de Almeida–Thouless line in vector spin glasses

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We consider the infinite-range spin glass in which the spins have $m>1$ components (a vector spin glass). Applying a magnetic field which is random in direction, there is a de Almeida-Thouless (AT) line below which the "replica symmetric" solution is unstable, just as for the Ising ($m=1$) case. We calculate the location of this AT line for Gaussian random fields for arbitrary m and verify our results by numerical simulations for $m=3$.

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I. INTRODUCTION

The infinite-range Ising spin glass, first proposed by Sherrington and Kirkpatrick [1], has been extensively studied. It was found by de Almeida and Thouless [2] (hereafter referred to as AT) that the simple "replica symmetric" (RS) ansatz for the spin-glass state becomes unstable below a line in the magnetic field-temperature plane, known as the AT line. While the Ising spin has $m=1$ components, the m-component vector spin glass for $m>1$ has received less attention. de Almeida *et al.* [3] (hereafter referred to as AJKT) found an instability in zero field but did not consider the effects of a magnetic field. The effects of a *uniform* field on a vector spin glass were first studied by Gabay and Toulouse (GT) [4]. They found a line of transitions (the GT line), which is of a different nature from the AT line. In a uniform field, a distinction has to be made between spin components longitudinal and transverse to the field, and the GT line is the spin-glass ordering of the transverse components, and these are effectively in *zero field* [5,6]. The AT line is different from the GT [4] line since it is a transition to a phase with replica symmetry breaking (RSB) but with *no change in spin symmetry*. The existence of the AT line is perhaps the most striking prediction of the mean field theory of spin glasses. The GT line occurs at a higher temperature than the putative AT line, which becomes a crossover [5,6] for a vector spin glass in a uniform field.

The main point of the present work is to argue that one should consider not a uniform field but a field which is random in *direction* (it will also be convenient to make it random in magnitude though this is not essential) and that, in this case, there *is* an AT line also for vector spin glasses. We will determine the location of this line for an arbitrary number of spin components.

The Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j - \sum_i \mathbf{h}_i\cdot\mathbf{S}_i, , \tag{1}$$

where the S_i^μ, ($\mu=1,\ldots,m$) are m-component spins of length $m^{1/2}$, i.e.,

$$\sum_{\mu=1}^{m} (S_i^\mu)^2 = m, \tag{2}$$

the interactions J_{ij} between all distinct pairs of spins $\langle i,j\rangle$ are independent Gaussian random variables with zero mean and variance given by

$$[J_{ij}^2]_{\rm av} = \frac{J^2}{N-1}, \tag{3}$$

and the h_i^μ are independent Gaussian random fields, uncorrelated between sites, with zero mean and which satisfy

$$[h_i^\mu h_j^\nu]_{\rm av} = h_r^2\delta_{ij}\delta_{\mu\nu}. \tag{4}$$

The notation $[\cdots]_{\rm av}$ indicates an average over the quenched disorder. The normalization of the spins in Eq. (2) is chosen so that the zero-field transition temperature is

$$T_c = J \tag{5}$$

for all m.

Consider first the Ising case ($m=1$). The spin-glass order parameter is

$$q \equiv \frac{1}{N}\sum_i [\langle S_i\rangle^2]_{\rm av}, \tag{6}$$

where $\langle\cdots\rangle$ denotes a thermal average. From linear response theory, if we make small additional random changes, δh_i, in the random fields, uncorrelated with each other and the original values of the fields, the change in $\langle S_i\rangle$ is given by

$$\delta\langle S_i\rangle = \frac{1}{T}\sum_j \chi_{ij}\delta h_j, \tag{7}$$

where the linear response function χ_{ij} is given by

$$\chi_{ij} = \langle S_iS_j\rangle - \langle S_i\rangle\langle S_j\rangle, \tag{8}$$

and for convenience, we have separated out the factor of $1/T$. Hence the change in q is given by

$$\delta q = \frac{1}{T^2}\frac{1}{N}\sum_{i,j,k}[\chi_{ij}\chi_{ik}]_{\rm av}[\delta h_j\delta h_k]_{\rm av} \tag{9}$$

$$= \frac{1}{T^2}\chi_{SG}\delta h_r^2, \tag{10}$$

where

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$$\chi_{SG} = \frac{1}{N}\sum_{i,j}[\chi_{ij}^2]_{\rm av}$$

$$= \frac{1}{N}\sum_{i,j}[(\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle)^2]_{\rm av} \quad (11)$$

is the spin-glass susceptibility.

The corresponding results for vector spins are easily obtained. The change in the spin-glass order parameter,

$$q \equiv \frac{1}{N}\sum_i \frac{1}{m}\sum_\mu [\langle S_i^\mu\rangle^2]_{\rm av}, \quad (12)$$

is given by

$$\delta q = \frac{1}{T^2}\frac{1}{N}\sum_{i,j,k}\frac{1}{m}\sum_{\mu,\nu,\eta}[\chi_{ij}^{\mu\nu}\chi_{ik}^{\mu\eta}]_{\rm av}[\delta h_j^\nu \delta h_k^\eta]_{\rm av} \quad (13)$$

$$= \frac{1}{T^2}\chi_{SG}\delta h_r^2, \quad (14)$$

where now

$$\chi_{SG} = \frac{1}{N}\sum_{i,j}\frac{1}{m}\sum_{\mu,\nu}[(\chi_{ij}^{\mu\nu})^2]_{\rm av} \quad (15)$$

$$= \frac{1}{N}\sum_{i,j}\frac{1}{m}\sum_{\mu,\nu}[(\langle S_i^\mu S_j^\nu\rangle - \langle S_i^\mu\rangle\langle S_j^\nu\rangle)^2]_{\rm av}. \quad (16)$$

For the Ising case, the sign of the field can be "gauged away" by the transformation $S_i \to -S_i$, and $J_{ij} \to -J_{ij}$ for all j. Hence the only difference between a uniform field and a Gaussian random field is that the latter varies in *magnitude*, and these magnitude fluctuations turn out to have only a minor effect [7]. However, for the vector case, the random *direction* of the Gaussian random field does make a big difference because there is no longer a distinction between longitudinal and transverse, and so there is *no longer a GT line to preempt the AT line*. As for the Ising case, there is only a small difference in the location of the AT line depending on whether the magnitude of the field is fixed or allowed to vary.

In zero field, χ_{SG} diverges at the transition temperature T_c given in Eq. (5), which is expected since χ_{SG} is the susceptibility corresponding to the order parameter. Surprisingly, AT showed for the Ising case ($m=1$) that it also diverges in a magnetic field (either uniform, as originally considered by AT, or random, as considered later by Bray [7]) along the AT line in the field-temperature plane. Below the AT line, χ_{SG} goes negative in the RS solution, indicating that the RS solution is incorrect and has to be replaced by the Parisi [8,9] RSB solution.

In this paper we calculate χ_{SG} for a vector spin glass in the presence of a random field and show that it also becomes negative below an AT line in the $h_r - T$ plane, whose location we calculate. This fact does not appear to be widely recognized. Although a field which is random in direction can presumably not be applied experimentally, we feel that there is theoretical interest in our result because a random field *can* be applied in simulations. Whether or not an AT line exists in finite-range spin glasses, is a crucial difference between the RSB picture [8–11] of the spin-glass state, where it does occur, and the droplet picture [12–15], where it does not. It has been found possible to simulate Heisenberg spin glasses for significantly larger sizes [16–18] than Ising spin glasses, so our results may give an additional avenue through which to investigate numerically the nature of the spin-glass state.

The plan of this paper is as follows. In Sec. II we compute the nonlinear susceptibility for the Ising spin glass following the lines of AT. In Sec. III we do the corresponding calculation for the vector spin glass. This is followed in Sec. IV by a numerical evaluation of the AT line for several values of m and a confirmation of the results by Monte Carlo simulations for the Heisenberg spin glass, $m=3$. We summarize our results in Sec. V. Many of the technical details are relegated to appendices. A somewhat longer version of this paper, with technical details on the calculation of the eigenvalues of the matrix Z in Eq. (B1) below, has been posted on the archive [19].

II. SPIN-GLASS SUSCEPTIBILITY FOR ISING SPIN GLASSES

In this section we review the calculation of the AT line for the Ising case. In the next section we shall use this approach to derive the AT line for vector spin glasses.

The standard way of averaging in random systems is the replica trick, which exploits the result

$$\ln Z = \lim_{n\to 0}\frac{Z^n - 1}{n}. \quad (17)$$

Applying this to the Ising ($m=1$) version of the Hamiltonian in Eq. (1), one has

$$[Z^n]_{\rm av} = \mathrm{Tr}\,\exp\left[\frac{(\beta J)^2}{2N}\sum_{\langle i,j\rangle}\sum_{\alpha,\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \frac{(\beta h_r)^2}{2}\sum_i\sum_{\alpha,\beta}S_i^\alpha S_i^\beta\right]. \quad (18)$$

We denote averages over the effective replica Hamiltonian in the exponential on the right-hand side (RHS) of Eq. (18) by $\langle\cdots\rangle$. Following standard steps, see, e.g., Refs. [1,20], one obtains (omitting an unimportant overall constant)

$$[Z^n]_{\rm av} = \int_{-\infty}^{\infty}\left(\prod_{(\alpha\beta)}\left(\frac{N}{2\pi}\right)^{1/2}(\beta J)dq_{\alpha\beta}\right) \times \exp\left(-N\frac{(\beta J)^2}{2}\sum_{(\alpha\beta)}q_{\alpha\beta}^2\right)(\mathrm{Tr}\,\exp L[q_{\alpha\beta}])^N, \quad (19)$$

where $L[q_{\alpha\beta}]$ is given by

$$L[q_{\alpha\beta}] = \beta^2\sum_{(\alpha\beta)}(J^2 q_{\alpha\beta} + h_r^2)S^\alpha S^\beta, \quad (20)$$

the trace is over the spins S^α, $\alpha=1,\ldots,n$, and $(\alpha\beta)$ denotes one of the $n(n-1)/2$ distinct pairs of replicas.

