Chimeras in networks of planar oscillators

Carlo R. Laing*
IIMS, Massey University, Private Bag 102-904 NSMC, Auckland, New Zealand
(Received 16 March 2010; revised manuscript received 7 June 2010; published 29 June 2010)

Chimera states occur in networks of coupled oscillators, and are characterized by having some fraction of the oscillators perfectly synchronized, while the remainder are desynchronized. Most chimera states have been observed in networks of phase oscillators with coupling via a sinusoidal function of phase differences, and it is only for such networks that any analysis has been performed. Here we present the first analysis of chimera states in a network of planar oscillators, each of which is described by both an amplitude and a phase. We find that as the attractivity of the underlying periodic orbit is reduced chimeras are destroyed in saddle-node bifurcations, and supercritical Hopf and homoclinic bifurcations of chimeras also occur.

DOI: 10.1103/PhysRevE.81.066221 PACS number(s): 05.45.Xt

Networks of coupled oscillators and their synchronization properties have been studied for many years [1,2]. One particular class of interest involves phase oscillators, where each oscillator is described by a single angular variable [3,4]. The use of such phase models is justified when the attraction to an underlying limit cycle is "strong" relative to the effects of other oscillators in the network [1,3,5]. Recently a number of investigators have studied "chimera" states in networks of phase oscillators [6–18], in which some fraction of the oscillators synchronize while the remainder run freely, even though the oscillators may be identical. Early analyses of these states [6,7,10,12–17] used a self-consistency argument which can be traced back to Kuramoto [5] to show existence of chimeras. Later work [8,9,11,19] used the remarkable ansatz of Ott and Antonsen [20,21] to derive differential equations governing the evolution of order parameters of the systems under study, allowing one to determine the stability of chimera states and the bifurcations they may undergo.

It has long been known that networks of identical phase oscillators, coupled through a sinusoidal function of phase differences, have nongeneric behavior [18,22–24]. Most chimera states have been observed in such idealized networks, and in order to determine whether chimeras might be observed in real physical systems one should investigate their robustness with respect to, for example, heterogeneity in intrinsic frequencies, or variations in oscillator amplitude. The first issue has already been addressed [8,9], and here we investigate the second.

Several authors have observed chimeras in networks of oscillators described by more than one variable [6,7,15,16,25,26], so they are known to exist, but these authors have either provided no analysis, or have reduced their (identical) oscillators to phase oscillators in order to analyze their dynamics using the approaches mentioned above. In this paper we give the first analysis of a chimera state in a network of planar oscillators in which the reduction to phase oscillators is not performed.

The model we consider is

$$\frac{dX_j}{dt} = i\omega X_j + \epsilon^{-1}\{1-(1+\delta\epsilon i)|X_j|^2\}X_j + e^{-i\alpha}\left[\frac{\mu}{N}\sum_{k=1}^{N} X_k + \frac{\nu}{N}\sum_{k=1}^{N} X_{N+k}\right] \tag{1}$$

for $j=1,\ldots N$ and

$$\frac{dX_j}{dt} = i\omega X_j + \epsilon^{-1}\{1-(1+\delta\epsilon i)|X_j|^2\}X_j + e^{-i\alpha}\left[\frac{\mu}{N}\sum_{k=1}^{N} X_{N+k} + \frac{\nu}{N}\sum_{k=1}^{N} X_k\right] \tag{2}$$

for $j=N+1,\ldots 2N$, where $X_j \in \mathbb{C}$, and $\omega,\epsilon,\alpha,\mu$ and ν are real parameters.

These equations describe a pair of populations of N Stuart-Landau oscillators with all-to-all coupling within each population of strength μ, and all-to-all coupling between the two populations of strength ν. Such oscillators are related to the normal form of a Hopf bifurcation, and are a specific example of $\lambda-\omega$ oscillators [1,5,27]. Such a pair of coupled populations of oscillators has been studied by several authors [11,28,29], and can be thought of as the simplest "network of networks" that one could study.

