Effect of anisotropy on the scaling of connectivity and conductivity in continuum percolation theory

S. Sadeghnejad,[1,*] M. Masihi,[1,†] P. R. King,[2,‡] A. Shojaei,[1,§] and M. Pishvaei[1,∥]
[1]*Department of Chemical and Petroleum Engineering, Sharif University of Technology, P.O. Box 11365-9465, Azadi Avenue, Tehran, Iran*
[2]*Department of Earth Science and Engineering, Imperial College, Exhibition Road, London SW7 2AZ, United Kingdom*
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We investigate the effects of anisotropy on the finite-size scaling of connectivity and conductivity of continuum percolation in three dimensions. We consider a system of size $X\times Y\times Z$ in which cubic bodies of size $a\times b\times c$ are placed randomly. We define two aspect ratios to request anisotropy then we expect that the displacement of average connected fraction P (averaged over the realizations), about the isotropic universal curves will be a function of the two aspect ratios. This is accounted by considering an apparent percolation threshold in each direction which leads to 50% of realizations connecting in that direction. We find the aspect ratios' dependency of the apparent threshold and investigate the finite-size scaling transformations for the mean connected fraction and its associated fluctuations. Moreover, we apply a single phase pressure solver to determine the conductivity of various realizations of the system. Finally we apply the same idea to account for the effect of anisotropy on the conductivity scaling.

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I. INTRODUCTION

The connectivity of permeable objects in space is important from flow point of view in many practical cases including in study of the oil recovery from channel reservoirs where essentially the connection of channels controls the flow.

The usual percolation theory was developed for infinite systems [1]. However, all systems are finite, and finite-size scaling has been proposed to deal with finite boundaries. Moreover continuum percolation has been found to be more appropriate in many approaches. Consider oil reservoirs that are very complex with geological heterogeneities appear on all scales. Continuum percolation theory is able to evaluate the connectivity and conductivity of these heterogeneities that is of great importance for decision making on various possible development scenarios including infill drilling projects.

There are numerous literatures on applying percolation theory to evaluate the connectivity and conductivity. For example, Balberg [2] considered universal percolation threshold for continuum systems. King [3] studied the connectivity and conductivity of overlapping sandbodies. Sahimi [4] described different applications of percolation theory including petroleum reservoirs. Lin and Hu [5] studied universal finite scaling of three dimensional lattices. Baker *et al.* [6] found percolation threshold of continuum systems for interpenetrating objects in two and three dimensions. Lee and Torquato [7] worked on correlated continuum percolation and found universal curves and percolation threshold of the system. Also connectivity was considered on fracture systems. Berkowitz [8] analyzed the connectivity of fracture networks. Adler and Thovert [9] studied the connectivity of fracture systems and fractured networks. Berkowitz [10] characterized the flow behavior of fractured geological media. Masihi *et al.* [11] worked on the fast estimation of connectivity in fractured reservoirs. Watson and Leath [12] and Pike and Seager [13] studied the conductivity behavior of the two-dimensional site percolation problems. Masihi *et al.* [14] worked on the conductivity of two-dimensional and three-dimensional lattice systems and found the universal exponent of percolation theory. Sadeghnjad *et al.* [15] analyzed the conductivity behavior of two-dimensional continuum systems in petroleum reservoirs.

Usual finite-size scaling assumes isotropic property for the system, whereas in most cases we need to deal with anisotropic reservoirs. Consequently, there is a need to extend the applicability of finite-size scaling to anisotropic systems. There are few studies on the anisotropic behavior in percolation theory. All of studies are related to two dimensions and to lattice systems. Monetti and Albano [16] performed Monte Carlo simulations to obtain the dependency of the horizontal and vertical finite-size percolation threshold to the aspect ratio of the two-dimensional lattice systems. Hovi and Aharony [17] used the renormalization-group theory and duality arguments to propose a correction to the scaling of spanning probability for aspect ratio in two-dimensional rectangular systems. The dependency of their corrected function to the aspect ratio of the system is in line with Cardy's analytical expression [18] derived from the conformal field theory. Marrink and Knackstedt [19] assumed an elongated lattice as a series of linked isotropic lattices to derive the effective percolation threshold. Watanabe *et al.* [20] studied the scaling behavior of the existence probability on the two-dimensional rectangular domains of different aspect ratios. Masihi *et al.* [21] investigated the effects of anisotropy on finite-size scaling of site percolation in two dimensions. They have defined an apparent percolation threshold and showed that standard finite-size scaling applies if one uses the proposed apparent threshold.

