

Noise-induced synchronization of a large population of globally coupled nonidentical oscillators

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We study a large population of globally coupled phase oscillators subject to common white Gaussian noise and find analytically that the critical coupling strength between oscillators for synchronization transition decreases with an increase in the intensity of common noise. Thus, common noise promotes the onset of synchronization. Our prediction is confirmed by numerical simulations of the phase oscillators as well as of limit-cycle oscillators.

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Synchronization of an ensemble of periodic oscillators has attracted considerable attention because of its broad applications in many fields ranging from physics to engineering [1–4]. In particular, synchronization plays an essential role in numerous biological functions, including the formation of pacemaker tissues of the heart and of the circadian master clock [5].

Because real systems are inevitably subject to noise, it is important to understand the effect of noise on the synchronization of periodic oscillators. Some types of noise, including thermal noise or intrinsic noise in cells, act independently on individual components, which usually inhibits synchronization [1,6]. However, there are many situations where a single noise process, such as that originating from environmental fluctuations, acts on an entire system. Whether such common noise enhances or inhibits synchronization is actually unclear. This issue is thought to be relevant to biological pacemaker tissues in that external noise could have a positive effect on synchronization. However, clarification of the outcome of fluctuating input is necessary in cases, such as that of deep brain stimulation for Parkinson disease [7], in which global external stimulation is used to destroy synchronization of dynamic components.

The effect of common noise on uncoupled oscillators or coupled-oscillator networks with small sizes has been extensively studied for both periodic and chaotic oscillators, and rigorous theoretical frameworks have been proposed [8]. In contrast, for a large population of coupled oscillators, despite a numerous body of numerical and experimental evidence [9], theoretical treatment is still an open and challenging problem.

In this paper, we investigate a large population of non-identical phase oscillators that are globally coupled and subject to common Gaussian white noise to clarify the effect of common noise on coupled oscillators. Utilizing the ansatz recently proposed by Ott and Antonsen [10], we analytically show that the addition of common noise leads to a decrease in the critical coupling strength for synchronization transition. Our prediction is corroborated by direct numerical simulations of the model. We also numerically confirm that globally coupled limit-cycle oscillators show the same de-

pendence on common noise. The employed phase model approximates many realistic systems with weak coupling and weak forcing. Thus, our results suggest that weak common noise generally promotes synchronization of oscillators with weak and global coupling.

Consider globally coupled phase oscillators, known as the Sakaguchi-Kuramoto model [11], subject to a common external force

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i + \beta) + p(t) \sin \theta_i, \quad (1)$$

where θ_i and ω_i are the phase and the natural frequency, respectively, of the i th oscillator, $K > 0$ is the coupling strength, β is a parameter of the coupling function ($-\pi/2 < \beta < \pi/2$), and $p(t)$ is a common external force. We assume that the natural frequency distribution is given by a Lorentzian function $f_{\text{freq}}(\omega) = \frac{1}{\pi} \frac{1}{(\omega - \omega_0)^2 + 1}$. We will further assume white Gaussian noise for $p(t)$ [12], but first we treat $p(t)$ as a general time-dependent function for a while.

Note that Eq. (1) approximates various realistic oscillators with weak coupling and weak forcing [1,3,13]. Note also that the common external force is multiplied by a function of the phase, $\sin \theta_i$, which is called a phase sensitivity function. The phase sensitivity function naturally appears in the phase description of limit-cycle oscillators [1,2,13]. We will later demonstrate these facts using a limit-cycle-oscillator model that generally appears near a Hopf bifurcation.

We examine the synchronization transition in the large- N limit. For a better presentation, we set $\beta = 0$ (corresponding to the Kuramoto model [1]). The extension to nonzero β is straightforward; we will only show a final result for nonzero β in the present paper. In the limit $N \rightarrow \infty$, Eq. (1) becomes

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta} \left\{ \left(\omega + K \frac{r e^{-i\theta} - r^* e^{i\theta}}{2i} + \frac{e^{i\theta} - e^{-i\theta}}{2i} p \right) f \right\} = 0, \quad (2)$$

where $f(\omega, \theta, t)$ is the distribution function for the phases of the oscillators with natural frequency ω , $r = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f e^{i\theta}$ is the Kuramoto order parameter, and $*$ represents the complex conjugate. For $p(t) = 0$, the synchronization transition (the so-called Kuramoto transition) occurs at $K = K_c = 2$, above which $|r|$ is nonvanishing [1].