Defining $X_j = r_j e^{i\theta_j}$, Eq. (1) can be written

$$\frac{dr_j}{dt} = \epsilon^{-1}(1-r_j^2)r_j + \frac{\mu}{N}\sum_{k=1}^{N} r_k\cos(\theta_k-\theta_j-\alpha) + \frac{\nu}{N}\sum_{k=1}^{N} r_{N+k}\cos(\theta_{N+k}-\theta_j-\alpha) \tag{3}$$

$$\frac{d\theta_j}{dt} = \omega - \delta r_j^2 + \frac{1}{r_j}\left[\frac{\mu}{N}\sum_{k=1}^{N} r_k\sin(\theta_k-\theta_j-\alpha) + \frac{\nu}{N}\sum_{k=1}^{N} r_{N+k}\sin(\theta_{N+k}-\theta_j-\alpha)\right] \tag{4}$$

and Eq. (2) can be written as a similar pair of equations. From Eq. (3) we see that as $\epsilon \to 0$, the rate of attraction to the

*c.r.laing@massey.ac.nz

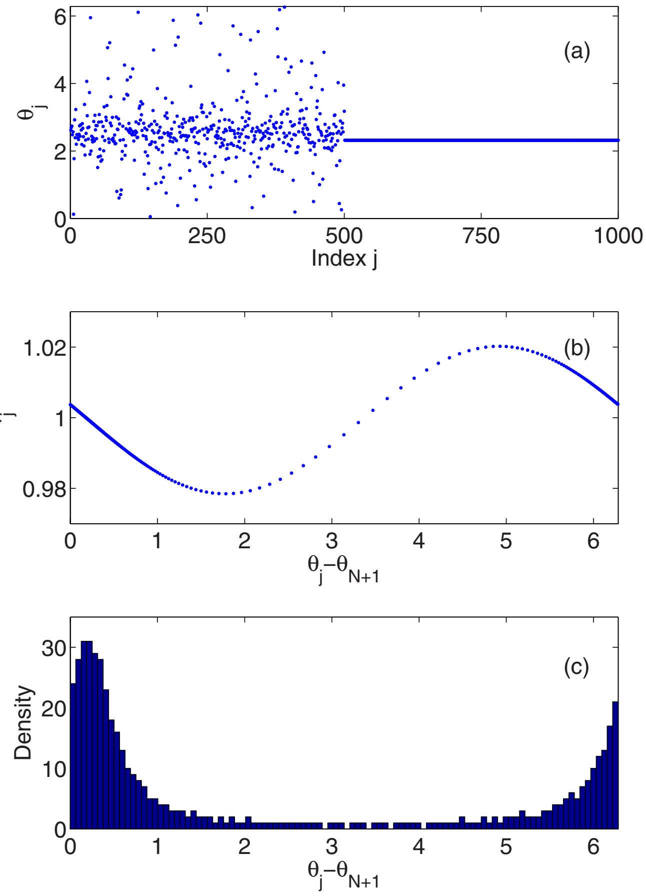

FIG. 1. (Color online) A chimera state for Eqs. (1) and (2). (a): A snapshot of the θ_j. (b): r_j as a function of θ_j (relative to θ_{N+1}) for $j=1,\ldots N$. (c): The density of the θ_j's, relative to θ_{N+1}, for $j=1,\ldots N$. Parameters: $N=500$, $\omega=0$, $\epsilon=0.05$, $\beta=0.08$, $A=0.2$, $\delta=-0.1$.

limit cycle $r_j=1 \quad \forall j$ becomes infinite, and Eq. (4) reduces to Eq. (1) of [11] (after a redefinition of ω), i.e., our system reduces to a previously-studied network of phase oscillators. We will investigate the dynamics of Eqs. (1) and (2) when $\epsilon \neq 0$. By allowing the radius r to vary, we expect a wider variety of behavior than that seen in networks of phase oscillators; for example, oscillator death and chaos [30]. For comparison with previous results we define $\beta=\pi/2-\alpha$ and we let $\mu=(1+A)/2$, $\nu=(1-A)/2$, where A is a parameter [11].

