

Phase coexistence in consolidating porous media

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The appearance of the fluid-rich phase in saturated porous media under the effect of an external pressure is investigated. For this purpose we introduce a two field second gradient model allowing the complete description of the phenomenon. We study the coexistence profile between poor and rich fluid phases and we show that for a suitable choice of the parameters nonmonotonic interfaces show up at coexistence.

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I. INTRODUCTION

When a porous medium is plunged in an infinite fluid reservoir, the solid matrix absorbs fluid until an equilibrium state is reached. Many interesting features of this swelling phenomenon have been demonstrated experimentally. The amount of swelled fluid can be controlled via different external parameters such as the fluid pressure in the reservoir [1,2], that is via its chemical potential, the fluid velocity [3], or a mechanical pressure exerted on the solid [4–6].

The solid-fluid segregation in consolidation is seen when, depending on the external pressure acting on the porous material, phases differing in fluid content are observed. This problem has been addressed [5] by the authors in the framework of continuum mechanics adopting in particular a first gradient model; the existence of two different phases depending on the external pressure has been proved. In that context the interesting question of the coexistence of the two phases could not be posed for the first gradient nature of the model. In this paper, we propose a more general two field one-dimensional second gradient model to study the profiles connecting two coexisting phases and the formation of critical droplets, if any, of one phase into the other.

The adopted approach in modeling the behavior of porous continua is essentially based on a pure solid Lagrangian description of motion, referring kinematics to the reference configuration of the porous skeleton (see Sec. II). The constitutive model is purely phenomenological, which means that the overall potential energy, regarded as a function of the strain of the skeleton and the fluid mass density (per unit volume, in the solid reference configuration), is built up in such a way to describe the existence of two states of equilibrium: the solid-rich and the fluid-rich phase. Thus no refined description of solid grain connectivity, as well as connection among regions with different porosity is available in such a model. Conversely the constitutive state parameters are selected so as to describe the showing up of the fluid-rich phase, which is possibly associated to the occurrence of fluid segregation.

II. POROMECHANICS SETUP

Let $B_s \subset \mathbb{R}$ be the *reference* configurations for the solid and fluid components [7]. The *solid placement* is a C^2 -diffeomorphism $\chi_s(\cdot, t): B_s \rightarrow \mathbb{R}$ such that $\chi_s(X_s, t)$ is the position occupied at time t by the solid particle X_s in the reference configuration B_s . Consider [8] $\phi(\cdot, t): B_s \rightarrow \mathbb{R}$ such that $\phi(X_s, t)$ is the fluid particle which at time t occupies the same position of the solid particle X_s . Assume also $\phi(\cdot, t)$ to be a C^2 -diffeomorphism, thus the map $\phi(\cdot, t)$ associate univocally a solid particle to a fluid one and vice versa. The *fluid placement* map $\chi_f(\cdot, t): \mathbb{R} \rightarrow \mathbb{R}$, giving the position of a fluid particle X_f , is defined as $\chi_f(X_f, t) := \chi_s(\phi^{-1}(X_f, t), t)$. The *current configuration* $\chi_s(B_s, t)$ at time t is the set of positions of the superposed solid and fluid particles.

Let $J_s(X_s, t) := |\partial \chi_s(X_s, t) / \partial X_s|$ be the Jacobian of the placement map $\chi_s(\cdot, t)$ measuring the ratio between current and reference volumes of the solid component; we let $\varepsilon(X_s, t) := (J_s(X_s, t)^2 - 1)/2$ be the *strain field*. Let $\varrho_{0,\alpha}(X_\alpha)$ with $\alpha = s, f$, be the solid and fluid reference *densities*; we define the *fluid mass density* field $m(X_s, t) := \varrho_{0,f}(\phi(X_s, t)) \partial \phi(X_s, t) / \partial X_s$. Assuming that the mass is conserved, it is not difficult to prove [8] that the field m can be interpreted as the fluid mass density measured with respect to the solid reference volume.

Assume, now, that the *Lagrangian density* $\mathcal{L}(\dot{\chi}_s, \dot{\phi}, \chi_s'', \phi'', \chi_s', \phi', \chi_s, \phi)$ of the system is in the form

$$\mathcal{L} = T(\dot{\chi}_s, \dot{\phi}, \chi_s, \phi) - \Phi(\chi_s'', \phi'', \chi_s', \phi', \chi_s, \phi), \quad (1)$$

where T is the *kinetic energy density* and Φ is the *overall potential energy density* accounting for both the internal and the external forces. In Eq. (1), we have denoted with the dot the derivative taken w.r.t. time and with the prime the derivative w.r.t. the solid reference space variable. The equation of motion for the two fields χ_s and ϕ can be derived assuming that the possible motions of the system in an interval of time $(t_1, t_2) \subset \mathbb{R}$ are those such that the fields χ_s and ϕ are stationary profiles for the *action functional*

$$A(\dot{\chi}_s, \dots, \phi) := \int_{B_s} dX_s \int_{t_1}^{t_2} dt \mathcal{L}(\dot{\chi}_s, \dots, \phi) \quad (2)$$

in correspondence of the independent variations of the two fields χ_s and ϕ on $B_s \times (t_1, t_2)$. In other words any possible

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motion of the system in the considered interval is a solution of the Euler-Lagrange equations associated to the variational principle $\delta A=0$.

If one is interested to find equilibrium profiles $\chi_s(X_s)$ and $\phi(X_s)$ of the system, namely, the solutions of the equations of motion independent of time, since the kinetic energy associated to those profiles is equal to zero, the Lagrangian density reduces to minus the potential energy density. In this case the action is given by Eq. (2) where the time integral gives a not essential multiplicative constant, and the variational principle associated to such an action gives the sought for equilibrium profiles.

