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Scaling and universality in the two-dimensional Ising model with a magnetic field

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The scaling function of the two-dimensional Ising model on the square and triangular lattices is obtained numerically via Baxter's variational corner transfer-matrix approach. The use of Aharony-Fisher nonlinear scaling variables allowed us to perform calculations sufficiently away from the critical point and to confirm all predictions of the scaling and universality hypotheses. Our results are in excellent agreement with quantum field theory calculations of Fonseca and Zamolodchikov as well as with many previously known exact and numerical calculations, including susceptibility results by Barouch, McCoy, Tracy, and Wu.

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The principles of scaling and universality (see, e.g., [1]) play important roles in the theory of phase transition and critical phenomena. The scaling assumption asserts that observable quantities exhibit power law singularities in the variable $\Delta T = T - T_c$ in the vicinity of the critical temperature T_c, with coefficients being functions of certain dimensionless combinations of available parameters, e.g., the magnetic field H and ΔT. The universality hypothesis states that the leading singular part of the free energy is a universal scaling function which is the same for all systems in a given "universality class." In two dimension classes of universal critical behavior are well understood—they are classified by conformal field theory (CFT) [2].

However, it appears that despite numerous analytical and numerical results (cited below), the full picture of scaling and universality has never been convincingly demonstrated through numerical calculations in lattice models. Our aim is to do this. Here we consider the planar nearest-neighbor Ising model on the regular square and triangular lattices, which has already played a prominent role in the development of the theory of phase transition and critical phenomena [2–8]. Its partition function reads

$$Z = \sum_{\sigma} \exp\left\{\beta \sum_{\langle ij\rangle} \sigma_i \sigma_j + H \sum_i \sigma_i\right\}, \quad \sigma_i = \pm 1, \tag{1}$$

where the first sum in the exponent is taken over all edges, the second over all sites, and the outer sum over all spin configurations $\{\sigma\}$ of the lattice. The constants H and β denote the (suitably normalized) magnetic field and inverse temperature. The free energy, magnetization, and magnetic susceptibility are defined as

$$F = -\lim_{N\to\infty} \frac{1}{N} \log Z, \quad M = -\frac{\partial F}{\partial H}, \quad \chi = -\frac{\partial^2 F}{\partial H^2}, \tag{2}$$

where N is the number of lattice sites. The model exhibits a second-order phase transition at $H=0$ and $\beta=\beta_c$, where

$$\beta_c^{(s)} = \tfrac{1}{2}\log(1+\sqrt{2}), \quad \beta_c^{(t)} = \tfrac{1}{4}\log 3 \tag{3}$$

for the square [3] and triangular [9] lattices, respectively.

The scaling and universality hypotheses predict that the leading singular part, $F_{sing}(\Delta\beta, H)$, of the free energy in the vicinity of the critical point, $\Delta\beta = \beta - \beta_c \sim 0$, $H \sim 0$, can be expressed through a universal function $\mathcal{F}(m,h)$,

$$F_{sing}(\Delta\beta, H) = \mathcal{F}[m(\Delta\beta,H), h(\Delta\beta,H)], \tag{4}$$

where $\Delta\beta$ and H enter the right-hand side only through nonlinear scaling variables [10],

$$m = m(\Delta\beta,H) = O(\Delta\beta) + O[(\Delta\beta)^3] + O(H^2) + \cdots,$$

$$h = h(\Delta\beta,H) = O(H) + HO(\Delta\beta) + O(H^3)\cdots, \tag{5}$$

which are analytic functions of $\Delta\beta$ and H. The coefficients in these expansions depend on the details of the microscopic interaction (for instance they are different for the square and triangular lattices), but the function $\mathcal{F}(m,h)$ is the same for all models in the two-dimensional (2D) Ising model universality class. It can be written as [11]

$$\mathcal{F}(m,h) = \frac{m^2}{8\pi}\log m^2 + h^{16/15}\Phi(\eta), \quad \eta = \frac{m}{h^{8/15}}, \tag{6}$$

where $\Phi(\eta)$ is a universal scaling function of a single variable η (the dimensionless scaling parameter), normalized such that

$$\mathcal{F}(m,0) = \frac{m^2}{8\pi}\log m^2. \tag{7}$$

The scaling function [Eq. (6)] is of much interest as it controls all thermodynamic properties of the Ising model in the critical domain. Although there are many exact results (obtained through exact solutions at $H=0$ and all β [3,4,12–14] and at $m=0$ and all h [8,15–21]; these data are collected in [22]) as well as much numerical data [23–28] about this function, its complete analytic characterization is still lacking. Recently [11] function (6) have been thoroughly studied in the framework of the "Ising field theory" (IFT). The authors of [11] made extensive numerical calculations of the scaling function $\Phi(\eta)$ for real and complex values of η.