First, we show a chimera state for Eqs. (1) and (2); see Fig. 1. Panel (a) shows a snapshot of all θ_j at an arbitrary time. We see that population two (with $N+1 \leq j \leq 2N$) has completely synchronized (all $r_j \approx 1.0019$), while oscillators in population one (with $1 \leq j \leq N$) remain incoherent. Panel (b) shows that oscillators in population one lie on a closed curve (a slight distortion of the unit circle) in the complex plane. Panel (c) shows the angular density of the oscillators in population one. It is nonuniform, i.e., these oscillators are not completely incoherent, and it was the dynamics of this density that Abrams *et al.* [11] studied, using the parametrization of Ott and Antonsen [20]. In this chimera state the oscillators in population two have a constant angular velocity and radius, and the distributions in panels (b) and (c) of Fig. 1 remain stationary. It is worth noting that the chimera state shown in Fig. 1 is attracting, i.e., nearby states are attracted to it, unlike the corresponding chimera states in networks of identical phase oscillators which are neutrally stable [11,18]. Allowing both the radius and the phase of the oscillators to vary seems to eliminate the nongeneric behavior seen in networks of identical, sinusoidally coupled phase oscillators, in the same way that making the oscillators non-identical does [8,9].

We briefly digress to analyze the chimera state shown in Fig. 1 in the limit $\epsilon \to 0$, i.e., $r_j=1 \, \forall j$. Let $\theta_j=\Theta$ for $N+1 \leq j \leq 2N$ and move to a coordinate frame rotating with angular velocity Ω in which Θ is constant. Using rotational invariance, set $\Theta=0$. Then, from the equation for population two,

$$0 = \omega - \Omega - \delta - \mu \sin \alpha + \nu S \tag{5}$$

and [using Eq. (5)] each oscillator in population one satisfies

$$\frac{d\theta}{dt} = \mu \sin \alpha - \nu S - \nu \sin(\theta + \alpha) + \mu S \cos \theta - \mu C \sin \theta, \tag{6}$$

where $S \equiv N^{-1}\sum_{k=1}^{N}\sin(\theta_k-\alpha)$ and $C \equiv N^{-1}\sum_{k=1}^{N}\cos(\theta_k-\alpha)$. In the limit $N \to \infty$, S and C are constant and can be replaced by the expected values of $\sin(\theta-\alpha)$ and $\cos(\theta-\alpha)$, respectively, calculated using the angular density, $\rho(\theta)$, which is proportional to the reciprocal of the velocity, $d\theta/dt$ [7,12]. Thus chimera states are described by the simultaneous solution of

$$S = \int_0^{2\pi} \sin(\theta - \alpha)\rho(\theta)d\theta \tag{7}$$

and

$$C = \int_0^{2\pi} \cos(\theta - \alpha)\rho(\theta)d\theta \tag{8}$$

where $\rho(\theta)=K(d\theta/dt)^{-1}$ and K is a normalization factor such that $\int_0^{2\pi}\rho(\theta)d\theta=1$. Following solutions of Eqs. (7) and (8) as parameters are varied one can find regions of parameter space in which chimera states exist, in agreement with the results of Abrams *et al.* [11] (results not shown). Equation (6) can be interpreted as saying that in a chimera state, each oscillator in population one follows a periodic orbit, and is nonlinearly driven by its own mean field. This effect is known to be capable of destroying completely synchronous behavior [31]. We now analyze the chimera state in Eqs. (1) and (2) for $\epsilon \neq 0$ using a similar argument, showing that it can be described by a single complex number.