Assume, now, that the effect of the internal forces exchanged by the solid and fluid particles and that of the conservative external fields can be described via a potential energy density $\Phi(m', \varepsilon', m, \varepsilon)$ depending on the kinematic fields χ_s and ϕ only through the strain and the fluid mass density fields. Note that the strain depends only on χ_s and the fluid mass density only on ϕ , hence the independent variations of those primitive fields reflect on independent variations of ε and m . Thus, limiting the study to boundary value problems expressed in terms of the fields ε and m , we can treat the fields ε and m as primitive, consider their independent variations and look for the equilibrium profiles $\varepsilon(X_s)$ and $m(X_s)$ starting from the variational principle

$$\delta \int_{B_s} dX_s \Phi[m'(X_s), \varepsilon'(X_s), m(X_s), \varepsilon(X_s)] = 0. \quad (3)$$

Finally, we can derive, starting from Eq. (3), the equations governing the equilibrium profiles ε and m . By computing the variation of the action functional on $B_s=(\ell_1, \ell_2)$, with $\ell_1, \ell_2 \in \mathbb{R}$, we get the Euler-Lagrange equations

$$\frac{\partial \Phi}{\partial \varepsilon} - \frac{d}{dX_s} \frac{\partial \Phi}{\partial \varepsilon'} = 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial m} - \frac{d}{dX_s} \frac{\partial \Phi}{\partial m'} = 0, \quad (4)$$

with boundary conditions ensuring that

$$\left[\frac{\partial \Phi}{\partial \varepsilon'} \delta \varepsilon + \frac{\partial \Phi}{\partial m'} \delta m \right]_{\ell_1}^{\ell_2} = 0, \quad (5)$$

where $\delta \varepsilon$ and δm are, respectively, the variations of the strain and fluid mass density fields. For instance Dirichelet boundary conditions would do the job, since we would have $\delta \varepsilon(\ell_1) = \delta \varepsilon(\ell_2) = 0$ and $\delta m(\ell_1) = \delta m(\ell_2) = 0$. But for potential energy densities Φ at least quadratic in the derivatives ε' and m' even Neumann boundary conditions would be acceptable. In the sequel we shall refer to any solution of the Euler-Lagrange Eq. (4) with suitable boundary conditions as an *equilibrium profile* of the system corresponding to the chosen boundary conditions.

The model Eq. (3) is called a *first gradient* model if the potential energy density Φ does not depend on the first derivatives of the strain and of the liquid density, otherwise the model is said a *second gradient* model. This way of classifying the models is related to the fact that both ε and m depends, by definition, on the gradient of the primitive kinetic fields χ_s and ϕ .

Second gradient theories are suitable to be developed for modeling stress/strain concentration due, for instance, to the presence of geometrical singularities (crack propagation in fracture mechanics [9]) or phase transitions as in the case of wetting [10,11]. In particular, second gradient poromechanics has been recently formulated [8,12] extending the standard arguments of the Biot theory [13]. Such a model addresses the description of those deformation phenomena which occur at the same length scale as that where high gradients in deformation can be detected. Classical poromechanics [7] is not able to describe these phenomena: the macroscopic model is regarded in that case as the average of a microscopic one where a kind of stationarity assumption [14] (spatial ergodicity) on the random field which characterizes the microscopic mechanical properties of the material has been formulated. This allows for replacing ensemble averages with volume averages insofar as the characteristic size of the heterogeneities is much smaller than the typical length scale of the reference volume element (RVE). If this is no more the case, the classical assumptions of uniform strain (stress) or periodic boundary conditions, for every reference volume, are no more valid, but, conversely, macroscopic strain gradient plays a crucial role in specifying the state of stress/strain inside the RVE itself.

The goal, here, is to formulate a second gradient poromechanical model for describing the transition from the standard Biot-like equilibrium, associated to a compacted solid, versus the fluid-segregated phase describing duct thinning in the matrix and, consequently, fluid mass concentration in the pores. Thus, we consider a model with total potential energy density in the form

$$\Phi(m', \varepsilon', m, \varepsilon) = K(m', \varepsilon', m, \varepsilon) + \Psi(m, \varepsilon), \quad (6)$$

where K is a polynomial quadratic function of ε' and m' , Ψ is a differentiable function positively diverging along any radial direction in the plane $\varepsilon-m$, having at least a local minimum, and whose stationary points are isolated. Since K is quadratic, we have that a constant solution of the Euler-Lagrange problem Eq. (4) must necessarily satisfy the equations $\Psi_\varepsilon=0$ and $\Psi_m=0$; in other words a constant profile must be constantly equal to an extremal point of the first gradient part Ψ of the total potential energy.

We then let a *phase* of the model to be a constant equilibrium profile equal to one of the local minima of the function Ψ . Note that the Euler-Lagrange problem Eqs. (4) and (5) for the first gradient model associated to Eq. (6), namely, the one obtained for $K=0$, is the system of algebraic equations $\Psi_m=0$ and $\Psi_\varepsilon=0$. Since the stationary points of the two variable function Ψ are isolated, we have that the equilibrium profiles for such a model are necessarily constant functions of $X_s \in B_s$ equal to the values of the stationary points of Ψ .

Hence, in the case of a first gradient model it is not possible to discuss phase coexistence, since there exist only continuous constant equilibrium profiles. On the other hand, in second gradient models, different (not constant) equilibrium profiles can exist. This fact allows us to pose the problem of the coexistence of two existing phases. Suppose that the model exhibits the two phases (m_1, ε_1) and (m_2, ε_2) ; a

connection [15] between those phases is an equilibrium profile m, ε of the action functional on $B_s = (-\infty, +\infty)$ satisfying the boundary conditions $m(-\infty) = m_1$, $m(+\infty) = m_2$, $\varepsilon(-\infty) = \varepsilon_1$, and $\varepsilon(+\infty) = \varepsilon_2$. We say that the two considered phases *coexist* if and only if a connection does exist.

III. MODEL

We study, now, a particular poroelastic model and in that framework we discuss the existence of the consolidation phase transition and prove the coexistence of the two phases for a particular value of the external pressure. More precisely we consider the poroelastic system with overall potential energy density Eq. (6) with

$$K(m', \varepsilon') := \frac{1}{2} [k_1(\varepsilon')^2 + 2k_2\varepsilon'm' + k_3(m')^2], \quad (7)$$

with $k_1, k_3 > 0$, $k_2 \in \mathbb{R}$ such that $k_1k_3 - k_2^2 \geq 0$, and

$$\Psi(m, \varepsilon, p) := \frac{\alpha}{12} m^2(3m^2 - 8b\varepsilon m + 6b^2\varepsilon^2) + \Psi_B(m, \varepsilon, p), \quad (8)$$

where

$$\Psi_B(m, \varepsilon; p) := p\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{2}a(m - b\varepsilon)^2 \quad (9)$$

is the Biot potential energy density [13], $a > 0$ is the ratio between the fluid and the solid rigidity, $b > 0$ is a coupling between the fluid and the solid component, $p > 0$ is the external pressure, and $\alpha > 0$ is a material parameter responsible for the showing up of the additional equilibrium. We remark that the condition $k_1k_3 - k_2^2 \geq 0$ ensures that the second gradient part K of the overall potential energy density is convex. Under this assumption there exists a minimizer for the action functional

$$\int_{\ell_1}^{\ell_2} dX_s \Phi(m', \varepsilon', m, \varepsilon)$$

on a bounded domain. As we will see later to ensure the existence of a connection profile, which is a Dirichlet problem on an unbounded domain, it will be necessary to assume $k_1k_3 - k_2^2 > 0$; in the limiting case $k_1k_3 - k_2^2 = 0$ the existence of the connection will depend on the choice of the parameter k_1 , k_2 , and k_3 .