*Corresponding author; s_sadeghnejad@mapsaeng.com
†masihi@sharif.edu
‡peter.king@imperial.ac.uk
§akbar.shojaei@sharif.edu
∥pishvaie@sharif.edu

In this paper we extend the idea of using apparent threshold in standard finite-size scaling of continuum systems in three dimensions. In particular, we extend the applicability of the percolation theory to study the connectivity and conductivity behavior of overlapping sand bodies. We show how the effect of anisotropy on the finite-size scaling of mean connected sand fraction P and mean conductivity K (averaged over the realizations) can be handled. Moreover, we find an appropriate finite-size scaling for the fluctuations about these average values.

Let first look at the definition of anisotropy. By anisotropy we mean that there will be an easy direction for connected paths to be formed an intermediate and a difficult direction. Consider an arbitrary external cubical region with size $X \times Y \times Z$ in which a number of cubical bodies of size $a \times b \times c$ are placed independently and uniformly. The effective system size is now defined by three dimensionless lengths in the X, Y, and Z directions as

$$L_x = \frac{X}{a}, \quad L_x = \frac{Y}{b}, \text{ and } \quad L_z = \frac{Z}{c}. \tag{1}$$

We expect many of the two-dimensional rules to be directly transferable to three dimensions and so a similar approach can be used [21]. However, the computations would have been very time consuming.

The same idea of apparent threshold dependence on the aspect ratio is used to account for the effects of anisotropy in three dimensions. However, there is an extra degree of freedom available. There are two aspect ratios $\omega_1 = L_x/L_y$ and $\omega_2 = L_x/L_z$. We shall consider a simple case where the two aspect ratios are the same (i.e., $b=c$). This enables us to compare the x-direction connectivity to the y-direction connectivity. We define the anisotropy of the system by the new aspect ratio $\omega = \omega_1 \omega_2$. As in two dimensions we expect that the main effect of anisotropy is to make the system no longer symmetric and so the connectivity and conductivity in each direction will be different. We only studied the impact of a moderate aspect ratio for large systems. As the aspect ratio increases, the system behaves as one dimensional system and we would expect to see a crossover to one-dimensional behavior. Under such extreme case the universal scaling exponents will be changed.

For simulation purposes free boundary conditions in the x, y, and z directions are considered and various clusters are identified using standard algorithms [22].

In order to determine the conductivity, we solve the single phase flow equation ($\Delta K \Delta P = 0$) on a fine grid covered on the system. The reader is referenced to Masihi *et al.* [14] and Sadeghnejad *et al.* [15] for the details. Then we use simulation using various realizations of the model to investigate connectivity and conductivity of the entire system as a function of the occupancy probability p. We fix the aspect ratios and investigate finite-size scaling using the effective system size (now defined by L_x). If $\omega_1 < 1$ then X direction is the shortest direction and we expect that connectivity is achieved faster in that direction in comparison with other directions. Then we repeat the calculations (from simulations) at different aspect ratios to study the impact of anisotropy. Having collected the necessary statistics from simulations, we analyze the effects of aspect ratio in anisotropic systems on the mean connected fraction P, mean conductivity of the system K, and their associated uncertainty Δ_P and Δ_K.

II. ASPECT RATIOS DEPENDENCY OF APPARENT PERCOLATION THRESHOLD

To examine the impact of aspect ratios on the apparent percolation threshold, we generated a relatively large number of realizations of various system sizes at four aspect ratios (i.e., five different sizes for each aspect ratio). The occupancy probability p, which leads to 50% of realizations connecting in the x direction we call the apparent x threshold $\tilde{P}_c^x$, with a similar definition in the other directions. This implies a special significance that the spanning probability of an infinite size system at the threshold p_c is equal to 0.5.

For anisotropic cases, we observed different apparent thresholds (when the system is for the first time spans in that direction) in each direction.