We take the RS saddle point, where all the $q_{\alpha\beta}$ are equal to the same value q. The spin traces at the RS saddle point are evaluated by writing

$$\begin{aligned} \mathrm{Tr}\, e^{L} &= \mathrm{Tr}\exp\left[\beta^2 \sum_{(\alpha\beta)} (J^2 q + h_r^2) S^\alpha S^\beta\right] \\ &= \mathrm{Tr}\exp\left\{\frac{\beta^2}{2}(J^2 q + h_r^2)\left[\left(\sum_\alpha S^\alpha\right)^2 - n\right]\right\} \\ &\propto \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2} dz \prod_{\alpha=1}^{n} [\mathrm{Tr}\, e^{\beta(J^2 q + h_r^2)^{1/2} z S^\alpha}], \end{aligned} \quad (21)$$

where, in the last line, we omitted the constant factor exp $[-(\beta^2/2)(J^2q+h_r^2)n]$, and decoupled the square in the exponential using the identity

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2+az} dz = e^{a^2/2}. \quad (22)$$

Consequently the replica spins S^α (without site label) are independent of each other and feel a Gaussian random field (the same for all replicas) with zero mean and variance given by

$$\Delta^2 \equiv \beta^2 (J^2 q + h_r^2). \quad (23)$$

We denote an average over the Gaussian random variable z in Eq. (21) by $[\cdots]_z$, i.e.,

$$[f(z)]_z = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2} f(z) dz. \quad (24)$$

It is straightforward to evaluate averages over the S^α since they are independent, so we will now express averages over the real spins S_i in terms of S^α averages.

One can show, see, e.g., Ref. [20], that each separate thermal average corresponds to a distinct replica, so, for example,

$$[\langle S_i S_j\rangle \langle S_k\rangle \langle S_l\rangle]_{\mathrm{av}} = \langle S_i^\alpha S_j^\alpha S_k^\beta S_l^\gamma\rangle \quad (25)$$

for α, β and γ all different. To evaluate averages of the form in the RHS of Eq. (25) we add fictitious fields $\Delta_{\alpha\beta}$ which couple the replicas [20], so Eq. (18) becomes

$$\begin{aligned} [Z^n]_{\mathrm{av}} = \mathrm{Tr}\exp\Bigg(& \frac{(\beta J)^2}{2N} \sum_{\langle i,j\rangle} \sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \frac{(\beta h_r)^2}{2} \sum_i \sum_{\alpha,\beta} S_i^\alpha S_i^\beta \\ & + \sum_{(\alpha\beta)} \Delta_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta \Bigg). \end{aligned} \quad (26)$$

Taking derivatives with respect to $\Delta_{\alpha\beta}$, one has, for $n \to 0$,

$$\sum_i \langle S_i^\alpha S_i^\beta\rangle = \frac{\partial}{\partial \Delta_{\alpha\beta}} [Z^n]_{\mathrm{av}}, \quad (27a)$$

$$\sum_{i,j} \langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta\rangle = \frac{\partial^2}{\partial \Delta_{\alpha\beta} \Delta_{\gamma\delta}} [Z^n]_{\mathrm{av}}. \quad (27b)$$

Now setting the $\Delta_{\alpha\beta}$ to zero we get, from Eq. (26), in the $n \to 0$ limit,

$$q \equiv \frac{1}{N}[\langle S_i\rangle^2]_{\mathrm{av}} = \frac{1}{N}\sum_i \langle S_i^\alpha S_i^\beta\rangle = [\langle S^\alpha S^\beta\rangle]_z, \quad (28)$$

for $\alpha \neq \beta$. We emphasize that, in the final average $[\langle \ldots \rangle]_z$, the inner brackets refer to averaging over the spins in a fixed value of the random field z in Eq. (21), and the outer brackets, $[\cdots]_z$, refer to averaging over z according to Eq. (24). Equation (28) leads to the well-known self-consistent expression [1,20] for the spin-glass order parameter q:

$$\begin{aligned} q = [\langle S^\alpha S^\beta\rangle]_z &= [\tanh^2[\beta(J^2 q + h_r^2)^{1/2} z]]_z \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2} \tanh^2 \\ &\quad \times [\beta(J^2 q + h_r^2)^{1/2} z] dz. \end{aligned} \quad (29)$$

It will be useful to express the average in Eq. (27a) in a different way. Including the fictitious fields $\Delta_{\alpha\beta}$ in the derivation which led from Eq. (18) to Eqs. (19) and (20) one finds an extra term, $\sum_{(\alpha\beta)} \Delta_{\alpha\beta} S^\alpha S^\beta$, in $L[q_{\alpha\beta}]$. Defining new integration variables by [20]

$$q_{\alpha\beta} + (\beta J)^{-2} \Delta_{\alpha\beta} \to q_{\alpha\beta}, \quad (30)$$

then $\Delta_{\alpha\beta}$ no longer appears in L, only in the quadratic term in Eq. (19). Using Eqs. (27), one then gets

$$q = \frac{1}{N}\sum_i \langle S_i^\alpha S_i^\beta\rangle = \langle q_{\alpha\beta}\rangle, \quad (31a)$$

$$\frac{1}{N}\sum_{i,j} \langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta\rangle = N\langle q_{\alpha\beta} q_{\gamma\delta}\rangle - (\beta J)^{-2} \delta_{(\alpha\beta),(\gamma\delta)}. \quad (31b)$$

Hence the spin-glass susceptibility, defined in Eq. (11), is given by [20,21]

$$\chi_{SG} = N(\langle \delta q_{\alpha\beta}^2\rangle - 2\langle \delta q_{\alpha\beta} \delta q_{\alpha\gamma}\rangle + \langle \delta q_{\alpha\beta} \delta q_{\gamma\delta}\rangle) - (\beta J)^{-2}, \quad (32)$$

where all replicas are different, and $\delta q_{\alpha\beta}$ is defined by

$$q_{\alpha\beta} = q + \delta q_{\alpha\beta}. \quad (33)$$

We now expand Eq. (19) about the saddle point to quadratic order in the $\delta q_{\alpha\beta}$. The result is that the exponential in Eq. (19) becomes

$$\exp\left(-Nf(q) - N\frac{(\beta J)^2}{2} \sum_{(\alpha\beta),(\gamma\delta)} A_{(\alpha\beta),(\gamma\delta)} \delta q_{\alpha\beta} \delta q_{\gamma\delta}\right), \quad (34)$$

where $f(q)$ is the value of the exponent at the saddle point. To obtain the elements of the $\frac{1}{2}n(n-1)$ by $\frac{1}{2}n(n-1)$ matrix A we take the logarithmic of Eq. (19) and write the coefficients in the expansion of ln Tr e^L in powers of the $\delta q_{\alpha\beta}$ in terms of spin averages, evaluated by the decoupling in Eq. (21). The result is

$$\begin{aligned} A_{(\alpha\beta),(\gamma\delta)} = \delta_{(\alpha\beta)(\gamma\delta)} - (\beta J)^2 \{ & [\langle S^\alpha S^\beta S^\gamma S^\delta\rangle]_z \\ & - [\langle S^\alpha S^\beta\rangle]_z [\langle S^\gamma S^\delta\rangle]_z \}. \end{aligned} \quad (35)$$

Equation (34) is the weight function used for averaging over the $\delta q_{\alpha\beta}$ in Eq. (32). Performing these Gaussian integrals gives

$$\chi_{SG} = \frac{1}{(\beta J)^2}[G_{(\alpha\beta),(\alpha\beta)} - 2G_{(\alpha\beta),(\alpha\gamma)} + G_{(\alpha\beta),(\gamma\delta)} - 1], \tag{36}$$

where G is the matrix inverse of A, i.e.,

$$GA = I, \tag{37}$$

where I is the identity matrix. Defining

$$G_{(\alpha\beta),(\alpha\beta)} = G_1, \tag{38a}$$

$$G_{(\alpha\beta),(\alpha\gamma)} = G_2, \tag{38b}$$

$$G_{(\alpha\beta),(\gamma\delta)} = G_3, \tag{38c}$$

we have

$$\chi_{SG} = \frac{1}{(\beta J)^2}(G_r - 1), \tag{39}$$

where

$$G_r = G_1 - 2G_2 + G_3 \tag{40}$$

is called the "replicon propagator" [22].