We have already studied [5] the associated first gradient model with overall potential energy density Ψ and we have proven the existence of a phase transition driven by the external pressure p . More precisely it has been shown that there exists a *critical pressure* $p_c = p_c(\alpha, a, b)$ such that for $0 < p \leq p_c$ the system admits the single *standard* phase $[m_s(p), \varepsilon_s(p)]$, while a second *fluid-rich* phase $[m_f(p), \varepsilon_f(p)]$, appears for $p > p_c$. The standard phase is similar to the unique phase described by the model with potential energy density [13] Ψ_B . In Fig. 1 the standard and the fluid-rich phases are depicted for $p \geq p_c$ and for a particular choice of the physical parameters α, a, b .

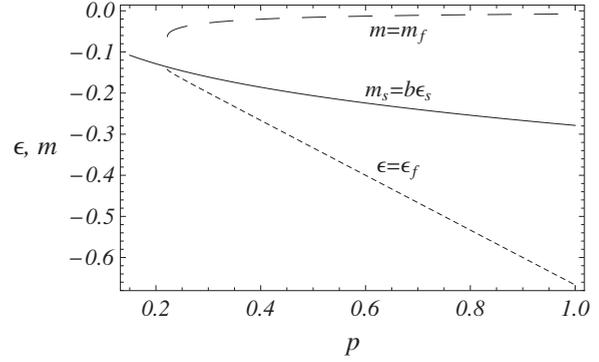


FIG. 1. From the bottom to the top the graphs of $\varepsilon_f(p)$, $m_s(p) = b\varepsilon_s(p)$, and $m_f(p)$ for $a=0.5$, $b=1$, and $\alpha=100$.

IV. COEXISTENCE

The second gradient model has the same phases as the associated first gradient model. The main result of this paper is the existence of $p_{co} = p_{co}(\alpha, a, b) > p_c$, called *coexistence pressure*, such that the standard and the fluid-rich phases coexist at the pressure $p = p_{co}$ and do not coexist at $p > p_c$ and $p \neq p_{co}$. The proof will be achieved in two steps: first we shall show that there exist a unique value of the pressure such that the total potential energy densities evaluated at the two phases are equal; the second step will consist in proving the existence of the connection, that is the equilibrium profile connecting the two phases.

A. Coexistence pressure

We first review some of the results in the previous paper [5,6]; there we have studied the equations $\Psi_\varepsilon = 0$ and $\Psi_m = 0$ looking for the minima of the function Ψ .

We have shown that the standard phase $[m_s(p), \varepsilon_s(p)]$ is the solution of the two equations $m = b\varepsilon$ and $p = f_1(\varepsilon)$, for any $p > 0$, where $f_1(\varepsilon) := -\varepsilon - ab^4\varepsilon^3/3$.

On the other hand the fluid-rich phase $[m_f(p), \varepsilon_f(p)]$ is the solution, with the smallest value of ε , of the two equations $m = m_+(\varepsilon)$ and $p = f_+(\varepsilon)$, where

$$m_+(\varepsilon) = \frac{b}{2} \left[\varepsilon + \sqrt{\varepsilon^2 - \frac{4a}{ab^2}} \right]$$

and

$$f_+(\varepsilon) := -\varepsilon + ab[m_+(\varepsilon) - b\varepsilon] - ab^2\varepsilon m_+(\varepsilon) + \frac{2}{3}abm_+(\varepsilon)^3.$$

For $\varepsilon \leq -2/(b\sqrt{\alpha/a})$ the function $f_+(\varepsilon)$ is positive, diverging to $+\infty$ for $\varepsilon \rightarrow -\infty$, and has a minimum at ε_c such that $f_+(\varepsilon_c) = p_c$; this explains why the fluidized phase is seen only for $p > p_c$. Moreover it has been proven that for any $p > p_c$ the point $[m_f(p), \varepsilon_f(p)]$ is a minimum of the two variable potential energy $\Psi(m, \varepsilon, p)$ with p fixed, while it is a saddle point for $p = p_c$.

We now prove the first step of the above stated coexistence result. For any $p > p_c$, we let $\Psi_s(p) := \Psi[m_s(p), \varepsilon_s(p), p]$ and $\Psi_f(p) := \Psi[m_f(p), \varepsilon_f(p), p]$ and prove that

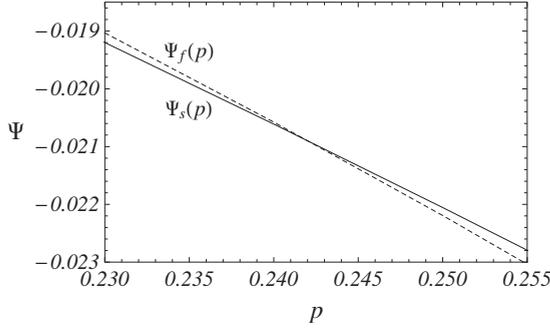


FIG. 2. Graph of the overall potential energy $\Psi_s(p)$ and $\Psi_f(p)$ for $a=0.5$, $b=1$, and $\alpha=100$.

$$\Psi_s(p) > \Psi_f(p) \quad \text{for } p > p_{co}$$

$$\Psi_s(p) = \Psi_f(p) \quad \text{for } p = p_{co}$$

$$\Psi_s(p) < \Psi_f(p) \quad \text{for } p_{co} > p \geq p_c, \quad (10)$$

that is the overall potential energy density of the standard and the fluid-rich phases are equal only at the coexistence pressure. This statement has been tested on numerical grounds, see Fig. 2 where the graphs of the functions $\Psi_s(p)$ and $\Psi_f(p)$ are depicted for a given set of physical parameters.

In order to prove Eq. (10) we first compute the derivative of the two functions $\Psi_s(p)$ and $\Psi_f(p)$ (with respect to p); by using Eq. (8), the chain rule, and the fact that $[m_s(p), \varepsilon_s(p)]$ and $[m_f(p), \varepsilon_f(p)]$ are solutions of the equations $\Psi_m(m, \varepsilon, p) = 0$ and $\Psi_\varepsilon(m, \varepsilon, p) = 0$, we have that $\Psi'_s(p) = \varepsilon'_s(p)$, $\Psi'_f(p) = \varepsilon'_f(p)$, $\Psi''_s(p) = \varepsilon''_s(p)$, and $\Psi''_f(p) = \varepsilon''_f(p)$.