Let $X_j=Y$ for $N+1 \leq j \leq 2N$ and go to a rotating coordinate frame such that Y is constant in this frame. Rotate the frame so that Y is real and positive. Then from Eq. (2) we have

$$0 = i(\omega - \Omega)Y + \epsilon^{-1}\{1 - (1 + \delta\epsilon i)Y^2\}Y + e^{-i\alpha}(\mu Y + \nu\hat{X}), \tag{9}$$

where $\hat{X} \equiv N^{-1}\sum_{k=1}^{N}X_k$, and each oscillator in population one satisfies

$$\frac{dX}{dt} = i(\omega - \Omega)X + \epsilon^{-1}\{1 - (1 + \delta\epsilon i)|X|^2\}X + e^{-i\alpha}[\mu\hat{X} + \nu Y]. \tag{10}$$

Given $\hat{X}$, the real part of Eq. (9) can be solved for Y, and the imaginary part of Eq. (9) can be used to show that each oscillator in population one satisfies

$$\frac{dX}{dt} = i[\delta Y^2 + \mu \sin\alpha - (\nu/Y)\mathrm{Im}\{e^{-i\alpha}\hat{X}\}]X + \epsilon^{-1}\{1 - (1 + \delta\epsilon i)|X|^2\}X + e^{-i\alpha}[\mu\hat{X} + \nu Y], \tag{11}$$

i.e., each oscillator in population one is driven in a nonlinear way by the mean field of population one. Thus our self-consistency equation, i.e., the analog of Eqs. (7) and (8), is

$$\hat{X} = \frac{1}{T(\hat{X})}\int_0^{T(\hat{X})} X(t;\hat{X})dt, \tag{12}$$

where $X(t;\hat{X})$ is a periodic solution of Eq. (11) with period $T(\hat{X})$. The main difference between Eqs. (7) and (8) and Eq. (12) is that $X(t;\hat{X})$ must be found by numerically integrating Eq. (11) to find a periodic solution, whereas the periodic solution of Eq. (6) need not be found—only the density, $\rho(\theta)$, proportional to the reciprocal of the angular velocity, is needed.

Having found a solution of Eq. (12), it can be numerically continued as parameters are varied. Typical results are shown in Fig. 2 where we vary ϵ. We see that for these parameter values the solution of Eq. (12) can be continued to $\epsilon \approx 0.109$, where it appears to undergo a saddle-node bifurcation. For ϵ small, points on the lower branch in panels (a)–(e) correspond to the stable chimera known to exist [11] when $\epsilon=0$, while the upper branch corresponds to the saddle chimera. A typical solution of Eq. (11) is shown in Fig. 2(f).

The saddle-node bifurcation seen in Fig. 2 can be followed as a second parameter, say δ, is varied. The result is shown in Fig. 3 (dashed curve). We see that as δ is increased, the range of values of ϵ for which a chimera state exists also increases. However, the curve of saddle-node bifurcations in Fig. 3 relates only to the *existence* of chimeras (found through a self-consistency argument similar to that of Kuramoto [5]) not their stability. Numerical simulations of Eqs. (1) and (2) show that a stable stationary chimera which exists to the right of the dashed curve in Fig. 3 can undergo a supercritical Hopf bifurcation as parameters are varied, leading to a "breathing" chimera [8,9,11]. These oscillatory states then seem to be destroyed in a homoclinic bifurcation as parameters are further varied. Numerically determined curves of Hopf and homoclinic bifurcations are shown in Fig. 3. These curves are conjectured to terminate at a Takens-Bogdanov bifurcation on the curve of saddle-node bifurcations, which seems to be the generic arrangement for chimera states [8,11,19]. Varying A or β rather than δ results in a similar arrangement of saddle-node, Hopf and homoclinic bifurcation curves (results not shown).

To the left of the dashed curve in Fig. 3 and above the curve of homoclinic bifurcations, the perfectly synchronous