The matrix inverse of A is evaluated in Appendix A. According to Eq. (A6) we can express Eq. (39) as

$$\chi_{SG} = \frac{1}{(\beta J)^2}\left(\frac{1}{\lambda_3} - 1\right), \tag{41}$$

where

$$\lambda_3 = P - 2Q + R, \tag{42}$$

and the quantities P, Q and R are defined in Eq. (A1). The eigenvalues of A were first worked out by AT and it turns out that λ_3 is an eigenvalue of A. We evaluate the relevant spin averages needed to determine λ_3 in Appendix C, and Eq. (C22) gives

$$\lambda_3 = 1 - (\beta J)^2\chi_{SG}^0, \tag{43}$$

or equivalently, from Eq. (41),

$$\chi_{SG} = \frac{\chi_{SG}^0}{1 - (\beta J)^2\chi_{SG}^0}, \tag{44}$$

where χ_{SG}^0 is a single-site spin-glass susceptibility, given for the Ising case by

$$\begin{aligned}\chi_{SG}^0 &= [(\langle SS\rangle - \langle S\rangle\langle S\rangle)^2]_z \\ &= [(1 - \langle S\rangle^2)^2]_z \\ &= [(1 - \tanh^2[\beta(J^2q + h_r^2)^{1/2}z])^2]_z \\ &= 1 - 2q + r,\end{aligned} \tag{45}$$

where q is given by Eq. (29) and r is given by

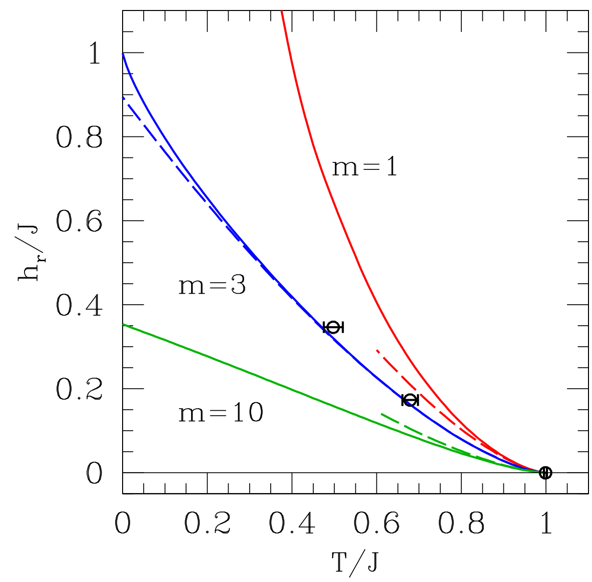

FIG. 1. (Color online) The solid lines indicate the location of the AT line for $m=1$, 3, and 10, according to Eq. (71), and χ_{SG}^0 given by Eq. (C24). For $m\to\infty$ the AT line collapses on to the horizontal axis. The dashed lines are the approximate form given in Eq. (62), which is valid close to $T=T_c=J$. Note that this approximation works remarkably well for the Heisenberg case, $m=3$, even down to quite low temperatures. Also shown are Monte Carlo results for the critical temperature for $h_r=0$, 0.173, and 0.346 for $m=3$.

$$r = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2}\tanh^4[\beta(J^2q + h_r^2)^{1/2}z]dz. \tag{46}$$

Hence, according to the RS ansatz, χ_{SG} is predicted to diverge where

$$(\beta J)^2\chi_{SG}^0 = 1, \tag{47}$$

which describes the location of the AT line. In particular, for small fields the AT line is given by

$$\left(\frac{h_r}{J}\right)^2 = \frac{4}{3}\left(\frac{T_c - T}{T_c}\right)^3 \quad (m=1), \tag{48}$$

see Eq. (C34). In fact, χ_{SG} turns out to be negative below this line since λ_3 is negative in this region, see Eq. (C33). These results were first found by AT. At low temperatures we get

$$\frac{h_r(T\to 0)}{J} = \sqrt{\frac{8}{9\pi}}\frac{J}{T} \quad (m=1), \tag{49}$$

see Eq. (C37), in agreement with Bray [7]. A plot of the AT line for $m=1$, obtained numerically, is shown in Fig. 1.

Although the derivation of Eq. (44) is rather involved, we note that the final answer is quite simple and has a familiar mean field form, i.e., a response function χ is equal to $\chi_0/(1-K\chi_0)$, where χ_0 is the noninteracting response function, and K [$=(\beta J)^2$ here], is a coupling constant. In the next section, we will see that χ_{SG} has precisely the same mean field form for the vector ($m>1$) case.

III. SPIN-GLASS SUSCEPTIBILITY FOR VECTOR SPIN GLASSES

Here we consider a vector spin glass in which the Ising spins are replaced by vector spins with m components. The fluctuations in zero field were first considered by AJKT and Ref. [23] and our approach follows closely that of the latter reference. However, we shall see that there are some differences between our results and those of AJKT and Ref. [23]. The derivation follows the lines of that for the Ising case in the previous section but with the burden of additional indices for the spin components. Hence we will not go through the details but just indicate the main steps and the results.

To avoid confusion in notation, we will use the Greek letters $\alpha,\beta,\gamma,\delta,\epsilon$ for replicas and μ,ν,κ,σ for spin indices. The auxiliary variables q will now involve four indices $(\alpha\beta),\mu\nu$, in which the order of the replica pair $(\alpha\beta)$ is unimportant, i.e., $(\beta\alpha)$ is the same as $(\alpha\beta)$, but the order of the spin indices does matter because $S_\alpha^\mu S_\beta^\nu$ is not the same as $S_\alpha^\nu S_\beta^\mu$. Another new feature which appears when we deal with vector spins is the appearance of terms with both replicas equal, $(\alpha\alpha)$. These do not appear for the Ising case because $(S_\alpha)^2$ is equal to 1, a constant. However, $(S_\alpha^\mu)^2$ is not a constant for $m>1$ and so we now need to include $(\alpha\alpha)$ terms in the analysis although they will not enter the final result for χ_{SG}.

The analogs of Eqs. (19) and (20) are

$$(Z^n)_{\rm av} = \int_{-\infty}^{\infty}\left[\prod_{(\alpha\beta),\mu,\nu}\left(\frac{N}{2\pi}\right)^{1/2}(\beta J)dq_{\alpha\beta}^{\mu\nu}\right] \times\left[\prod_{\alpha,\mu,\nu}\left(\frac{N}{2\pi}\right)^{1/2}(\beta J)dq_{\alpha\alpha}^{\mu\nu}\right] \times\exp\left\{-N\frac{(\beta J)^2}{2}\left[\sum_{(\alpha\beta),\mu,\nu}(q_{\alpha\beta}^{\mu\nu})^2+\sum_{\alpha,\mu,\nu}(q_{\alpha\alpha}^{\mu\nu})^2\right]\right\} \times[\mathrm{Tr}\exp L(q_{\alpha\beta}^{\mu\nu},q_{\alpha\alpha}^{\mu\nu})]^N, \tag{50}$$

where $L[q_{\alpha\beta}^{\mu\nu},q_{\alpha\alpha}^{\mu\nu}]$ is given by

$$L[q_{\alpha\beta}^{\mu\nu},q_{\alpha\alpha}^{\mu\nu}] = \beta^2\sum_{(\alpha\beta),\mu,\nu}(J^2q_{\alpha\beta}^{\mu\nu}+h_r^2\delta_{\mu\nu})S_\mu^\alpha S_\nu^\beta + \frac{(\beta J)^2}{\sqrt{2}}\sum_{\alpha,\mu,\nu}q_{\alpha\alpha}^{\mu\nu}S_\mu^\alpha S_\nu^\alpha, \tag{51}$$

where we ignored a term $\frac{1}{2}(\beta h_r)^2\Sigma_{\mu,\alpha}(S_\mu^\alpha)^2$ since it is a constant.

We take the RS saddle point, where

$$q_{\alpha\beta}^{\mu\nu}=q\delta_{\mu\nu},\quad q_{\alpha\alpha}^{\mu\nu}=x\delta_{\mu\nu}. \tag{52}$$

We then have, ignoring overall constant factors,

$$e^L \propto \mathrm{Tr}\exp\left[\beta^2\sum_{(\alpha\beta),\mu,\nu}(J^2q+h_r^2)S_\mu^\alpha S_\mu^\beta\right] = \mathrm{Tr}\exp\left\{\frac{\beta^2}{2}(J^2q+h_r^2)\sum_\mu\left[\left(\sum_\alpha S_\mu^\alpha\right)^2\right]-nm\right\} \propto \int_{-\infty}^{\infty}\left(\prod_\mu\frac{dz_\mu}{\sqrt{2\pi}}\right)e^{-\Sigma_\mu z_\mu^2/2}\prod_{\alpha=1}^{n}\left[\mathrm{Tr}\,e^{\beta(J^2q+h_r^2)^{1/2}\Sigma_\mu z_\mu S_\mu^\alpha}\right], \tag{53}$$

where, to get the last line, we decoupled the square in the exponent using Eq. (22). As for the Ising case, we denote an average over the Gaussian random variables z_μ by $[\cdots]_z$.