Now, since $\varepsilon_s(p)$ and $\varepsilon_f(p)$ are negative functions of the pressure, we have that both $\Psi_s(p)$ and $\Psi_f(p)$ are decreasing functions of the pressure on the interval $(p_c, +\infty)$. Moreover, noted that both f_1 and f_+ are decreasing functions (of the strain) on $(-\infty, \varepsilon_c]$, we have that $\varepsilon_s(p)$ and $\varepsilon_f(p)$ decrease when p increases. It then follows that $\varepsilon'_s(p)$ and $\varepsilon'_f(p)$ are negative and therefore $\Psi_s(p)$ and $\Psi_f(p)$ are concave on the interval $(p_c, +\infty)$.

Since the two functions $\Psi_s(p)$ and $\Psi_f(p)$ are decreasing concave functions on the interval $(p_c, +\infty)$, in order to prove Eq. (10) it is sufficient to show that $\Psi_s(p_c) < \Psi_f(p_c)$ and $\Psi_s(p) > \Psi_f(p)$ for some p sufficiently large. The proof of the first remark is easy: at $p=p_c$ the two variable function $\Psi(m, \varepsilon, p_c)$ has just the two stationary points [5] $(m_s(p_c), \varepsilon_s(p_c))$ and $(m_f(p_c), \varepsilon_f(p_c))$. Since $(m_s(p_c), \varepsilon_s(p_c))$ is a local minimum of $\Psi(m, \varepsilon, p_c)$, which tends to $+\infty$ along every direction on the plane $m-\varepsilon$, the single local minimum must be the absolute minimum; hence, $\Psi_s(p_c) < \Psi_f(p_c)$. The second remark follows from the asymptotic behavior of the two functions $\Psi_s(p)$ and $\Psi_f(p)$; as proven in the Appendix A, for $p \rightarrow \infty$ we have

$$\Psi_s(p) = -\frac{3}{4} 3^{1/3} \left(\frac{1}{\alpha b^4} \right) p^{4/3} + O(p^{2/3}) \quad (11)$$

and

$$\begin{aligned} \Psi_f(p) &= -\frac{1}{1+ab^2} p^2 + \frac{1}{2(1+ab^2)} p^2 + O(p) \\ &= -\frac{1}{2} (1+ab^2) p^2 + O(p). \end{aligned} \quad (12)$$

By comparing the two asymptotic formulas (11) and (12) we get immediately that for p large enough $\Psi_s(p) > \Psi_f(p)$.

B. Connection profile

It is worth remarking that the variational problem Eq. (3) for profiles with fixed values at the end points ℓ_1 and ℓ_2 of the interval B_s , is nothing but the Hamilton principle for a two degree of freedom mechanical system with Lagrangian coordinates ε and m , kinetic energy T and potential energy U respectively given by

$$T(m', \varepsilon') = \frac{1}{2} [k_1(\varepsilon')^2 + 2k_2\varepsilon'm' + k_3(m')^2] \quad (13)$$

and

$$U(m, \varepsilon) = -\Psi(m, \varepsilon), \quad (14)$$

and the space variable X_s interpreted as time. In other words, the function Φ defined by Eqs. (6)–(8) is the Lagrangian for such a two degree of freedom equivalent mechanical system.

It is important to remark that the mechanical interpretation is correct only when T is a positive definite quadratic form. It is easy to prove that this is the case provided $k_1k_3 - k_2^2 > 0$. In the limiting case $k_1k_3 - k_2^2 = 0$ the form T is positive semidefinite, indeed if we substitute $k_1 = k_2^2/k_3$ in Eq. (13) the function T becomes

$$T(m', \varepsilon') = K(m', \varepsilon') = \frac{1}{2} k_3 (k\varepsilon' + m')^2, \quad (15)$$

where we have set $k := k_2/k_3$, and is equal to zero when $k\varepsilon' + m' = 0$.

We study now the case $k_1k_3 - k_2^2 > 0$ and postpone the degenerate $k_1k_3 - k_2^2 = 0$ to the following section. Let us denote X_s by t and the derivative taken with respect to t by the dot. By using Eq. (4) with $\Phi = T - U$ and recalling Eq. (13), we have that the equations of motion are

$$k_2\ddot{m} + k_1\ddot{\varepsilon} = -\frac{\partial U}{\partial \varepsilon} \quad \text{and} \quad k_3\ddot{m} + k_2\ddot{\varepsilon} = -\frac{\partial U}{\partial m}. \quad (16)$$

We note that the mechanical energy of the associated mechanical problem $E(\dot{m}, \dot{\varepsilon}, m, \varepsilon) := T(\dot{m}, \dot{\varepsilon}) + U(m, \varepsilon)$ is a constant of the motion.

First note that the two points $[m_s(p), \varepsilon_s(p)]$ and $[m_f(p), \varepsilon_f(p)]$, with $p > p_c$, are maxima of the potential energy $U(m, \varepsilon)$ of the equivalent mechanical system. The problem of the existence of a connection between the standard and the fluid-rich phase can be rephrased as follows: look for a solution of the Eqs. (16), namely, a motion $[m_p(t), \varepsilon_p(t)]$ of the equivalent mechanical system, on \mathbb{R} connecting the phase space point $[m_p(-\infty), \varepsilon_p(-\infty)] = [m_s(p), \varepsilon_s(p)]$ and $[\dot{m}_p(-\infty), \dot{\varepsilon}_p(-\infty)] = (0, 0)$ to the phase space point

$[m_p(+\infty), \varepsilon_p(+\infty)] = [m_f(p), \varepsilon_f(p)]$ and $[\dot{m}_p(+\infty), \dot{\varepsilon}_p(+\infty)] = (0, 0)$. The connection we are seeking for is an heteroclinic solution of the equation of motion tending to two fixed points in the phase space for $t \rightarrow -\infty$ and $t \rightarrow +\infty$.

Recall that the mechanical energy E is a constant of the motion and remark that at the equilibrium points it is equal to $E[0, 0, m_s(p), \varepsilon_s(p)] = -\Psi[m_s(p), \varepsilon_s(p)]$ and $E[0, 0, m_f(p), \varepsilon_f(p)] = -\Psi[m_f(p), \varepsilon_f(p)]$. From the results in Sec. IV it follows that those two energies are equal only for $p = p_{co}$. This remark yields that for any $p > p_c$ and $p \neq p_{co}$ the standard and the fluid-rich phases do not coexist.