As in two dimensions [21], an effective threshold can be defined which we would expect to be dependent on both the system size and the two aspect ratios,

$$\tilde{p}_c^i(\omega_1, \omega_2, L) - p_c^\infty = \Lambda_i(\omega_1, \omega_2) L_x^{-1/\nu}, \tag{2}$$

where i denotes the x, y, or z direction and p_c^∞ is the infinite percolation threshold in three dimensions [$p_c^\infty = 0.274$ (Ref. [6])]. To find the aspect ratios dependence of the shift in apparent threshold Λ_i, we expect to see certain symmetries. For example, keeping L_z fixed and interchanging two lengths L_x and L_y; ω_1 goes to $1/\omega_1$ and ω_2 goes to ω_2/ω_1. This simply means that we have swapped the labeling of the axes. Therefore the x-direction connectivity before this interchange should be equivalent to the connectivity in the y direction after the interchange, $\tilde{p}_c^x(\omega_1, \omega_2) = \tilde{p}_c^y(1/\omega_1, \omega_2/\omega_1)$. However this does not alter the connectivity in the z direction, $\tilde{p}_c^z(\omega_1, \omega_2) = \tilde{p}_c^z(1/\omega_1, \omega_2/\omega_1)$. Using the scaling relationship for the apparent threshold, these properties give

$$\Lambda_x(\omega_1, \omega_2) = \omega_1^{1/\nu} \Lambda_y(1/\omega_1, \omega_2/\omega_1), \tag{3a}$$

$$\Lambda_z(\omega_1, \omega_2) = \omega_1^{1/\nu} \Lambda_z(1/\omega_1, \omega_2/\omega_1). \tag{3b}$$

Using other possible rotations of L_x, L_y, and L_z give the following symmetry relations:

$$\Lambda_x(\omega_1, \omega_2) = \omega_2^{1/\nu} \Lambda_y(\omega_1/\omega_2, 1/\omega_2), \tag{3c}$$

$$\Lambda_y(\omega_1, \omega_2) = \omega_2^{1/\nu} \Lambda_y(\omega_1/\omega_2, 1/\omega_2), \tag{3d}$$

$$\Lambda_x(\omega_1, \omega_2) = \Lambda_x(\omega_2, \omega_1), \tag{3e}$$

$$\Lambda_y(\omega_1, \omega_2) = \Lambda_y(\omega_2, \omega_1). \tag{3f}$$

In two dimensions, the numerical results have shown that the apparent thresholds were symmetrically placed about the isotropic case [21]. However, this is not the case in three dimensions as can be seen in Fig. 1.

Moreover, the amount of the finite-size shift in threshold in two dimensions was negligible [21] in comparison with

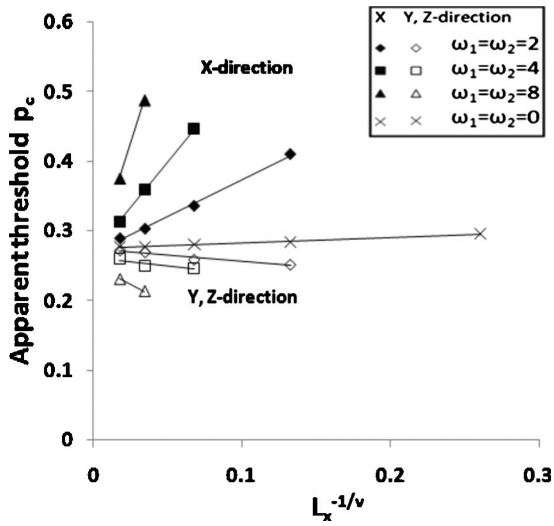

FIG. 1. Illustration of apparent threshold, $\widetilde{P}_c$ in Eq. (2), in the x and the y (or z) directions as a function of $L_x^{-1/\nu}$.

the effect of anisotropy but the numerical results show that this shift is considerable in three dimensions, with the scaling:

$$\widetilde{p}_c^{isotropic} - p_c^{\infty} \approx 0.078 L_x^{-1/\nu}. \quad (4)$$

Hence, it is not possible to define a single function for the proportionality coefficient Λ as in two dimensions. In three dimensions we have three functions of Λ which are interrelated through Eqs. (3). By determining one of these Λ functions, the others can be obtained through the symmetry relations.

