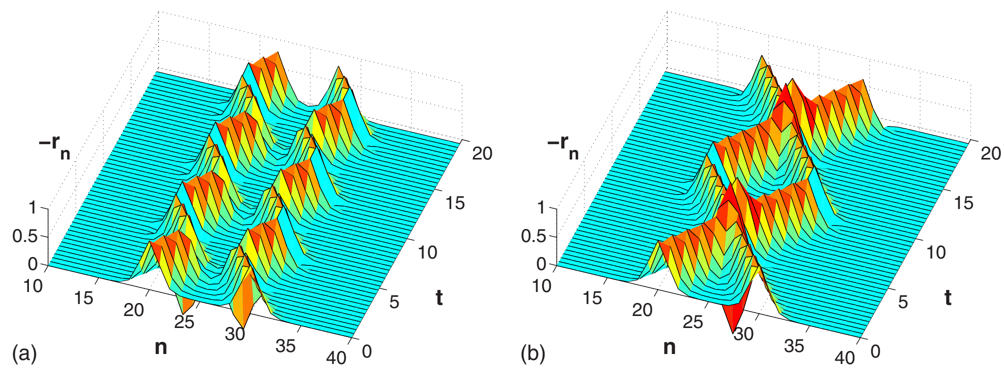

FIG. 6. (Color online) Evolution of two breathers corresponding to Toda solitons with $\alpha=1$ with $T_u^{(1)}=T_u^{(2)}=2\pi/\omega_0$ and $T_{+c}=T_{-c}=2$ (a), $T_{+c}=T_{-c}=4$ (b).

velocity and collision control. The results can be readily applied to a larger class of lattices with pulsating coupling, where the dynamics of the coupled lattice system are described by models being different from Toda's, which can be either integrable or nonintegrable. Finally, it is worth mentioning that the same arguments and results of this work apply not only for solitons and solitary waves but also for the cases of periodic solutions.

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