We are left with the case $p = p_{co}$. In principle an heteroclinic solution can exist, but to prove its existence is an highly not trivial problem which has been solved, under suitable hypotheses on the potential energy, in the recent paper [15] whose main results have been summarized in the Appendix B. Since in the not degenerate case the form T is positive definite, it is possible to find an orthogonal transformation of the coordinates in the plane $m - \varepsilon$, which diagonalize the form itself. Then, performing this transformation and subtracting to the potential energy U of the equivalent mechanical system the constant term $U[m_s(p_{co}), \varepsilon_s(p_{co})] = U[m_f(p_{co}), \varepsilon_f(p_{co})]$, the problem of finding a connection between the standard and fluid-rich phase is transformed in a problem in the form Eq. (B1) with $n=2$ and W replaced by $-[U - U(m_s(p_{co}), \varepsilon_s(p_{co}))]$. Since this function satisfies the hypotheses of the Theorem 3.6 by Alikakos and Fusco [15] (see the Appendix B) we can then conclude that in the case $p = p_{co}$ there exists a connection between the standard and the fluid-rich phase and hence the two phases coexist.

V. DEGENERATE CASE

Consider the case $k_2 = \pm \sqrt{k_1 k_3}$ and the change of variables

$$x := \frac{m + k\varepsilon}{\sqrt{1+k^2}} \quad \text{and} \quad y := \frac{-km + \varepsilon}{\sqrt{1+k^2}},$$

where we recall $k = k_2/k_3 = \pm \sqrt{k_1/k_3}$, which amounts to perform a rotation of the Cartesian reference system in the plane $m - \varepsilon$. Using the new variables the two functions T and U become respectively

$$K(\dot{x}, \dot{y}) = T(\dot{m}(\dot{x}, \dot{y}), \dot{\varepsilon}(\dot{x}, \dot{y})) = \frac{1}{2}k_3(1+k^2)\dot{x}^2 \quad (17)$$

and

$$V(x, y) = U[m(x, y), \varepsilon(x, y)] \quad (18)$$

where we have used Eqs. (15) and (14) with

$$m = \frac{x - ky}{\sqrt{1+k^2}} \quad \text{and} \quad \varepsilon = \frac{kx + y}{\sqrt{1+k^2}}.$$

The expression Eq. (18) of V is awful, but this will not be a problem since V is precisely the two variable functions $-\Psi$, which we have already deeply studied [5], written via a rotation of the Cartesian reference system.

We apply, now, the variational principle Eq. (3) to the total poroelastic potential energy density $\mathcal{L}(\dot{x}, \dot{y}, x, y)$

$:= K(\dot{x}, \dot{y}) - V(x, y)$ and get the analogous (indeed it is a particularization) of the Eqs. (4)

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = 0 \quad (19)$$

which must be solved with the boundary conditions Eq. (5). By using the definition of \mathcal{L} the above equations become

$$k_3(1+k^2)\ddot{x} = -\frac{\partial V}{\partial x}(x, y) \quad \text{and} \quad \frac{\partial V}{\partial y}(x, y) = 0. \quad (20)$$

We remark that the second of the equations above is an algebraic equation involving the two variable x and y ; provided it can be solved w.r.t. y , the first one becomes a second order ordinary differential equation in the unique unknown function x . More precisely, the root locus of $\partial V(x, y)/\partial y = 0$ is made of a certain number of maximal components such that each of them is the graph of a function $x \in \mathbb{R} \rightarrow y(x) \in \mathbb{R}$; for each of them the first of the two Eq. (20) becomes a standard one dimensional conservative mechanical system with potential energy $V[x, y(x)]$.

A. Degenerate case: heteroclinic

The function V is obtained by flipping the sign of the function Ψ and rotating the coordinate axes. This implies that the function V , at $p = p_{co}$, has the two absolute maximum points

$$[x_s(p), y_s(p)] = \left[\frac{m_s(p) + k\varepsilon_s(p)}{\sqrt{1+k^2}}, \frac{-km_s(p) + \varepsilon_s(p)}{\sqrt{1+k^2}} \right]$$

and

$$[x_f(p), y_f(p)] = \left[\frac{m_f(p) + k\varepsilon_f(p)}{\sqrt{1+k^2}}, \frac{-km_f(p) + \varepsilon_f(p)}{\sqrt{1+k^2}} \right]$$

corresponding, respectively, to the standard and to the fluid-rich phases.

Since $[m_s(p), \varepsilon_s(p)]$ and $[m_f(p), \varepsilon_f(p)]$ satisfy the equations $\Psi_m(m, \varepsilon) = 0$ and $\Psi_\varepsilon(m, \varepsilon) = 0$, we have that the two points $[x_s(p), y_s(p)]$ and $[x_f(p), y_f(p)]$ are solutions of the constraint equation $\partial V(x, y)/\partial y = 0$ and hence they belong to the constraint curve.

We consider, now, the case in which at $p = p_{co}$ the two points above fall on the same maximal component of the constraint equation (see Fig. 3 and 4). Using the conservation of the mechanical energy of the equivalent one dimensional conservative system allows for reducing the computation of the coexistence profile (heteroclinic) to the evaluation of a definite integral. Since the function V has two isolated absolute maximum points which, by hypothesis, belong to the same maximal component of the constraint curve, we have that the function $V[x, y(x)]$ of the real function x has two absolute isolated maxima in $x_s(p_{co})$ and $x_f(p_{co})$ (see Fig. 5). Consider the motion of the equivalent one dimensional system corresponding to the energy level $V_{\max} := V(x_s(p_{co}), y_s(p_{co}))$. The conservation of the mechanical energy implies

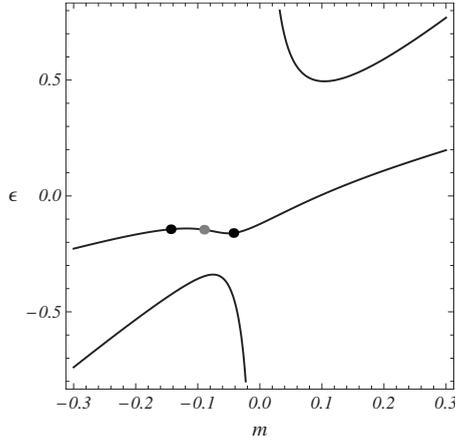


FIG. 3. Graph of the constraint curve in the plane $m-\epsilon$. The three disks represent the fluid-rich phase, the standard phase and the saddle (gray disk) of the potential energy Ψ . Parameters: $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=1$, and $p=p_{co}=0.24218$.

$$\frac{1}{2}k_3(1+k^2)\dot{x}^2 + V[x,y(x)] = V_{\max}.$$

Hence, the heteroclinic connecting the two maxima is given by

$$t = \int_{x_0}^x \sqrt{\frac{k_3(1+k^2)}{2\{V_{\max} - V[x',y(x')]\}}} dx', \quad (21)$$

for any $x \in \{\min[x_s(p_{co}), x_f(p_{co})], \max[x_s(p_{co}), x_f(p_{co})]\}$ and for some fixed x_0 in the same interval. By changing x_0 it is found a family of heteroclinic orbits which are the same curve up to a time translation.