Proceeding as in Sec. II, the spin-glass susceptibility, defined in Eq. (16), is given by

$$\chi_{SG} = \frac{N}{m}\left(\sum_{\mu,\nu}\langle\delta q_{\alpha\beta}^{\mu\mu}\delta q_{\alpha\beta}^{\nu\nu}\rangle - 2\langle\delta q_{\alpha\beta}^{\mu\mu}\delta q_{\alpha\gamma}^{\nu\nu}\rangle + \langle\delta q_{\alpha\beta}^{\mu\mu}\delta q_{\gamma\delta}^{\nu\nu}\rangle\right) - (\beta J)^{-2} \tag{54}$$

(with α, β, γ and δ all different), where the averages over the δq are with respect to the following Gaussian weight [analogous to that in Eq. (34) for the Ising case]:

$$\exp\left(-N\frac{(\beta J)^2}{2}\left\{\sum_{(\alpha\beta),(\gamma\delta)}Z_{(\alpha\beta),(\gamma\delta)}^{\mu\nu,\kappa\sigma}\delta q_{\alpha\beta}^{\mu\nu}\delta q_{\gamma\delta}^{\kappa\sigma} + \sum_{\alpha,(\gamma\delta)}Z_{(\alpha\alpha),(\gamma\delta)}^{\mu\nu,\kappa\sigma}\frac{\delta q_{\alpha\alpha}^{\mu\nu}}{\sqrt{2}}\delta q_{\gamma\delta}^{\kappa\sigma} + \sum_{\alpha,\gamma}Z_{(\alpha\alpha),(\gamma\gamma)}^{\mu\nu,\kappa\sigma}\frac{\delta q_{\alpha\alpha}^{\mu\nu}}{\sqrt{2}}\frac{\delta q_{\gamma\gamma}^{\kappa\sigma}}{\sqrt{2}}\right\}\right) \tag{55}$$

and

$$Z_{(\alpha\beta),(\gamma\delta)}^{\mu\nu,\kappa\sigma} = \delta_{(\alpha\beta)(\gamma\delta)}\delta_{\mu\kappa}\delta_{\nu\sigma} - (\beta J)^2\{[\langle S_\mu^\alpha S_\nu^\beta S_\kappa^\gamma S_\sigma^\delta\rangle]_z - [\langle S_\mu^\alpha S_\nu^\beta\rangle] \times[\langle S_\kappa^\gamma S_\sigma^\delta\rangle]_z\}. \tag{56}$$

Note that the annoying factors of $1/\sqrt{2}$ and 1/2 in Eq. (55) can be removed simply by incorporating a factor of $1/\sqrt{2}$ into the $q_{\alpha\alpha}^{\mu\nu}$. Doing the averages in Eq. (54) using the Gaussian weight in Eq. (55) gives

$$\chi_{SG} = \frac{1}{(\beta J)^2}\left\{\frac{1}{m}\sum_{\mu,\nu}[G_{(\alpha\beta),(\alpha\beta)}^{\mu\mu\nu\nu} - 2G_{(\alpha\beta),(\alpha\gamma)}^{\mu\mu\nu\nu} + G_{(\alpha\beta),(\gamma\delta)}^{\mu\mu\nu\nu}] - 1\right\}, \tag{57}$$

where $G=Z^{-1}$. Using the definitions in Eqs. (B7), we have

$$\chi_{SG} = \frac{1}{(\beta J)^2}(G_r-1), \tag{58}$$

where the "replicon" propagator is given by

$$G_r = G_{1L}+(m-1)G_{1T}-2[G_{2L}+(m-1)G_{2T}] + G_{3L}+(m-1)G_{3T}. \tag{59}$$

The matrix inverse of Z is evaluated in Appendix B. According to Eq. (B10), we can express Eq. (58) as

$$\chi_{SG} = \frac{1}{(\beta J)^2}\left(\frac{1}{\lambda_{3S}} - 1\right), \tag{60}$$

where

$$\lambda_{3S} = P_L + (m-1)P_T - 2[Q_L + (m-1)Q_T] + R_L + (m-1)R_T. \tag{61}$$

The eigenvalues of Z are evaluated in the longer version of this paper on the archive [19], where it is shown that λ_{3S} is an eigenvalue.

From Eq. (C22), we see that Eq. (58) can be written in the same form as for the Ising case, Eq. (44), where, for the case of general m, the single-site spin-glass susceptibility χ_{SG}^0 is evaluated in Appendix C, and given by Eq. (C24).

The AT line is where $(\beta J)^2\chi_{SG}^0=1$. Near T_c this is given by

$$\left(\frac{h_r}{J}\right)^2 = \frac{4}{m+2}t^3, \tag{62}$$

see Eq. (C34). The same expression was obtained by Gabay and Toulouse [4] but for a uniform field, in which case it refers to a crossover rather than a sharp transition. Note that $h_r=0$ for $m=\infty$, as expected since AJKT showed that the replica symmetric solution is stable in this limit. In the opposite limit, $T\to 0$, we find that the value of the AT field is finite for $m>2$,

$$\frac{h_r(T=0)}{J} = \frac{1}{\sqrt{m-2}} \quad (m>2), \tag{63}$$

see Eq. (C36), while $h_r(T\to 0)$ diverges for $m\le 2$. The location of the AT line, obtained numerically, is plotted in Fig. 1 for several values of m.

Below the AT line, χ_{SG} is predicted to be negative, see Eq. (C33), which is impossible and shows that the RS solution (which we have assumed) is wrong in this region.

For $h_r=0$, Eq. (C33) gives $\lambda_{3S}=-4t^2/(m+2)$, which disagrees with the unstable eigenvalue $-8t^2/(m+2)^2$ given by AJKT and Ref. [23]. However, we note that the replicon propagator in Eq. (59) corresponds precisely to Eq. (3.5) of Ref. [24], and Eq. (62) also appears in the paper by Gabay and Toulouse [4], so we are confident that Eq. (C33) is correct. Note too that we obtained the spin-glass susceptibility, the divergence of which indicates the AT line, *directly* from the inverse of the matrix Z, the calculation of which is fairly simple, see Appendix B. The extra information that χ_{SG} is related to an eigenvalue, λ_{3S}, is not strictly needed to locate the AT line.

IV. NUMERICAL RESULTS

We have determined the location of the AT line numerically for $m=1$, 3, and 10. For a given T and assumed value of h_r we solve for q self-consistently from Eq. (C18) and substitute into Eq. (C24) which gives λ_{3S} from Eq. (C22). The value of h_r is then adjusted until $\lambda_{3S}=0$. The results are shown by the solid lines in Fig. 1. Also shown, by the dashed lines, is the approximate form in Eq. (62) which is valid close to the zero-field transition temperature. For $m=3$ this approximation actually works well down to rather low temperatures.

If the spins are normalized to have length 1 rather than $m^{1/2}$ one divides the horizontal scale in Fig. 1 by m and the vertical scale by $1/m^{1/2}$, so the zero-field transition temperature would be $T_c=J/m$ and the zero temperature limit of the AT field would be $h_r=J/\sqrt{m(m-2)}$, for $m>2$ [compare with Eq. (63)].

We have also checked these results by Monte Carlo simulations for the Heisenberg case, $m=3$. The method has been discussed elsewhere [16,18], so here we just give a few salient features. We use three types of moves: heatbath, over-relaxation, and parallel tempering [25,26]. We perform one heatbath sweep and one parallel tempering sweep for every ten over-relaxation sweeps. The parameters of the simulations are given in Table I. In calculating the spin-glass susceptibility in Eq. (16), each thermal average is run in a separate copy of the system to avoid bias. Hence we simulate four copies at each temperature.

When the quenched random disorder variables are Gaussian, as here, the following identity is easily shown to hold by integrating by parts the expression for the average energy U with respect to the disorder variables [16,17],

$$-\frac{U}{m} \equiv \frac{[\langle\mathcal{H}\rangle]_{\rm av}}{m} = \frac{J^2}{2T}(q_s - q_l) + \frac{h_r^2}{T}(1-\bar{q}), \tag{64}$$

where

$$q_s = \frac{1}{Nm}\sum_{i\neq j}[\langle(\mathbf{S}_i\cdot\mathbf{S}_j)^2\rangle]_{\rm av}, \tag{65}$$

$$q_l = \frac{1}{Nm}\sum_{i\neq j}[\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle^2]_{\rm av}, \tag{66}$$

$$\bar{q} = \frac{1}{Nm}\sum_{i}[\langle\mathbf{S}_i\rangle\cdot\langle\mathbf{S}_i\rangle]_{\rm av}, \tag{67}$$

in which $\bar{q}$ is the expectation value of the spin-glass order parameter and q_l is called the "link" overlap.

While Eq. (64) is true in equilibrium, it is not true before equilibrium is reached, and indeed, the two sides of the equation approach the equilibrium value from opposite directions [16,27]. Hence we only accept the results of a simulation if Eq. (64) is satisfied with small error bars. (Note that this equation refers to an average over samples; the connection between the energy and the spin correlations is not true sample by sample.)

According to finite-size scaling the spin-glass susceptibility in a finite, infinite-range system, should vary as [28–31]

$$\chi_{SG} = N^{1/3}\tilde{X}\{N^{1/3}[T - T_c(h_r)]\}, \tag{68}$$

so plots of $\chi_{SG}/N^{1/3}$ should intersect at the transition temperature $T_c(h_r)$. Data for $\chi_{SG}/N^{1/3}$ for $m=3$ for random field values $h_r=0$, 0.173, and 0.346 are shown in Figs. 2–4. The data do indeed intersect, indicating a transition, although the data for different sizes do not intersect at exactly the same temperature which indicates the presence of corrections to finite-size scaling.