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To investigate the transition in Eq. (2), we first derive a dynamical equation for the order parameter r . For this, we employ the Ott-Antonsen ansatz for the phase distribution

$$f = \frac{f_{\text{freq}}(\omega)}{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} [(\alpha e^{i\theta})^n + (\alpha^* e^{-i\theta})^n] \right\}, \quad (3)$$

where $\alpha(\omega, t)$ is a certain function [10]. Substituting Eq. (3) into Eq. (2), we obtain a dynamical equation for α

$$\frac{\partial \alpha}{\partial t} + \frac{K}{2}(r\alpha^2 - r^*) + i\omega\alpha + \frac{p}{2}(1 - \alpha^2) = 0. \quad (4)$$

Note that $r(t) = \alpha^*(\omega_0 - i, t)$ because $r = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta f e^{i\theta} = \int_{-\infty}^{\infty} d\omega f_{\text{freq}} \alpha^*$ and $f_{\text{freq}}(\omega) = \frac{1}{\pi} \frac{1}{(\omega - \omega_0 + i)(\omega - \omega_0 - i)}$. Thus, by setting $\omega = \omega_0 - i$ in Eq. (4), we obtain

$$\frac{dr}{dt} = \left(-1 + \frac{K}{2} + i\omega_0 \right) r - \frac{K}{2} |r|^2 r - \frac{p}{2} (1 - r^2). \quad (5)$$

By letting $r = \sqrt{A} e^{i(\omega_0 t + \phi)}$, we further obtain

$$\frac{dA}{dt} = h(A) + g_A(A, \omega_0 t + \phi) p, \quad (6)$$

$$\frac{d\phi}{dt} = g_{\phi}(A, \omega_0 t + \phi) p, \quad (7)$$

where $h(A) = (K-2)A - KA^2$, $g_A(A, \omega_0 t + \phi) = -\sqrt{A}(1-A)\cos(\omega_0 t + \phi)$, and $g_{\phi}(A, \omega_0 t + \phi) = \frac{(1+A)}{2\sqrt{A}}\sin(\omega_0 t + \phi)$.

Now we assume that $p(t)$ is white Gaussian noise with $\langle p(t) \rangle = 0$ and $\langle p(t)p(s) \rangle = 2D\delta(t-s)$, and interpret Eq. (7) as a Stratonovich differential equation. Then we obtain the Fokker-Planck equation for the probability distribution $Q(A, \phi, t)$, given by

$$\begin{aligned} \frac{\partial Q}{\partial t} = & -\frac{\partial}{\partial A} \left[\left[h + D \left(g_A \frac{\partial g_A}{\partial A} + g_{\phi} \frac{\partial g_A}{\partial \phi} \right) \right] Q \right] \\ & - \frac{\partial}{\partial \phi} \left[D \left(g_{\phi} \frac{\partial g_{\phi}}{\partial \phi} + g_A \frac{\partial g_{\phi}}{\partial A} \right) Q \right] \\ & + D \left(\frac{\partial^2}{\partial A^2} (g_A^2 Q) + 2 \frac{\partial^2}{\partial A \partial \phi} (g_A g_{\phi} Q) + \frac{\partial^2}{\partial \phi^2} (g_{\phi}^2 Q) \right). \end{aligned} \quad (8)$$

Because h , g_A , g_{ϕ} , and q are 2π -periodic functions, integrating of both sides of Eq. (8) over ϕ from 0 to 2π yields

$$\begin{aligned} \frac{\partial Q}{\partial t} = & -\frac{\partial}{\partial A} \left\{ \int_0^{2\pi} \left[h + D \left(g_A \frac{\partial g_A}{\partial A} + g_{\phi} \frac{\partial g_A}{\partial \phi} \right) \right] q d\phi \right\} \\ & + D \frac{\partial^2}{\partial A^2} \left(\int_0^{2\pi} g_A^2 q d\phi \right) \end{aligned} \quad (9)$$

where $Q(A, t) = \int_0^{2\pi} q d\phi$.