We shall start by determining Λ_x. From Eqs. (3) we expect that Λ_x is a function of the product of the two aspect ratios ω_1 and ω_2. Note that, it is unlikely for the sum (in instead of product) to be important as the aspect ratios are basically defined by the ratios of the lengths. Plotting Λ_x

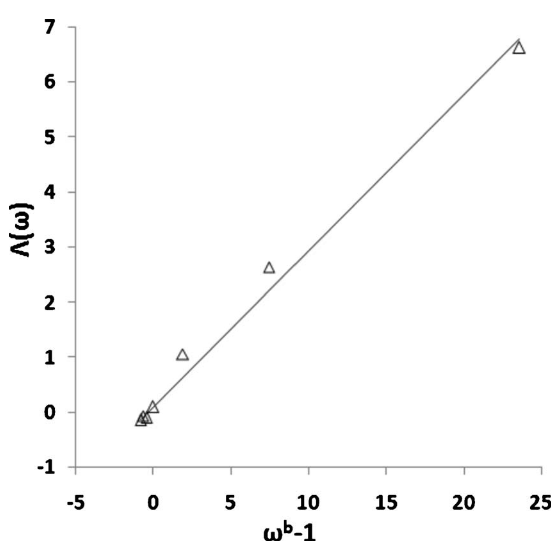

FIG. 2. Illustration of the shift in the apparent threshold, Λ in Eq. (5), as a function of aspect ratio, $\omega = \omega_1 \omega_2$.

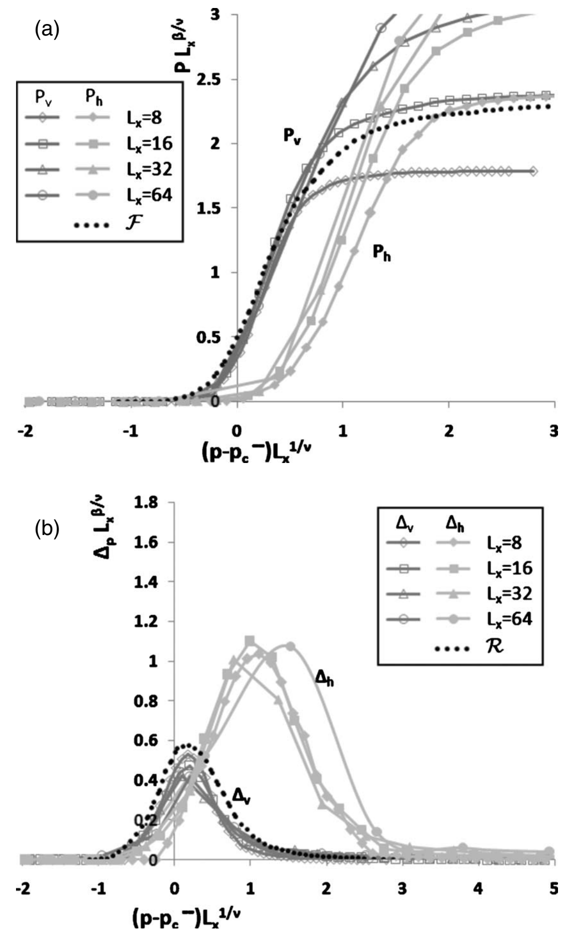

FIG. 3. (a) Rescaled horizontal P_h and vertical P_v mean connectivity curves using infinite threshold for the aspect ratio of 4 (b) with its associated uncertainty, where $\mathcal{F}$ and $\mathcal{R}$ (dotted lines) are the isotropic mean connectivity curve and its associated uncertainty curve, respectively.

against $\omega = \omega_1 \omega_2$ which gives a single curve from which we find the following functional form for Λ_x (Fig. 2):

$$\Lambda_x(\omega_1, \omega_2) = a[(\omega_1 \omega_2)^b - 1] + c, \quad (5)$$

where $a \approx 0.25$, $b \approx 0.77$, and $c \approx 0.1$.

Having obtained the aspect ratio dependency of the apparent threshold, we are now able to investigate the finite-size scaling transformations of such anisotropic system.

III. SCALING OF MEAN CONNECTIVITY AND ITS ASSOCIATE UNCERTAINTY

So far we have determined how the apparent threshold varies as a function of system size and the aspect ratios in three dimensions. We can use this within the usual finite-size scaling rules instead of infinite threshold. We expect the average connected fraction P to follow standard finite-size scaling [22] as was observed in two dimensions [21]. If we rescale the mean connectivity results $P(p,L)$ with L_x, we see that again there is a data collapse in each direction. Plot of $PL^{\beta/\nu}$ and $\Delta_P L^{\beta/\nu}$ against $(p-p_c^{\infty})^{1/\nu}$ for a fixed value of the aspect ratio of 4 is shown in Fig. 3.