TABLE I. Parameters of the simulations for different values of h_r. Here $N_{\rm samp}$ is the number of samples, $N_{\rm sweep}$ is the number of over-relaxation Monte Carlo sweeps, $T_{\rm min}$ and $T_{\rm max}$ are the lowest and highest temperatures simulated, and N_T is the number of temperatures.

h_r	N	$N_{\rm samp}$	$N_{\rm sweep}$	$T_{\rm min}$	$T_{\rm max}$	N_T
0	64	8000	256	0.30	1.50	40
0	128	8000	512	0.30	1.50	40
0	256	8000	1024	0.30	1.50	40
0	512	8000	2048	0.30	1.50	40
0	1024	2078	4096	0.30	1.50	40
0.173	64	8000	1024	0.30	1.50	40
0.173	128	8000	2048	0.30	1.50	40
0.173	256	8000	4096	0.30	1.50	40
0.173	512	4279	8192	0.30	1.50	40
0.173	1024	1494	16384	0.39	1.50	40
0.346	64	8000	1024	0.15	1.20	40
0.346	128	8000	2048	0.15	1.20	40
0.346	256	8000	4096	0.15	1.20	40
0.346	512	4293	8192	0.15	1.20	40
0.346	1024	3037	16384	0.15	1.20	40

There are both singular and analytic corrections to scaling. In the mean field limit the leading correction to χ_{SG} is analytic [32], in fact just a constant, so we replace Eq. (68) by

$$\chi_{SG} = N^{1/3}\tilde{X}\{N^{1/3}[T - T_c(h_r)]\} + c_0. \tag{69}$$

We compute the intersection temperature $T^\star(N,2N)$ between data for $\chi_{SG}/N^{1/3}$ for sizes N and $2N$. It is then easy to see from Eq. (69) that the $T^\star(N,2N)$ converge to the transition temperature like

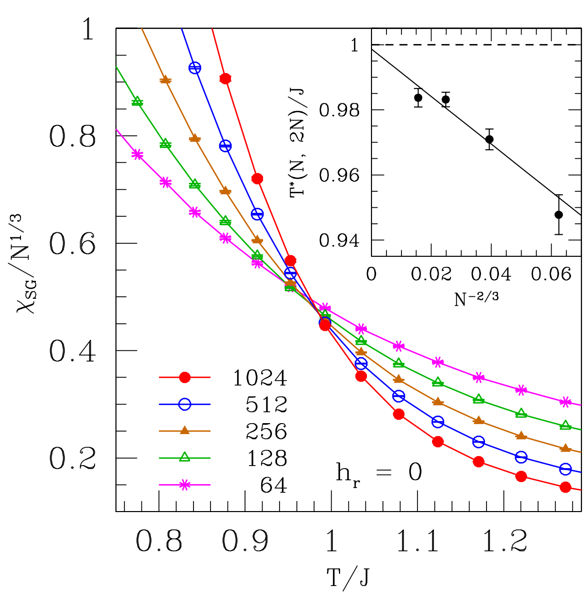

FIG. 2. (Color online) Zero-field Monte Carlo data for the spin-glass susceptibility for the $m=3$ (Heisenberg) infinite-range spin glass, divided by $N^{1/3}$, for different sizes. According to finite-size scaling, the data should intersect at the transition temperature T_c in the absence of corrections to scaling. Allowing for the leading corrections, the inset shows intersection temperatures $T^\star(N,2N)$ for sizes N and $2N$ and the extrapolation to $N=\infty$ according to Eq. (70). This leads to the estimate $T_c=0.9987\pm0.0036$ (see Table II), which agrees well with the exact value of 1, shown as the dashed line in the inset.

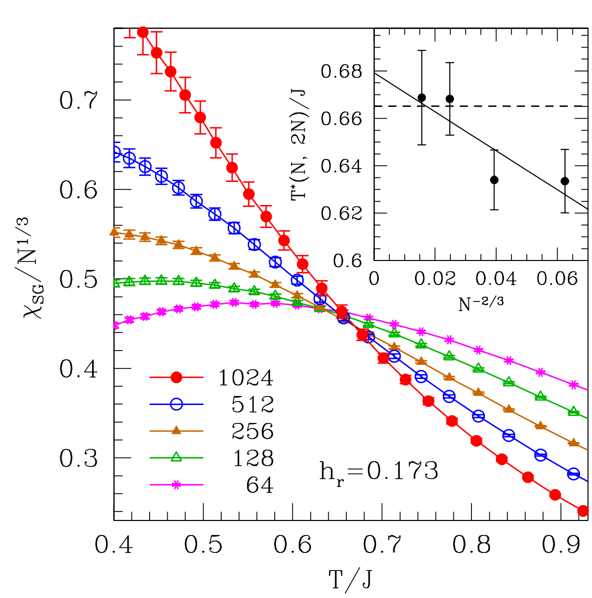

FIG. 3. (Color online) Same as Fig. 2 but for random field strength $h_r=0.173$. The final estimate of $T_c(h_r)$ is 0.685 ± 0.019 which is to be compared with the exact value of 0.6652, see Table II, which is shown as the dashed line in the inset.

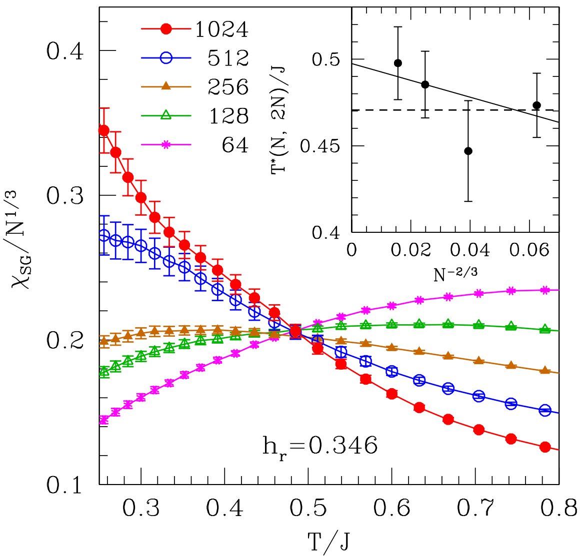

FIG. 4. (Color online) Same as Fig. 2 but for random field strength h_r=0.346. The final estimate of $T_c(h_r)$ is 0.497 ± 0.023 to be compared with the exact value of 0.4706, see Table II, which is shown as the dashed line in the inset.

$$T^{\star}(N,2N) - T_c(h_r) = \frac{A}{N^{2/3}}, \tag{70}$$

where the constant A is related to c_0 and $\widetilde{X}'(0)$. We determine $T^{\star}(N,2N)$ by a bootstrap analysis and show the results both in Table II and in the insets to Figs. 2–4. Fitting a straight line through $T^{\star}(N,2N)$ against $N^{-2/3}$ according to Eq. (70) gives estimates of T_c which shown both in Fig. 1 and Table II.

We see that, in zero field, the numerics accurately gives the exact value for T_c of 1, and for nonzero h_r, the numerics gives the correct answer to within about one sigma. Hence our analytical predictions for the AT line in Heisenberg spin glasses are well confirmed by simulations.

TABLE II. Intersection temperatures $T^{\star}(N,2N)$ and extrapolated values of $T_c(h_r)$ determined from fits to Eq. (70). Also shown is the exact value for $T_c(h_r)$ obtained as described in the text.

h_r	N	$T^{\star}(N,2N)$	$T_c(h_r)$	$T_c(h_r)$ (exact)
0	64	0.9478(61)		
0	128	0.9709(32)		
0	256	0.9832(22)		
0	512	0.9837(29)		
0	∞		0.9987(36)	1
0.173	64	0.633(13)		
0.173	128	0.634(13)		
0.173	256	0.668(15)		
0.173	512	0.680(18)		
0.173	∞		0.679(19)	0.6652
0.346	64	0.473(19)		
0.346	128	0.447(29)		
0.346	256	0.485(19)		
0.346	512	0.498(21)		
0.346	∞		0.497(23)	0.4706

V. CONCLUSIONS

We have emphasized that the appropriate symmetry breaking field for a spin glass is a random field and that, for a vector spin glass, the crucial ingredient is the random *direction* of the field. Incorporating a random field, there is a line of transitions (AT line) in vector spin glasses, just as there is in the Ising spin glass, a fact which does not seem to be widely recognized. The AT line is different from the GT [4] line since it is a transition to a phase with replica symmetry breaking but *no change in spin symmetry*.

The location of the AT line for vector spin glasses with Gaussian random fields is given by

$$\left(\frac{T}{J}\right)^2 = \chi_{SG}^0, \tag{71}$$

where χ_{SG}^0 is given by Eq. (C24). For the important case of the Heisenberg (m=3) spin glass, the simpler expression for χ_{SG}^0 is given in Eq. (C26). We have plotted the AT line numerically for several values of m in Fig. 1 and confirmed these results numerically by simulations for the case of m =3.

For the Ising case, we note that Bray and Moore [33] have obtained Eq. (44) for the spin-glass susceptibility without replicas, starting from the local mean-field equations of Thouless, Anderson, and Palmer [34] (the TAP equations). It would be interesting to see if one could derive, along similar lines, a more straightforward and nonreplica calculation of χ_{SG} for the vector spin case too.

Although it is not possible experimentally to apply a field which is random in direction to a vector spin glass, so the AT line seems to be experimentally inaccessible (except for the Ising case), one *can* detect the AT line for vector spin glasses in simulations. Whether or not at AT line exists in finite-range spin glasses, it is a crucial difference between the replica symmetry breaking (RSB) picture, where it does occur, and the droplet picture, where it does not. It has been found possible to simulate Heisenberg spin glasses for significantly larger sizes [16–28] than Ising spin glasses, so our results may give an additional avenue through which to investigate the nature of the spin-glass state.