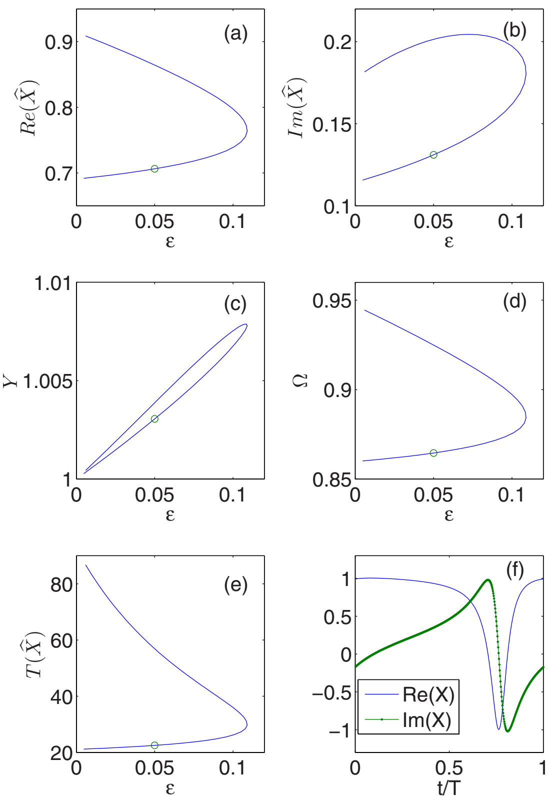

FIG. 2. (Color online) The solution of Eq. (12). (a): $\mathrm{Re}(\hat{X})$; (b): $\mathrm{Im}(\hat{X})$; (c): Y; (d): Ω and (e): $T(\hat{X})$, as functions of ϵ. (f): Real and imaginary parts of the self-consistent solution of Eq. (11) for parameter values shown with a circle in panels (a)–(e). Parameters: $\beta=0.08$, $A=0.2$, $\delta=-0.01$.

state ($X_j = X_k \forall j,k$) is stable. Despite the radii of our oscillators being able to vary, we have not been able to find oscillator death or more exotic dynamics by varying parameters. Perhaps this is not too surprising, since nonidentical oscillators (which we have not considered here) and strong coupling relative to the attraction to the limit cycle (i.e., the

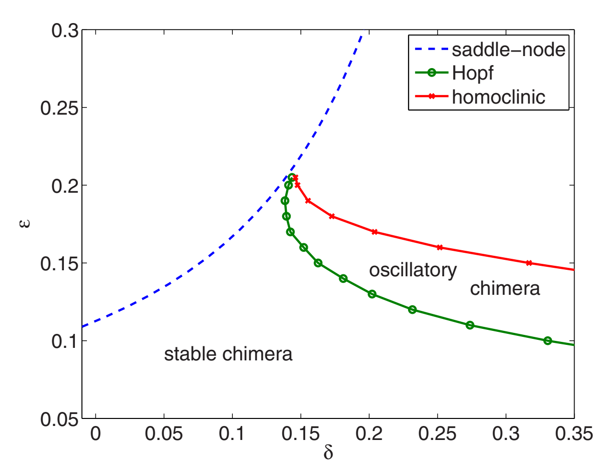

FIG. 3. (Color online) Bifurcation curves in the $\delta-\epsilon$ plane for chimera solutions of Eqs. (1) and (2). Hopf and homoclinic bifurcations were found by direct simulation of Eqs. (1) and (2). $A=0.2$, $\beta=0.08$, $N=500$.

opposite limit from that considered here) seem to be required to observe oscillator death [30,32].

In principle, the stability of the chimera states studied here, and thus the location of the Hopf bifurcation seen in Fig. 3, could be determined using the ideas presented in Sec. 6 of Matthews *et al.* [30]. However, a difficulty arises because we do not have an analytic expression for the chimera state around which to linearize—the density, $\rho(r,\theta)$, of oscillators in population one can only be found indirectly by numerically solving Eq. (11). (Note that the stability or otherwise of the periodic solution of Eq. (11) that we find is not related to the stability of the chimera state. Solving Eq. (11) is just a convenient way of finding the invariant density for population one.)

For chimeras to be observable in a physical system they must be generic, and not only occur in networks of identical phase oscillators with all-to-all coupling via a sinusoidal function of phase differences, which are known to have unusual properties [18,22–24]. Their persistence when phase oscillators are made nonidentical has been characterized previously [8,9], and in this paper we have shown that chimeras also persist (within limits) when both the amplitude and phase of the oscillators are allowed to vary.