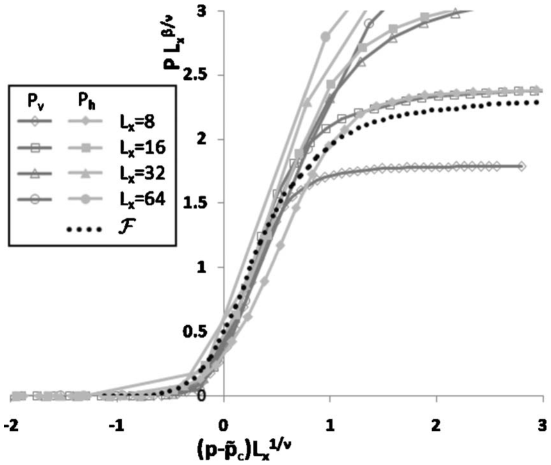

FIG. 4. Illustration of data collapses of the mean connectivity results using the finite-size apparent threshold for the aspect ratio of 4 where $\mathcal{F}$ (dotted line) is the isotropic mean connectivity curve.

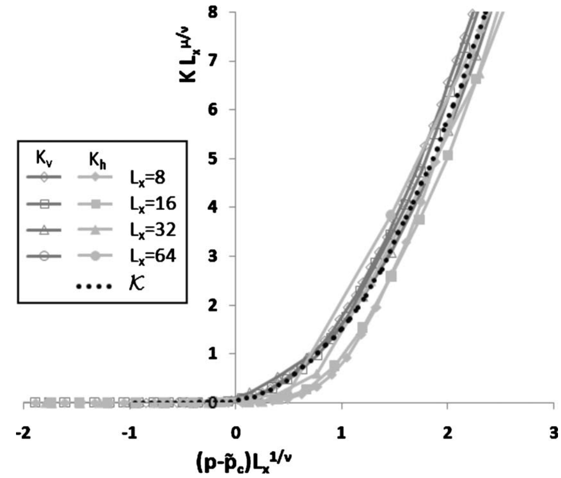

FIG. 6. Illustration of data collapse of the mean conductivity results using the finite-size apparent threshold for the aspect ratio of 4 where κ (dotted line) is the isotropic mean conductivity curve.

However, as can be seen in Fig. 3(b) there is a change in magnitude of standard deviation of the connectivity [i.e., $\Delta_P = \sqrt{(p-\bar{p})^2}$] in various directions in addition to the shift of these curves.

To bring the standard deviation of the connectivity curves back to the isotropic curve we have to use the finite-size apparent threshold as discussed above as well as a change in magnitude, which can be accounted for by rescaling with the geometric mean length, $(L_x L_y)^{0.5}$ [21]. This leads to the following scaling law for the standard deviation in connectivity as

$$\Delta_P(p,L,\omega) = \omega^{1/2} L_x^{-\beta/\nu} \Re[(p - p_c^{\infty}) L_x^{1/\nu}]. \qquad (6)$$

The results of data collapse are shown in Fig. 5, which indicates that this improves the fit in the standard deviation of connectivity results. These results (Figs. 4 and 5) enable us to use the same three dimensions isotropic universal curves ($\mathcal{F}$ and $\mathcal{R}$) for predicting the connectivity of anisotropic continuum systems, which is a good enough approximation for engineering purposes.