ACKNOWLEDGMENTS

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APPENDIX A: MATRIX INVERSE FOR ISING CASE

Because the theory is invariant under permutation of the replicas, there are only three distinct values for the matrix elements:

$$A_{(\alpha\beta),(\alpha\beta)} = P, \tag{A1a}$$

$$A_{(\alpha\beta),(\alpha\gamma)} = Q, \tag{A1b}$$

$$A_{(\alpha\beta),(\gamma\delta)} = R, \tag{A1c}$$

in which α, β, γ and δ are all different. Recall that $(\alpha\beta)$ takes $n(n-1)/2$ distinct values, i.e., the pair $(\beta\alpha)$ is the same as the pair $(\alpha\beta)$.

Consider the matrix G which is the inverse of A, i.e.,

$$AG = I, \tag{A2}$$

where I is the identity matrix. We assume that G has the same structure as A and define, see Eq. (38),

$$G_{(\alpha\beta),(\alpha\beta)} = G_1, \tag{A3a}$$

$$G_{(\alpha\beta),(\alpha\gamma)} = G_2, \tag{A3b}$$

$$G_{(\alpha\beta),(\gamma\delta)} = G_3. \tag{A3c}$$

Evaluating the $(\alpha\beta),(\alpha\beta)$, the $(\alpha\beta),(\alpha\gamma)$, and the $(\alpha\beta),(\gamma\delta)$ elements of Eq. (A2) gives, respectively,

$$PG_1 + 2(n-2)QG_2 + \frac{1}{2}(n-2)(n-3)RG_3 = 1, \tag{A4a}$$

$$QG_1 + [P + (n-2)Q + (n-3)R]G_2 + \left[(n-3)Q + \frac{1}{2}(n-3)(n-4)R\right]G_3 = 0, \tag{A4b}$$

$$RG_1 + [4Q + 2(n-4)R]G_2 + \left[P + 2(n-4)Q + \frac{1}{2}(n-4)(n-5)R\right]G_3 = 0. \tag{A4c}$$

Taking $1\times(\mathrm{A4a}) - 2\times(\mathrm{A4b}) + 1\times(\mathrm{A4c})$ gives

$$(P-2Q+R)(G_1 - 2G_2 + G_3) = 1, \tag{A5}$$

so the "replicon propagator" is given by

$$G_r \equiv G_1 - 2G_2 + G_3 = \frac{1}{P-2Q+R}. \tag{A6}$$

The spin-glass susceptibility is determined from G_r according to Eq. (39). Note that Eqs. (A5) and (39) determine χ_{SG} without needing to diagonalize the matrix A. However, since the diagonalization has been done by AT, it is instructive to see that G_r is the inverse of the replicon eigenvalue λ_3, see also Ref. [19].

If we accept that λ_3 is an eigenvalue, then Eq. (A6) is obvious since the eigenvectors of A and G are the same, and the corresponding eigenvalues are the inverses of each other. Furthermore, since the inverse matrix G has the same structure as that of the original matrix A, the expressions for the eigenvalues of A in terms of the parameters P, Q, and R, are the same as the expressions for the eigenvalues of G in terms of the corresponding parameters G_1, G_2, and G_3.

APPENDIX B: MATRIX INVERSE FOR VECTOR CASE

We now have additional indices for the spin components, and to avoid confusion in notation, we will use Greek letters $\alpha,\beta,\gamma,\delta,\epsilon$ for replicas and μ,ν,κ,σ for spin indices. A row or column of the matrix will then involve four indices $(\alpha\beta),\mu\nu$, in which the order of the replica pair $(\alpha\beta)$ is unimportant, i.e., $(\beta\alpha)$ is the same as $(\alpha\beta)$, but the order of the spin indices does matter because $S_\alpha^\mu S_\beta^\nu$ is not the same as $S_\alpha^\nu S_\beta^\mu$.

Another new feature which appears when we deal with vector spins is the appearance of terms with both replicas equal, $(\alpha\alpha)$. These do not appear for the Ising case because $(S_\alpha)^2$ is equal to 1, a constant. However, $(S_\alpha^\mu)^2$ is not a constant for $m>1$ and so we now need to include $(\alpha\alpha)$ terms in the analysis.

Hence we shall inverse of a matrix Z of size $\frac{1}{2}n(n+1)m^2$ whose elements are given by

$$Z^{\mu\nu,\kappa\sigma}_{(\alpha\beta),(\gamma\delta)} = \delta_{(\alpha\beta)(\gamma\delta)}\delta_{\mu\kappa}\delta_{\nu\sigma} - (\beta J)^2\{[\langle S_\mu^\alpha S_\nu^\beta S_\kappa^\gamma S_\sigma^\delta\rangle]_z - [\langle S_\mu^\alpha S_\nu^\beta\rangle] \times[\langle S_\kappa^\gamma S_\sigma^\delta\rangle]_z\}. \tag{B1}$$

Ignoring for now the spin indices (which will be put back later) we consider the following matrix of dimension $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)$,

$$Z = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{B2}$$

in which A is the matrix of dimension $\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)$ with rows and columns labeled by two distinct replicas $(\alpha\beta)$ defined in Eq. (A1), C is an $n\times n$ matrix with rows and columns labeled by a single replica $(\alpha\alpha)$, and B is a matrix with $\frac{1}{2}n(n-1)$ rows and n columns.

We now discuss each of these matrices in turn.

(i) The elements of A are given by Eq. (A1).

(ii) The elements of B are

$$B_{(\alpha\beta),(\alpha\alpha)} = S, \tag{B3a}$$

$$B_{(\alpha\beta),(\gamma\gamma)} = T, \tag{B3b}$$

in which α, β and γ are all different.

(iii) The elements of C are

$$C_{(\alpha\alpha),(\alpha\alpha)} = U, \tag{B4a}$$

$$C_{(\alpha\alpha),(\beta\beta)} = V, \tag{B4b}$$

in which α and β are different.

Now we add the Cartesian spin indices. The result is that each element of the matrix Z in Eq. (B2) becomes an $m^2\times m^2$ matrix with rows and columns labeled by a pair of spin component indices μ and ν, each of which runs over values from 1 to m.

A simplification is that the only nonzero elements are those where each Cartesian spin component occurs an even number of times (combining the row and column indices). Hence each $m^2\times m^2$ matrix breaks up into different blocks.

There is one $m\times m$ block, $(\mu\mu,\nu\nu)$, where $\mu=1,\ldots,n$, $\nu=1,\ldots,m$, and $m(m-1)/2$ blocks of size 2, $(\mu\nu,\mu\nu)$ and $(\mu\nu,\nu\mu)$, where μ and $\nu(\neq\mu)$ are fixed.

Consider, for example, one of the elements in A with value P, see Eq. (A1). This is now expanded into an $m^2\times m^2$ matrix which is block diagonalized into:

(i) One $m\times m$ matrix, with rows and columns labeled by $\mu\mu(\mu=1,\ldots,m)$,

$$\begin{pmatrix} P_L & P_T\cdots P_T \\ P_T & P_L\cdots P_T \\ \vdots & \vdots\ddots\vdots \\ P_T & P_T\cdots P_L \end{pmatrix}, \tag{B5}$$

where the diagonal elements (to which we give the subscript L) are different from the off-diagonal elements (to which we give the subscript T).

(ii) $m(m-1)/2$ identical matrices of size 2×2, with rows and columns labeled by $\mu\nu$ and $\nu\mu$ (for fixed μ and ν with $\mu\neq\nu$),

$$\begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix}, \tag{B6}$$

in which we give the subscript "1" to the (equal) diagonal elements and the subscript "2" to the off-diagonal elements.

The R, S, T, U, and V elements of the replica matrix, in Eqs. (A1), (B3), and (B4), expand out into the same block structure in spin-component space.

For Q, there are some differences involving the Q_1 and Q_2 elements [19], but these will not be needed here.