At this stage, we additionally assume that K and D are sufficiently small compared to a typical natural frequency ω_0 . It is natural to assume this because this is the condition under which Eq. (1) approximates coupled limit-cycle oscillators. Under this assumption, Q evolves sufficiently slowly

compared to a typical oscillation time scale, i.e., $2\pi/\omega_0$. Thus, to a good approximation, the right-hand side of Eq. (9) can be time-averaged over the duration of $2\pi/\omega_0$, leading to

$$\begin{aligned} \frac{\partial Q}{\partial t} = & -\frac{\partial}{\partial A} \left\{ \left[\frac{D}{2} + (K-2-D)A - \left(K - \frac{D}{2} \right) A^2 \right] Q \right\} \\ & + \frac{\partial^2}{\partial A^2} \left\{ \frac{D}{2} A(1-A)^2 Q \right\}. \end{aligned} \quad (10)$$

Letting $\partial Q/\partial t = 0$, we obtain the stationary distribution $Q_{\infty}(A)$ as

$$Q_{\infty}(A) = C \exp \left[\frac{2}{D} \left\{ -\frac{2A}{1-A} - (K+D)\log(1-A) \right\} \right], \quad (11)$$

where $C = 1/\int_0^1 dA \exp[\frac{2}{D}\{-\frac{2A}{1-A} - (K+D)\log(1-A)\}]$. In stochastic systems, the maximum of the probability distribution function is often adopted as the order parameter characterizing a transition [14]. From Eq. (11), it follows that $Q_{\infty}(A)$ assumes its maximum at

$$A_{\text{max}} = \begin{cases} 0 & (K+D < 2) \\ \frac{K+D-2}{K+D} & (K+D \geq 2) \end{cases}. \quad (12)$$

Thus we find that the critical coupling strength at which A_{max} becomes nonvanishing is $K_c = 2 - D$; the common noise decreases the critical coupling strength by D as compared to that in the original Kuramoto transition.

For nonzero β , one can show that K in Eq. (12) is replaced by $K \cos \beta$. Thus, the critical condition is given by $K_c = \frac{2-D}{\cos \beta}$.

We confirmed our prediction by numerical simulation of Eq. (1) with $N=10\,000$ and $\beta=0$. The Lorentzian distribution for the natural frequency was given by [15]

$$\omega_i = \omega_0 + \tan \left\{ i \frac{\pi}{N} - (N+1) \frac{\pi}{2N} \right\} \quad (1 \leq i \leq N). \quad (13)$$

We set $\omega_0 = 100$ to ensure that K and D are much smaller than ω_0 . We employed random initial conditions and numerical data were obtained from $t=10\,000$ to $t=60\,000$. As shown in Fig. 1(a), the phase distribution did not cluster for $K=1.99$ and $D=0$ ($K+D < 2$). In contrast, a cluster of oscillators was observed for $K=1.99$ and $D=0.02$ ($K+D > 2$). To estimate A_{max} from the numerical data, the logarithm of the histogram of A around the peak was fitted to the logarithm of Eq. (11), i.e., $a + \frac{2}{b} [-\frac{2A}{1-A} - (c+b)\log(1-A)]$ with fitting parameters a , b , and c . The obtained data were well fitted [Fig. 1(b)]. The numerically identified values of $A_{\text{max}} = \frac{b+c-2}{b+c}$ were plotted in Fig. 2, which shows excellent agreement with the theoretical prediction of Eq. (12). In our preliminary numerical simulations, we also confirmed that a similar transition behavior occurs in the case of the Gaussian distribution for the natural frequency (data not shown).