Results are depicted in the Fig. 6. The ϵ coexistence profile shows a bump [17,18] close both to the standard and the fluid-rich phase. This behavior is due to the two-dimensionality of the problem: in Fig. 7 we have depicted the heteroclinic on the graph of the function $U(m,\epsilon) = -\Psi(m,\epsilon)$. From the picture it is clear that the optimal path

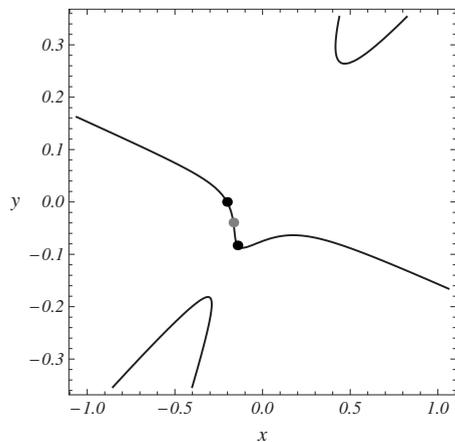


FIG. 4. Graph of the constraint curve in the plane $x-y$. The three disks represent the fluid-rich phase, the standard phase and the saddle (gray disk) of the potential energy Ψ . Parameters: $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=1$, and $p=p_{co}=0.24218$.

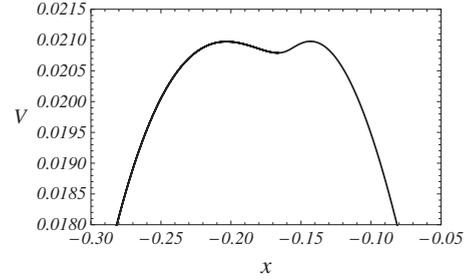


FIG. 5. Function $V[x,y(x)]$ in the case $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=1$, and $p=p_{co}=0.24218$.

climbs the two hills going around the hills themselves. In other words the existence of the bump in the connecting ϵ profile is due to the shape of the constraint curve in the plane $m-\epsilon$. Since the problem has been reduced to the computation of the heteroclinic of a one dimensional conservative mechanical system in the x variable, it is obvious that no bump can exist in the x profile. On the other hand by looking closely at the picture in Figs. 3 and 4, it emerges that the constraint curve is monotonic w.r.t. y and m ; this implies the monotonicity of the y and m profiles. However, it is possible to find values of the parameters such that the y profile presents a bump.

In Sec. IV B, we have proven that in the not degenerate case the connecting profile does exist for any proper choice of the parameters. A similar result does not hold true in the degenerate case, indeed it is possible to find the connection if and only if the two maxima of the function U lie on the same maximal component of the constraint curve. We have that this is not the case for $k>0$ large enough, see the dashed curve in Fig. 8, which is associated to the value $k=1.9$. It is immediate to remark that the two maxima do not lie on the same connected component, hence in this case it is not possible to find a connection between the fluid-rich and the standard phase. It is worth remarking that no evidence of this pathologic behavior is found in the case $k<0$; see Fig. 9 where the constraint curve and the stationary point of U

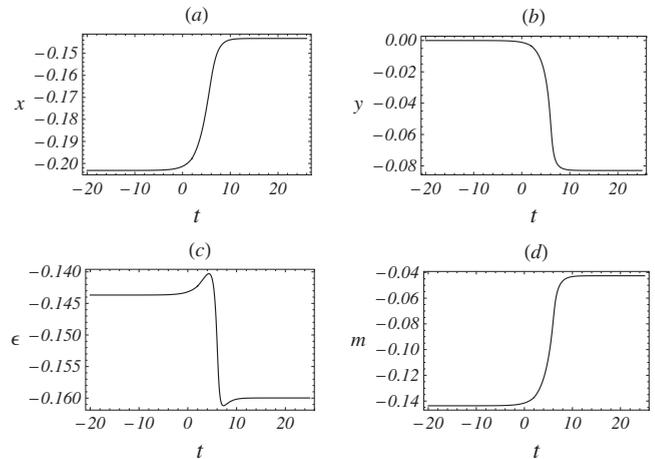


FIG. 6. Heteroclinic (coexistence profile) in the case $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=1$, and $p=p_{co}=0.24218$. Time (space in the original model) on the horizontal axis and x , y , ϵ , and m on the vertical axis, respectively, in (a), (b), (c), and (d).

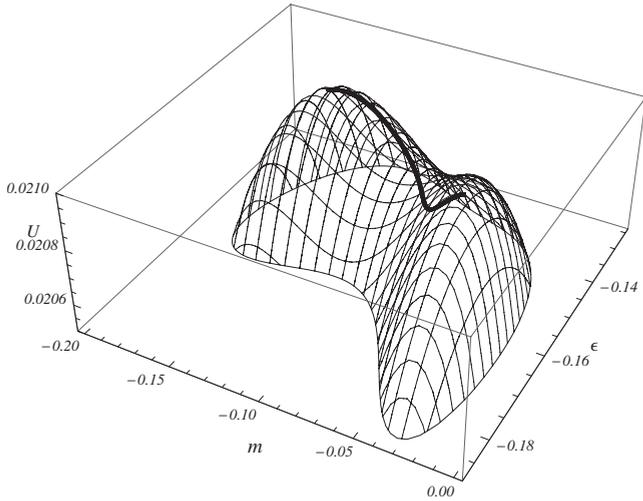


FIG. 7. Graph of the function $U(m, \epsilon)$ and the heteroclinic connecting the two maxima. Parameters: $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=1$, and $p=p_{c0}=0.24218$.

depicted for $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=-0.3, -0.4, -1.0$, and $p=p_{c0}=0.24218$. This case is the most interesting one from the physical point of view, indeed for $k_2 < 0$ the coupling between ϵ' and m' is negative, hence the preferred states are such that the two fields $\epsilon(X_s)$ and $m(X_s)$ are both increasing or decreasing.

B. Degenerate case: homoclinic

Consider the degenerate model and suppose that the pressure p is larger than p_c but different from p_{c0} . Suppose that the two local minima of the potential energy Ψ lie on the same connected component of the constraint curve whose equation is Eq. (20). Consider the function $V(x, y(x))$ as in Sec. V A and note that the two local maxima are not equal.