As for the Ising case, we assume that G, the matrix inverse of Z, has the same structure as Z itself. In particular, we define

$$G^{\mu\mu,\mu\mu}_{(\alpha\beta),(\alpha\beta)}=G_{1L},\quad G^{\mu\mu,\nu\nu}_{(\alpha\beta),(\alpha\beta)}=G_{1T}(\mu\neq\nu), \tag{B7a}$$

$$G^{\mu\mu,\mu\mu}_{(\alpha\beta),(\alpha\gamma)}=G_{2L},\quad G^{\mu\mu,\nu\nu}_{(\alpha\beta),(\alpha\gamma)}=G_{2T}(\mu\neq\nu), \tag{B7b}$$

$$G^{\mu\mu,\mu\mu}_{(\alpha\beta),(\gamma\delta)}=G_{3L},\quad G^{\mu\mu,\nu\nu}_{(\alpha\beta),(\gamma\delta)}=G_{3T}(\mu\neq\nu), \tag{B7c}$$

$$G^{\mu\mu,\mu\mu}_{(\alpha\beta),(\alpha\alpha)}=G_{4L},\quad G^{\mu\mu,\nu\nu}_{(\alpha\beta),(\alpha\alpha)}=G_{4T}(\mu\neq\nu), \tag{B7d}$$

$$G^{\mu\mu,\mu\mu}_{(\alpha\beta),(\gamma\gamma)}=G_{5L},\quad G^{\mu\mu,\nu\nu}_{(\alpha\beta),(\gamma\gamma)}=G_{5T}(\mu\neq\nu), \tag{B7e}$$

where α, β, γ, and δ are all different. Considering various matrix elements of both sides of $ZG=I$ we get

$$\begin{aligned}&P_LG_{1L}+(m-1)P_TG_{1T}+2(n-2)[Q_LG_{2L}+(m-1)Q_TG_{2T}]\\&\quad+\frac{1}{2}(n-2)(n-3)[R_LG_{3L}+(m-1)R_TG_{3T}]+2[S_LG_{4L}\\&\quad+(m-1)S_TG_{4T}]+(n-2)[T_LG_{5L}+(m-1)T_TG_{5T}]=1,\end{aligned} \tag{B8a}$$

$$\begin{aligned}&P_LG_{1T}+P_TG_{1L}+(m-2)P_TG_{1T}+2(n-2)[Q_LG_{2T}+Q_TG_{2L}\\&\quad+(m-2)Q_TG_{2T}]+\frac{1}{2}(n-2)(n-3)[R_LG_{3T}+R_TG_{3L}\\&\quad+(m-2)R_TG_{3T}]+2[S_LG_{4T}+S_TG_{4L}+(m-2)S_TG_{4T}]\\&\quad+(n-2)[T_LG_{5T}+T_TG_{5L}+(m-2)T_TG_{5T}]=0,\end{aligned} \tag{B8b}$$

$$\begin{aligned}&Q_LG_{1L}+(m-1)Q_TG_{1T}+[P_L+(n-2)Q_L+(n-3)R_L]G_{2L}\\&\quad+(m-1)[P_T+(n-2)Q_T+(n-3)R_T]G_{2T}+\Bigg[(n-3)Q_L\\&\quad+\frac{1}{2}(n-3)(n-4)R_L\Bigg]G_{3L}+(m-1)\Bigg[(n-3)Q_T+\frac{1}{2}\\&\quad\times(n-3)(n-4)R_T\Bigg]G_{3T}+2[S_LG_{4L}+(m-1)S_TG_{4T}]\\&\quad+(n-2)[T_LG_{5L}+(m-1)T_TG_{5T}]=0,\end{aligned} \tag{B8c}$$

$$\begin{aligned}&Q_LG_{1T}+Q_TG_{1L}+(m-2)Q_TG_{1T}+[P_L+(n-2)Q_L\\&\quad+(n-3)R_L]G_{2T}+[P_T+(n-2)Q_T+(n-3)R_T]G_{2L}\\&\quad+(m-2)[P_T+(n-2)Q_T+(n-3)R_T]G_{2T}\\&\quad+\Bigg[(n-3)Q_L+\frac{1}{2}(n-3)(n-4)R_L\Bigg]G_{3T}+\Bigg[(n-3)Q_T\\&\quad+\frac{1}{2}(n-3)(n-4)R_T\Bigg]G_{3L}+(m-2)\Bigg[(n-3)Q_T+\frac{1}{2}\\&\quad\times(n-3)(n-4)R_T\Bigg]G_{3T}+2[S_LG_{4T}+S_TG_{4L}\\&\quad+(m-2)S_TG_{4T}]+(n-2)\\&\quad\times[T_LG_{5T}+T_TG_{5L}+(m-2)T_TG_{5T}]=0,\end{aligned} \tag{B8d}$$

$$\begin{aligned}&R_LG_{1L}+(m-1)R_TG_{1T}+[4Q_L+2(n-4)R_L]G_{2L}+(m-1)\\&\quad\times[4Q_T+2(n-4)R_T]G_{2T}+\Bigg[P_L+2(n-4)Q_L+\frac{1}{2}(n-4)\\&\quad\times(n-5)R_L\Bigg]G_{3L}+(m-1)\Bigg[P_T+2(n-4)Q_T+\frac{1}{2}(n-4)\\&\quad\times(n-5)R_T\Bigg]G_{3T}+2[S_LG_{4L}+(m-1)S_TG_{4T}]+(n-2)\\&\quad\times[T_LG_{5L}+(m-1)T_TG_{5T}]=0,\end{aligned} \tag{B8e}$$

$$\begin{aligned}&R_LG_{1T}+R_TG_{1L}+(m-2)R_TG_{1T}+[4Q_L+2(n-4)R_L]\\&\quad\times G_{2T}+[4Q_T+2(n-4)R_T]G_{2L}+(m-2)[4Q_T+2\\&\quad\times(n-4)R_T]G_{2T}+\Bigg[P_L+2(n-4)Q_L+\frac{1}{2}(n-4)\\&\quad\times(n-5)R_L\Bigg]G_{3T}+\Bigg[P_T+2(n-4)Q_T+\frac{1}{2}(n-4)\\&\quad\times(n-5)R_T\Bigg]G_{3L}+(m-2)\Bigg[P_T+2(n-4)Q_T+\frac{1}{2}(n-4)\end{aligned}$$

$$\times(n-5)R_T\Big]G_{3T}+2[S_LG_{4T}+S_TG_{4L}+(m-2)S_TG_{4T}]$$
$$+(n-2)[T_LG_{5T}+T_TG_{5L}+(m-2)T_TG_{5T}]=0. \quad \text{(B8f)}$$

Forming appropriate linear combinations of Eqs. (B8) gives

$$\{[G_{1L}+(m-1)G_{1T}]-2[G_{2L}+(m-1)G_{2T}]+[G_{3L}+(m-1)G_{3T}]\}\times\{[P_L+(m-1)P_T]-2[Q_L+(m-1)Q_T]+[R_L+(m-1)R_T]\}=1, \quad \text{(B9)}$$

so the "replicon" propagator is given by

$$G_r\equiv[G_{1L}+(m-1)G_{1T}]-2[G_{2L}+(m-1)G_{2T}]+[G_{3L}+(m-1)G_{3T}]$$
$$=([P_L+(m-1)P_T]-2[Q_L+(m-1)Q_T]+[R_L+(m-1)R_T])^{-1}. \quad \text{(B10)}$$

APPENDIX C: AVERAGES OVER SPIN DIRECTIONS

To evaluate the spin-glass susceptibility we need to compute averages over spin directions. Consider, for example,

$$Z=\int d\Omega_m \exp[\mathbf{H}\cdot\mathbf{e}], \quad \text{(C1)}$$

where the integral is over the surface, Ω_m, of a sphere of unit radius, **e** is a unit vector whose direction is to be integrated over, and **H** is a fixed vector.

Working in polar coordinates, with the polar axis along the direction of the fixed vector **H**, the integral in Eq. (C1) can be expressed entirely in terms of the polar angle θ since $\exp[\mathbf{H}\cdot\mathbf{e}]=\exp[H\cos\theta]$ and $\int d\Omega_m=C_m\int_0^\pi \sin^{m-2}\theta$ for a constant C_m. To determine C_m we note the following results [35,36]:

$$\Omega_m\equiv\int d\Omega_m=\frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}, \quad \text{(C2)}$$

$$\int_0^\pi \sin^{m-2}\theta d\theta=\sqrt{\pi}\frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}, \quad \text{(C3)}$$

where Γ is the usual Gamma function, which gives

$$C_m=\frac{2\pi^{(m-1)/2}}{\Gamma\left(\frac{m-1}{2}\right)}. \quad \text{(C4)}$$

Hence Z can be written as

$$Z=\frac{2\pi^{(m-1)/2}}{\Gamma\left(\frac{m-1}{2}\right)}\int_0^\pi \exp[H\cos\theta]\sin^{m-2}\theta d\theta. \quad \text{(C5)}$$

The integral is given in terms of a modified Bessel function [36], and we have

$$Z=(2\pi)^{m/2}\frac{I_{m/2-1}(H)}{H^{m/2-1}}. \quad \text{(C6)}$$

Of greater interest are averages of the spins. Consider first

$$\langle S_\mu\rangle=m^{1/2}\langle e_\mu\rangle \quad \text{(C7)}$$

$$=m^{1/2}\frac{1}{Z}\frac{\partial Z}{\partial H^\mu} \quad \text{(C8)}$$

$$=m^{1/2}\frac{1}{Z}\frac{H^\mu}{H}\frac{\partial Z}{\partial H}. \quad \text{(C9)}$$

Using [36]

$$\frac{d}{dx}\left[\frac{I_{m/2-1}(x)}{x^{m/2-1}}\right]=\frac{I_{m/2}(x)}{x^{m/2-1}}, \quad \text{(C10)}$$

we get

$$\langle S_\mu\rangle=m^{1/2}\frac{H^\mu}{H}\frac{I_{m/2}(H)}{I_{m/2-1}(H)}. \quad \text{(C11)}$$

We shall also need

$$\langle S_\mu S_\nu\rangle=m\frac{1}{Z}\frac{H^\mu}{H}\frac{\partial}{\partial H}\left(\frac{H^\nu}{H}\frac{\partial Z}{\partial H}\right) \quad \text{(C12)}$$

$$=\frac{m}{I_{m/2-1}(H)}\left[\delta_{\mu\nu}\frac{I_{m/2}(H)}{H}+\frac{H_\mu H_\nu}{H^2}I_{m/2+1}(H)\right], \quad \text{(C13)}$$

in which we again used Eq. (C10).