One caveat is that the system studied here has all-to-all coupling, both within and between populations. It would be interesting to determine whether this is necessary in order to observe chimeras. Indeed, this raises a more general question as to which network topologies support chimeras. Also, the system [Eqs. (1) and (2)] is invariant under the phase shift $X_j \mapsto e^{i\gamma}X_j \, \forall j$, where γ is a real constant. This seems to be the reason that, in a chimera state, the synchronous population undergoes uniform rotation at fixed radius in the complex plane, and we can describe the incoherent population as having a stationary distribution in a uniformly rotating coordinate frame. It would be of interest to study chimeras in networks for which this is not the case. Addressing these two issues would help determine the general robustness of chimeras, and thus the likelihood of them having relevance to the physical world.

I thank Steve Strogatz for useful correspondence concerning the work presented here.

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge University Press, Cambridge, England, 2003).

[2] S. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hyperion, New York, 2003).

[3] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005).

[4] S. Strogatz, Physica D **143**, 1 (2000).

[5] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).

[6] Y. Kuramoto and D. Battogtokh, Nonlinear Phenom. Complex Syst. (Dordrecht, Neth.) **5**, 380 (2002).

[7] S. I. Shima and Y. Kuramoto, Phys. Rev. E **69**, 036213 (2004).

[8] C. R. Laing, Physica D **238**, 1569 (2009).

[9] C. R. Laing, Chaos **19**, 013113 (2009).

[10] D. M. Abrams and S. H. Strogatz, Phys. Rev. Lett. **93**, 174102 (2004).

[11] D. Abrams, R. Mirollo, S. Strogatz, and D. Wiley, Phys. Rev. Lett. **101**, 084103 (2008).

[12] D. M. Abrams and S. H. Strogatz, Int. J. Bifurcation Chaos Appl. Sci. Eng. **16**, 21 (2006).

[13] G. C. Sethia, A. Sen, and F. M. Atay, Phys. Rev. Lett. **100**, 144102 (2008).

[14] O. E. Omel'chenko, Y. L. Maistrenko, and P. A. Tass, Phys. Rev. Lett. **100**, 044105 (2008).

[15] Y. Kawamura, Phys. Rev. E **75**, 056204 (2007).

[16] Y. Kuramoto, S. Shima, D. Battogtokh, and Y. Shiogai, Prog. Theor. Phys. **161**, (Suppl.), 127 (2006).

[17] E. A. Martens, C. R. Laing, and S. H. Strogatz, Phys. Rev. Lett. **104**, 044101 (2010).

[18] A. Pikovsky and M. Rosenblum, Phys. Rev. Lett. **101**, 264103 (2008).

[19] E. Martens, e-print arXiv:1003.2827.

[20] E. Ott and T. M. Antonsen, Chaos **18**, 037113 (2008).

[21] E. Ott and T. M. Antonsen, Chaos **19**, 023117 (2009).

[22] S. Watanabe and S. H. Strogatz, Phys. Rev. Lett. **70**, 2391 (1993).

[23] S. Watanabe and S. Strogatz, Physica D **74**, 197 (1994).

[24] S. A. Marvel, R. E. Mirollo, and S. H. Strogatz, Chaos **19**, 043104 (2009).

[25] H. Sakaguchi, Phys. Rev. E **73**, 031907 (2006).

[26] Y. Kuramoto and S. Shima, Prog. Theor. Phys. **150**, 115 (2003).

[27] J. M. Greenberg, SIAM J. Appl. Math. **39**, 301 (1980).

[28] E. Barreto, B. Hunt, E. Ott, and P. So, Phys. Rev. E **77**, 036107 (2008).

[29] E. Montbrió, J. Kurths, and B. Blasius, Phys. Rev. E **70**, 056125 (2004).

[30] P. Matthews, R. Mirollo, and S. Strogatz, Physica D **52**, 293 (1991).

[31] M. Rosenblum and A. Pikovsky, Phys. Rev. Lett. **98**, 064101 (2007).

[32] G. Ermentrout, Physica D **41**, 219 (1990).