We consider the homoclinic solution corresponding to the lowest maximum \bar{x} . In analogy with the discussion of the

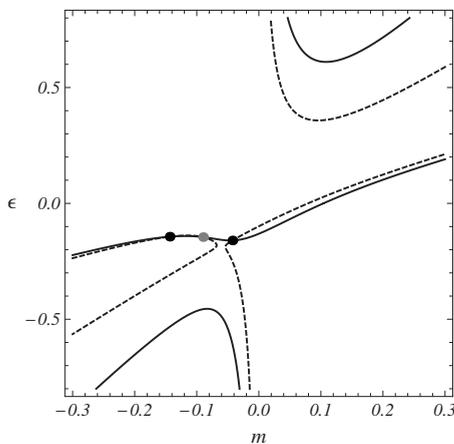


FIG. 8. The constraint curves and the points representing the fluid-rich phase, the standard phase and the saddle (gray disk) of the potential energy Ψ . Parameters: $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=0.7, 1.9$ (solid, dashed), and $p=p_{c0}=0.24218$.

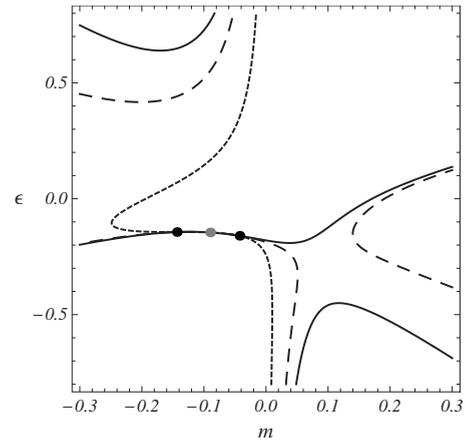


FIG. 9. The constraint curves and the points representing the fluid-rich phase, the standard phase and the saddle (gray disk) of the potential energy Ψ . Parameters: $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=-0.3, -0.4, -1.0$ (solid, dashed, dotted), and $p=p_{c0}=0.24218$.

above section, the homoclinic equilibrium profile can be found, see Fig. 10, by computing the integral

$$t = \pm \int_{\hat{x}}^x \sqrt{\frac{k_3(1+k^2)}{2\{V[\bar{x}, y(\bar{x})] - V[x', y(x')]\}}} dx' \quad (22)$$

for any $x \in (\min\{\hat{x}, \bar{x}\}, \max\{\hat{x}, \bar{x}\})$ with \hat{x} the unique (inversion) point in the interval $(\min\{x_{st}, x_{f1}\}, \max\{x_{st}, x_{f1}\})$ such that $V[\bar{x}, y(\bar{x})] = V[\hat{x}, y(\hat{x})]$.

The homoclinic solution corresponding to the lowest maximum is often interpreted as a “critical nucleus.” In the sense that, if a dynamic evolution would be taken into account, one would expect that an initial condition close to the critical nucleus would evolve into the standard or the fluid-rich phase (subcritical and supercritical behavior). This behavior depends on the size of the droplet in the neighborhood of $t=0$. Indeed in $t=0$ the profile has the value \hat{x} which, for

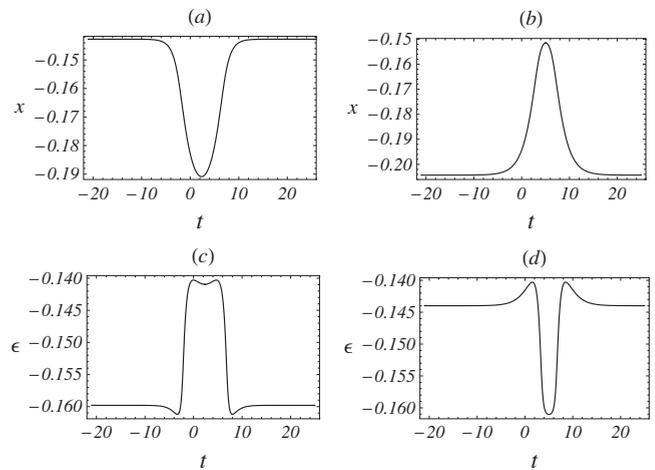


FIG. 10. Homoclinic profile (critical nucleus) in the case $a=0.5$, $b=1$, $\alpha=100$, $k_3=1$, $k=1$, and $p=p_{c0}-10^{-2}p_{c0}$ (left) and $p=p_{c0}+10^{-2}p_{c0}$ (right) with $p_{c0}=0.24218$. Time (space in the original model) on the horizontal axis and x [(a) and (b)] and ϵ [(c) and (d)] on the vertical one.

p close to p_{co} , is a good approximation of the phase corresponding to the largest maximum of the function V . In other words the critical nucleus can be seen as a droplet of the phase corresponding to the smallest value of the potential energy Ψ plunged into the other phase.

VI. CONCLUSIONS

In conclusion, we have studied the phase transition between the fluid poor and rich phases in the context of consolidating completely fluid saturated porous media. A second gradient model to study the existence of such a transition has been proposed. Moreover, coexistence between the two phases at the pressure p_{co} , defined as the pressure such that the total potential energy of the two phases is the same, has been established. We have also shown that at different values of the pressure the two phases cannot coexist. For a particular choice of the parameters of the model it is possible to reduce the problem of finding the coexistence profile to the computation of a definite integral. We have studied the coexistence profile for different values of the physical parameters of the model and shown that nonmonotonic interfaces exist.

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF POTENTIAL ENERGY

In this appendix we discuss the asymptotic behavior of the two functions $\Psi_s(p)$ and $\Psi_f(p)$, see Sec. IV A, for p large and, in particular, prove the Eqs. (11) and (12). We first note that by using $m_s(p) = b\varepsilon_s(p)$, we get

$$\Psi_s(p) = p\varepsilon_s(p) + \frac{1}{2}[\varepsilon_s(p)]^2 + \frac{1}{12}ab^4[\varepsilon_s(p)]^4.$$