To apply these results, we note that, in the presence of a external random field, the replica symmetric solution predicts that **H** is given by

$$\mathbf{H}=\beta m^{1/2}(J^2q+h_r^2)\mathbf{z}, \quad \text{(C14)}$$

where each component of the variable **z** is a Gaussian random variable with zero mean and standard deviation unity. To see this, compare Eq. (53) with Eq. (C1) and note that the spins are of length $m^{1/2}$ according to Eq. (2). Hence each component of **H** has zero mean and standard deviation given by

$$\Delta=\beta m^{1/2}(J^2q+h_r^2)^{1/2}. \quad \text{(C15)}$$

As for the Ising case, we denote averages over **H** or equivalently over **z** [**H** and **z** are related by Eq. (C14)], by $[\cdots]_z$ and so, for example, in situations which only involve the magnitude of **H**, we have

$$[f(H)]_z=\int_{-\infty}^{\infty}\left(\prod_\mu \frac{dH_\mu}{(2\pi)^{1/2}\Delta}\right)e^{-\Sigma_\mu H_\mu^2/2\Delta^2}f(H)dH$$

$$=\frac{\Omega_m}{(2\pi)^{m/2}\Delta^m}\int_0^\infty H^{m-1}\exp\left(-\frac{H^2}{2\Delta^2}\right)f(H)dH$$

$$= \frac{2^{1-m/2}}{\Delta^m \Gamma\left(\frac{m}{2}\right)} \int_0^\infty H^{m-1} \exp\left(-\frac{H^2}{2\Delta^2}\right) f(H) dH \tag{C16}$$

$$= \frac{2^{1-m/2}}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty z^{m-1} e^{-z^2/2} f(\Delta z) dz, \tag{C17}$$

where we used the result for Ω_m in Eq. (C2), and Δ is given by Eq. (C15).

Using these results we now calculate the spin-glass order parameter q, which is given by

$$\begin{aligned} q &= \frac{1}{m}\left[\sum_{\mu=1}^{m} \langle S_\mu \rangle^2\right]_z \\ &= \sum_{\mu=1}^{m} \frac{(H^\mu)^2}{H^2}\left[\left(\frac{I_{m/2}(H)}{I_{m/2-1}(H)}\right)^2\right]_z \\ &= \left[\left(\frac{I_{m/2}(H)}{I_{m/2-1}(H)}\right)^2\right]_z \\ &= \frac{2^{1-m/2}}{\Delta^m \Gamma\left(\frac{m}{2}\right)} \int_0^\infty dH H^{m-1} \exp\left(-\frac{H^2}{2\Delta^2}\right) \\ &\quad \times \left(\frac{I_{m/2}(H)}{I_{m/2-1}(H)}\right)^2 \\ &= \frac{2^{1-m/2}}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty dz z^{m-1} e^{-z^2/2} \left(\frac{I_{m/2}(\Delta z)}{I_{m/2-1}(\Delta z)}\right)^2, \end{aligned} \tag{C18}$$

where we used Eq. (C11). Equation (C18), with Δ given by Eq. (C15), is the self-consistent equation which determines q. As an example, for $m=1$, $I_{m/2}(H)/I_{m/2-1}(H)=\tanh(H)$ $=\tanh(\Delta z)$, and we recover the result for q in Eq. (29). For general m, expanding the Bessel functions for small argument [36], we get

$$q = \left[\frac{1}{m^2}H^2 - \frac{2}{m^3(m+2)}H^4 + \frac{5m+12}{m^4(m+2)^2(m+4)}H^6 + O(H^8)\right]_z. \tag{C19}$$

If we do the Gaussian integrals, set $h_r=0$, and solve for q, we find

$$q = t + \frac{1}{m+2}t^2 + O(t^3), \quad (h_r=0), \tag{C20}$$

where t, the reduced temperature, is given by $t=(T_c-T)/T_c$, and the zero-field transition temperature is $T_c=J$, see Eq. (5).

Our main goal is to compute the eigenvalue λ_{3S} since this determines the spin-glass susceptibility, the divergence of which indicates the location of the AT line. From Eqs. (61), (B1), (B2), and (B5), we find the fairly simple expression

$$\lambda_{3S} = 1 - (\beta J)^2 \frac{1}{m}\sum_{\mu,\nu} [\langle S_\mu S_\nu\rangle^2 - 2\langle S_\mu S_\nu\rangle\langle S_\mu\rangle\langle S_\nu\rangle + \langle S_\mu\rangle^2\langle S_\nu\rangle^2]_z, \tag{C21}$$

which is instructive to write in the following form:

$$\lambda_{3S} = 1 - (\beta J)^2 \chi_{SG}^0, \tag{C22}$$

where χ_{SG}^0 is a single-site spin-glass susceptibility,

$$\chi_{SG}^0 = \frac{1}{m}\sum_{\mu,\nu}[(\langle S_\mu S_\nu\rangle - \langle S_\mu\rangle\langle S_\nu\rangle)^2]_z. \tag{C23}$$

Evaluating the spin averages in Eq. (C23) using Eqs. (C11) and (C13) gives

$$\begin{aligned} \chi_{SG}^0 = m\Bigg[& \frac{1}{I_{m/2-1}^2(H)}\left\{\frac{m}{H^2}I_{m/2}^2(H) + \frac{2}{H}I_{m/2}(H)I_{m/2+1}(H) + I_{m/2+1}^2(H)\right\} - \frac{2}{I_{m/2-1}^3(H)}\left\{\frac{1}{H}I_{m/2}^3(H) \right. \\ & \left. + I_{m/2}^2(H)I_{m/2+1}(H)\right\} + \left\{\frac{I_{m/2}(H)}{I_{m/2-1}(H)}\right\}^4\Bigg]_z. \end{aligned} \tag{C24}$$

We recall that the average over H is evaluated according to Eq. (C16). For the Ising case, $m=1$, Eq. (C24) simplifies to

$$\chi_{SG}^0 = [1 - 2\tanh^2 H + \tanh^4 H]_z, \tag{C25}$$

in agreement with Eq. (45). For the Heisenberg case, $m=3$, Eq. (C24) becomes

$$\chi_{SG}^0 = 3\left[\frac{3 + 2H^2 - 4H\coth(H)}{H^4} + \frac{1}{\sinh^4(H)}\right]_z, \tag{C26}$$

which, together with Eqs. (C16) and (C22), gives λ_{3S}. Equations (C24) and (C26) appear to be a new results. Expanding the Bessel functions [36] for small H gives

$$\chi_{SG}^0 = \left[1 - \frac{2}{m^2}H^2 + \frac{5m+16}{m^3(m+2)^2}H^4 + O(H^6)\right]_z. \tag{C27}$$

Let us evaluate q and λ_{3S} near $T=T_c(=J)$, the zero-field transition temperature, and for small h_r. Using Eqs. (C19) and (C27), and doing the Gaussian integrals, we find

$$q = \tilde{\Delta}^2 - 2\tilde{\Delta}^4 + \frac{5m+12}{m+2}\tilde{\Delta}^6 + \cdots, \tag{C28}$$

$$\lambda_{3S} = 1 - (\beta J)^2\left[1 - 2\tilde{\Delta}^2 + \frac{5m+16}{m+2}\tilde{\Delta}^4 + \cdots\right], \tag{C29}$$

where

$$\tilde{\Delta}^2 \equiv \frac{\Delta^2}{m} = \beta^2(J^2 q + h_r^2). \tag{C30}$$

Combining Eqs. (C29) and (C28) and assuming

$$h_r \ll t \equiv (T_c - T)/T_c \ll 1, \tag{C31}$$

which will be valid at and below the AT line near T_c, we get

$$\lambda_{3S} = \left(\frac{h_r}{J}\right)^2 \frac{1}{q} - \frac{4}{m+2} q^2. \quad \text{(C32)}$$

In the limits of Eq. (C31), we have $q = t + O(t^2)$, see Eqs. (C20) and (C35), and so

$$\lambda_{3S} = \left(\frac{h_r}{J}\right)^2 \frac{1}{t} - \frac{4}{m+2} t^2, \quad (h_r \ll t), \quad \text{(C33)}$$

which changes sign for

$$\left(\frac{h_r}{J}\right)^2 = \frac{4}{m+2} t^3. \quad \text{(C34)}$$

Equation (C34) gives the location of the AT line for an m-component spin glass near the zero-field transition. The replica symmetric solution is unstable at lower temperatures and fields since $\lambda_{3S} < 0$ in that region according to Eq. (C33). Note that Eq. (C34) correctly gives the AT result that $h_r^2 = (4/3)t^3$ for $m=1$ (This is a valid comparison even though AT used a uniform field since, to lowest order in t, the location of the AT line in the Ising case is the same [7] for random and uniform fields.) On the AT line we find that the spin-glass order parameter is given by

$$q = t + \frac{3}{m+2} t^2 + O(t^3), \quad \text{(on AT line)}. \quad \text{(C35)}$$

In the opposite limit, $T \to 0$, we find, using properties of the Bessel functions, that

$$\frac{h_r(T=0)}{J} = \frac{1}{\sqrt{m-2}} \quad (m > 2), \quad \text{(C36)}$$

while $h_r(T \to 0)$ diverges for $m \le 2$. For the Ising case, we get

$$\frac{h_r(T \to 0)}{J} = \sqrt{\frac{8}{9\pi}\frac{J}{T}} \quad (m=1), \quad \text{(C37)}$$

in agreement with Bray [7].

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