IV. SCALING OF MEAN CONDUCTIVITY AND ITS ASSOCIATE UNCERTAINTY

For many systems flow is strongly controlled by the connectivity of the flow units. Percolation theory can then be used to model the conductivity of such systems. Let concentrate on the conductivity of such systems by studying how the flow can move through the spanning cluster. In practice, we need to setup the flow equations on a fine grid covered on the continuum system and solve the resulting equations to find the conductivity of the system. Previous investigations show that above the threshold p_c, connectivity increases rapidly whereas the conductivity is extended to increase slowly. From percolation theory the conductivity of an infinite system has a power-law behavior near the threshold [1],

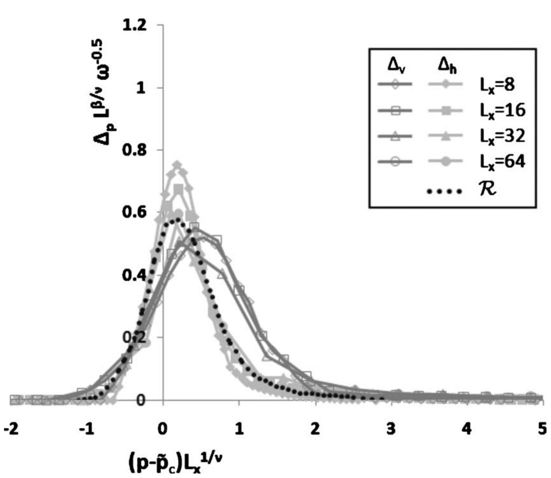

FIG. 5. Illustration of data collapse of the standard deviation of the connectivity results using the finite-size apparent threshold for the aspect ratio of 4 where $\mathcal{R}$ (dotted line) is the isotropic standard deviation of the connectivity curve.

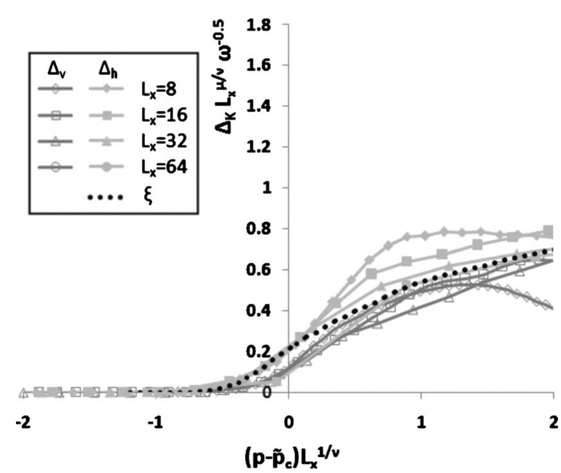

FIG. 7. Illustration of data collapse of the standard deviation of the conductivity results using the finite-size apparent threshold for the aspect ratio of 4 where ξ (dotted line) is the isotropic standard deviation of the conductivity curve.

$$K(p) = (p - p_c^{\infty})^{\mu}, \tag{7}$$

where μ is a universal exponent called the conductivity exponent. The numerically reported value of this exponent is 1.3 and 1.8 in two and three dimensions, respectively [1,14,15].

The finite-size scaling laws for the conductivity results are expected to be the same as the scaling of connectivity results except with new universal exponents of conductivity as

$$K(p,L,\omega) = L_x^{-\mu/\upsilon}\kappa[(p - \tilde{p}_c)L_x^{1/\upsilon}], \tag{8}$$

$$\Delta_K(p,L,\omega) = \omega^{1/2}L_x^{-\mu/\upsilon}\xi[(p - \tilde{p}_c)L_x^{1/\upsilon}], \tag{9}$$

where κ and ξ are two universal master curves for respectively the effective permeability and its associated standard deviation. For computing the conductivity of system we used an up-scaling method based on renormalization theory [23]. We used 500 realizations to generate various conductivity curves for different system sizes. The same approach used to account for the effect of anisotropy in the connectivity results can be directly used here to analyze the conductivity behavior. In Fig. 6 we plot $KL_x^{-\mu/\upsilon}$ against $(p-\tilde{p}_c)L_x^{1/\upsilon}$ for a fixed value of the aspect ratio of 4. Figure 7 shows the data collapse for the fluctuations about the mean conductivity.

The results of data collapse in Figs. 6 and 7 indicate that using the proposed scaling laws [Eqs. (8) and (9)] is a good enough approximate which enable us to use the same isotropic universal curves for predicting the conductivity of anisotropic systems.

V. CONCLUSION

It is shown that we can account for moderate anisotropy in finite-size scaling within percolation by first considering the apparent threshold in the principal coordinate directions of the anisotropy in three-dimensional continuum systems. We then use this within the usual finite-size scaling rules. We also have shown how the same idea can be extended to the conductivity results. Moreover, the approximate scaling for the fluctuations about the mean percolation properties (e.g., conductivity) has been investigated.

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