We also observed the distribution of the averaged frequencies ω_i^{ave} defined as the long-time average of $\dot{\theta}_i$. Numerical results are shown in Fig. 3. Without noise, the distribution had a delta-function peak at ω_0 , whereas for $D \neq 0$, this

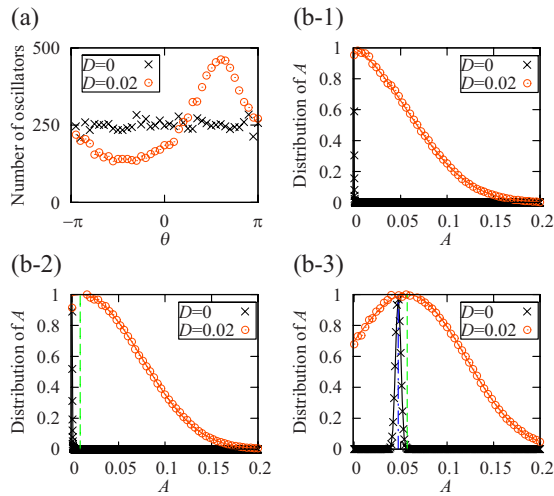


FIG. 1. (Color online) Numerical results for the phase model given by Eq. (1). Crosses (black) and open circles (orange) represent numerical data for $D=0$ and $D=0.02$, respectively. (a) Snapshot of the phase distribution for $K=1.99$. (b) Distribution of A for (b-1) $K=1.96$, (b-2) $K=1.99$, and (b-3) $K=2.1$. Lines on the points are fitting curves. Histograms and curves are normalized for the maximum of curves to be 1. Point-dashed line (blue) and dashed line (green) represent the numerically identified A_{\max} for $D=0$ and $D=0.02$, respectively.

peak disappeared and the distribution was continuous. This qualitative difference can be explained as follows. Using the Kuramoto order parameter r , Eq. (1) can be written as $\dot{\theta}_i = \omega_i + K|r|\sin(\omega_0 t + \phi - \theta_i) + p(t)\sin\theta_i$. For $D=0$, $|r|$ and ϕ are time-independent after transient [1]. Then, oscillators with $|\omega_i - \omega_0| < |r|$ are phase locked to the mean field, so that their actual frequencies are exactly the same as that of the mean field, which is ω_0 . However, for $D \neq 0$, $|r|$ fluctuates with time and becomes vanishingly small with a finite probability [see Eq. (11) and Fig. 1(b)]. This implies that any oscillator except that with $\omega_i = \omega_0$ cannot be phase-locked to the mean field for an infinitely long time. Therefore, oscillators

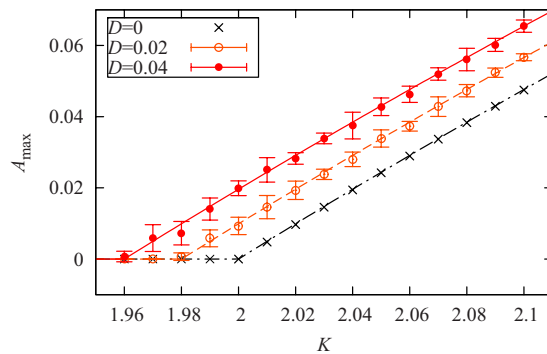


FIG. 2. (Color online) A_{\max} for the phase model as a function of K . Crosses (black), open circles (orange) and filled circles (red) indicate the numerically identified A_{\max} for $D=0$, $D=0.02$, and $D=0.04$, respectively. Error bars represent the variance of A_{\max} for ten trials with different initial conditions and different noise processes. Point-dashed line (black), dashed line (orange), and a solid line (red) represent Eq. (12) for $D=0$, $D=0.02$, and $D=0.04$, respectively.