The equation $f_1(\varepsilon) = p$ is a cubic equation in the form $\varepsilon^3 + \lambda\varepsilon + \lambda p = 0$, with $\lambda = 3/(ab^4)$; by Cardano's formula, since $D := (\lambda/3)^3 + (\lambda p/2)^2 > 0$, there exists a single real solution given by

$$\begin{aligned} \varepsilon_s(p) &= \left[-\frac{1}{2}\lambda p + \sqrt{D} \right]^{1/3} + \left[-\frac{1}{2}\lambda p - \sqrt{D} \right]^{1/3} \\ &= \left[-\frac{1}{2}\frac{3}{ab^4}p + \sqrt{\left(\frac{1}{3}\frac{3}{ab^4}\right)^3 + \left(\frac{1}{2}\frac{3}{ab^4}p\right)^2} \right]^{1/3} \\ &\quad + \left[-\frac{1}{2}\frac{3}{ab^4}p - \sqrt{\left(\frac{1}{3}\frac{3}{ab^4}\right)^3 + \left(\frac{1}{2}\frac{3}{ab^4}p\right)^2} \right]^{1/3}. \end{aligned}$$

By using the Taylor series $(1+x)^\alpha = \sum_{n=0}^{\infty} C_n(\alpha)x^n$, with $C_n(\alpha) = \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$ being the binomial coefficient, which is convergent for $-1 < x < +1$, it is not difficult to prove that $\varepsilon_s(p) = -(3/(ab^4))^{1/3}p^{1/3} + (3/ab^4)^{2/3}(1/p)^{1/3}/3 + O(p^{-5/3})$ for p large. By inserting this expression in the expansion for Ψ_s , we get Eq. (11).

We can perform a similar computation for $\Psi_f(p)$. Accounting in particular for the qualitative of $\varepsilon_f(p)$, which tends to $-\infty$ when p is increased, we shall study the asymptotic behavior of $m_+(\varepsilon)$ for $\varepsilon \rightarrow -\infty$ and that of $\varepsilon_f(p)$, which is the solution of the equation $f_+(\varepsilon) = p$ when $p \rightarrow \infty$. The result of this analysis will provide us with the asymptotic behavior of $\Psi_f(p)$. First of all we note that

$$m_+(\varepsilon) = \frac{1}{2}b\varepsilon \left[\frac{2a}{ab^2\varepsilon^2} + \frac{2a^2}{\alpha^2b^4\varepsilon^4} + O(\varepsilon^{-6}) \right],$$

for $\varepsilon \rightarrow -\infty$. By using Eq. (8) we then have

$$\Psi_f(p) = p\varepsilon_f(p) + \frac{1}{2}(1+ab^2)[\varepsilon_f(p)]^2 - \frac{a^2}{2\alpha} + O\{[\varepsilon_f(p)]^{-2}\}$$

for $p \rightarrow \infty$, where we have used that $\varepsilon_f(p) \rightarrow -\infty$ for $p \rightarrow \infty$. The function $\varepsilon_f(p)$ is implicitly defined by the equation $f_+(\varepsilon) = p$ which is pretty complicated. By expanding f_+ for $\varepsilon \rightarrow -\infty$ the equation becomes $-\varepsilon(1+ab^2) + h(\varepsilon) = p$ with $h(\varepsilon)$ a function having limit 0 for $\varepsilon \rightarrow -\infty$. Suppose p is large enough and let $\varepsilon_f(p)$ be the solution of the equation above; by the qualitative study we get that $\varepsilon_f(p) \rightarrow -\infty$ for $p \rightarrow \infty$. It is then easy to show that $g(p) := \varepsilon_f(p) - [-p/(1+ab^2)]$ tends to zero for $p \rightarrow \infty$, indeed, since ε_f is the solution of the equation above, we have that

$$g(p) = \frac{(1+ab^2)\varepsilon_f(p) + p}{1+ab^2} = \frac{h[\varepsilon_f(p)]}{1+ab^2} \rightarrow 0,$$

for $p \rightarrow \infty$, where we have used that $h(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow -\infty$ and $\varepsilon_f(p) \rightarrow -\infty$ for $p \rightarrow \infty$. By inserting the obtained expression of $\varepsilon_f(p)$ in the above expansion of $\Psi_f(p)$ we get Eq. (12).

APPENDIX B: GENERAL RESULT ON THE EXISTENCE OF CONNECTIONS

In this appendix we briefly review the main results by Alikakos and Fusco [15] on the existence of connections. Let $W: \mathbb{R}^n \rightarrow \mathbb{R}$, with $n \geq 1$, be a $C^2(\mathbb{R}^n)$ positive function satisfying the following hypotheses: (1) W has two distinct local minima $a_-, a_+ \in \mathbb{R}^n$ such that $W(a_-) = W(a_+) = 0$, (2) $W(u) > 0$ for any $u \neq a_-, a_+$, (3) $\liminf_{|u| \rightarrow \infty} W(u) > 0$, and (4) there exists r_0 in the open interval $(0, |a_- - a_+|)$ such that for any $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$ the two maps $r \mapsto W(a_\pm + r\xi)$ have a strictly positive derivative for every $r \in (0, r_0)$. Conditions (1)–(3) are quite natural and physically obvious; condition (4) is a mild technical requirement allowing for potential energies with C^∞ contact at zeroes.

Consider the ordinary differential equation problem

$$\begin{aligned} u_{xx} &= \nabla W(u) \\ u(-\infty) &= a_- \quad \text{and} \quad u(+\infty) = a_+ \end{aligned} \quad (\text{B1})$$

where $u: \mathbb{R} \rightarrow \mathbb{R}^n$. Solutions to the problem Eq. (B1) are known in the literature as *heteroclinic* motions of the mechanical system or *connection* solutions in the context of phase transitions. The Theorem 3.6 by Alikakos and Fusco [15] states that, under the hypotheses discussed above, the problem Eq. (B1) admits a solution. In other words the theorem states the existence of a connection under very general

and mild requirements on the potential W . The proof of the theorem is based on a direct variational computation. More precisely the authors prove the existence of a critical profile of the action functional

$$A(u) := \int_{-\infty}^{\infty} \left\{ \frac{1}{2} |\dot{u}(x)|^2 + W[u(x)] \right\} dx,$$

on the Sobolev space $W_{\text{loc}}^{1,2}(\mathbb{R}, \mathbb{R}^n)$ of functions $u: \mathbb{R} \rightarrow \mathbb{R}^n$ such that u and its weak derivative are in $L^2(\Omega, \mathbb{R}^n)$ for any

bounded subsets $\Omega \subset \mathbb{R}$. Such a critical profile is the solution of the ordinary differential equation problem Eq. (B1).

Compared to the standard variational calculus, see for instance the paragraph 8.2 in Evan's classical book [16], the authors have to face the lack of compactness due to the infinite domain \mathbb{R} on which the solution of the variational problem is defined. This problem is overcome by using suitable constraints that are successively removed. It is also worth noting that in the Theorem 3.7 the authors state that the connection is a minimizer of the action functional $A(u)$.

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