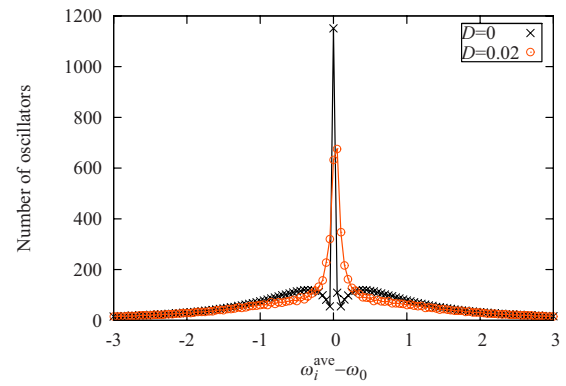


FIG. 3. (Color online) Distribution of the long-time averaged frequencies of the phase oscillators for $K=2.02$. Crosses (black) and open circles (orange) with connecting lines are the numerical results for $D=0$ and $D=0.02$, respectively.

tors with $\omega_i > \omega_0$ ($\omega_i < \omega_0$) tend to have a larger (smaller) averaged frequency than that for $D=0$, so that the delta-function peak vanishes.

Finally, we demonstrate the validity of our prediction in limit-cycle oscillators. We introduce the following model

$$\frac{dW_i}{dt} = (1 + i\omega_i)W_i - |W_i|^2 W_i + \frac{\epsilon K}{N} \sum_{j=1}^N (W_j - W_i) + \sqrt{\epsilon} p(t), \quad (14)$$

where W_i is the complex state variable of the i th oscillator, ϵ is a small parameter to denote that the coupling strength and the noise strength smaller than both the relaxation rate of the amplitude dynamics and the natural frequencies of oscillators, and $p(t)$ is a common white Gaussian noise with strength D . Each individual oscillator is called a Stuart-Landau oscillator, which generically appears when the system is near a Hopf bifurcation [1]. Equation (14) is approximated by Eq. (1) with $\beta=0$ for small ϵ [1], so similar behavior is expected.

We numerically simulated Eq. (14) with $N=1000$. We defined A as $|\sum_{j=1}^N e^{i\theta_j}/N|^2$ with $\theta_j = \arg W_j$ and estimated A_{\max} in the same manner as for the phase oscillators. The numeri-

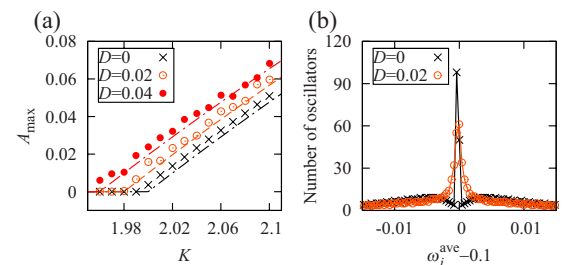


FIG. 4. (Color online) Numerical results for the limit-cycle oscillators given by Eq. (14). Legends for (a) and (b) are the same as those in Figs. 2 and 3, respectively. (a) Order parameter A_{\max} as a function of K . Lines represent Eq. (12). (b) Distribution of the long-time averaged frequencies for $K=2.02$. We used the data from $t=5 \times 10^6$ to $t=10 \times 10^6$. $N=1000$, $\epsilon=0.01$, and $\omega_i = 0.1 + \epsilon \tan\{i\frac{\pi}{N} - (N+1)\frac{\pi}{2N}\}$.

cally determined A_{\max} values are shown in Fig. 4(a), which agrees reasonably well with the prediction of Eq. (12). We also observed that the distribution of ω_i^{ave} was continuous for $D \neq 0$ [Fig. 4(b)].

To conclude, we have studied the Sakaguchi-Kuramoto model subject to common noise and analytically showed that the critical coupling strength for the synchronization-desynchronization transition decreases with an increase in the strength of the common noise. The prediction has been numerically corroborated. We have also found that the distribution of the averaged frequencies is continuous when

common noise is present. Our results suggest that weak common noise generally promotes synchronization of weakly coupled oscillators. Nevertheless there are few reports on synchronization induced by common noise in biological systems or chemical systems. Experimental observation of the effect of common noise on such systems would be of great interest.

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