One of the motivations of our work was to confirm and extend the field theory results of [11] through *ab initio* calculations, directly from the original lattice formulation [Eq. (1)] of the Ising model. Here we give a brief summary of our

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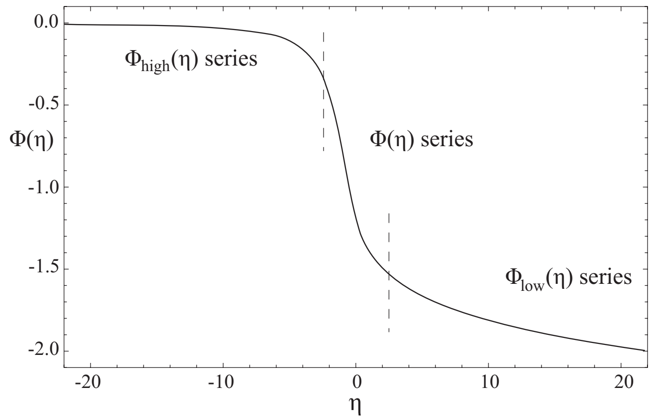

FIG. 1. The scaling function $\Phi(\eta)$ in the three regions, separated by the dashed lines at $\eta \simeq \pm 2.43$, can be approximated with relative precision 10^{-4} by the series (9), (10), and (8). The required coefficients to achieve this precision given in Table I.

results for the triangular and square lattices (the latter were previously reported in [29]). We used Baxter's variational approach based on the corner transfer-matrix (CTM) method [30,31]. The original approach was enhanced by an improved iteration scheme, known as the CTM renormalization group (CTMRG) [32]. The main advantage of this approach over other numerical schemes (e.g., the row-to-row transfer matrix method) is that it is formulated directly in the limit of an *infinite* lattice. Its accuracy depends on the magnitude of truncated eigenvalues of the CTM, which is at our control, rather than the size of the lattice. The use of the nonlinear scaling variables [Eq. (5)] allowed us to perform calculations sufficiently away from the critical point with a reliable convergence of the algorithm. In total we have calculated about 10 000 data points for different values of the temperature and magnetic field on the square lattice and more than 5000 for the triangular one.

The results for the scaling function $\Phi(\eta)$ are shown in Fig. 1. As seen from the picture all points collapse on a smooth curve, shown by the solid line (as expected, the curve is the *same* for the square and triangular lattices). The spread of the points at any fixed value η does not exceed $10^{-7}-10^{-6}$ relative accuracy. This gives a convincing demonstration of the scaling and universality in the 2D Ising model. Furthermore, our numerical results for $\Phi(\eta)$ remarkably confirm the field theory calculations [11] to within all six significant digits presented therein.

For further reference we write down asymptotic expansions of the function $\Phi(\eta)$ for large values of η on the real line

$$\Phi_{low}(\eta) = \eta^2 \sum_{k=1}^{\infty} \tilde{G}_k \eta^{-15k/8}, \quad \eta \to +\infty, \tag{8}$$

$$\Phi_{high}(\eta) = \eta^2 \sum_{k=1}^{\infty} G_{2k} |\eta|^{-30k/8}, \quad \eta \to -\infty, \tag{9}$$

and convergent series for small values of η,

$$\Phi(\eta) = -\frac{\eta^2}{8\pi} \log \eta^2 + \sum_{k=0}^{\infty} \Phi_k \eta^k. \tag{10}$$

Several first coefficients of the above expansion are known exactly. The coefficient $\tilde{G}_1$ has a simple explicit expression [5]; the coefficients G_2 and $\tilde{G}_2$ have integral expressions [12,13] involving solutions of the Painlevé III equation. They were numerically evaluated to very high precision (50 digits) in [33]. The coefficients Φ_0 and Φ_1 were analytically calculated in [15,34], respectively. The numerical value of Φ_1 (which requires certain quadratures) was found in [29]. The above values are quoted in the last column of Table I.

In what follows we exclude the temperature variable β in favor of a new variable

$$\tau = \begin{cases} (1-\sinh^2 2\beta)/(2\sinh 2\beta), & \text{(square lattice)} \\ (e^{-\beta} - e^{\beta}\sinh 2\beta)/(\sinh 2\beta)^{1/2}, & \text{(triangular lattice)}, \end{cases} \tag{11}$$

which is vanishing for $\beta = \beta_c$ and positive for $\beta < \beta_c$ (above the critical temperature). Another useful variable

$$k^2 = \begin{cases} 16e^{8\beta}/(e^{4\beta}-1)^4, & \text{(square lattice)} \\ 16e^{4\beta}/[(e^{4\beta}-1)^3(e^{4\beta}+3)], & \text{(triangular lattice)}. \end{cases} \tag{12}$$

The lattice free energy for $\tau, H \to 0$,

$$F(\tau,H) = F_{sing}(\tau,H) + F_{reg}(\tau,H) + F_{sub}(\tau,H), \tag{13}$$

contains leading universal part [Eq. (4)], regular terms $F_{reg}(\tau,H)$, which are analytic in τ and H, and subleading singular terms $F_{sub}(\tau,H)$, which are nonanalytic, but less singular than the first term in Eq. (13). Therefore, to extract the universal scaling function from the lattice calculations one should be able to isolate and subtract the regular and subleading singular terms. Moreover, one needs to know the explicit form of the nonlinear scaling variables [Eq. (5)]. In principle, all this information can be determined entirely from numerical calculations (provided one assumes the values of exponents of the subleading terms, predicted by the analysis [33,35] of the CFT irrelevant operators, contributing to the free energy [Eq. (13)]). Much more accurate results can be obtained if the numerical work is combined with known exact results. Namely, the zero-field free energy reads [3,9]

$$F^{(s)}(\tau,0) = -\frac{1}{2}\log(4\sinh 2\beta) - \frac{1}{8\pi^2}\int\int_0^{2\pi} d\phi_1 d\phi_2 \times \log(2\sqrt{1+\tau^2} - \cos\phi_1 - \cos\phi_2),$$

$$F^{(t)}(\tau,0) = -\frac{1}{2}\log(4\sinh 2\beta) - \frac{1}{8\pi^2}\int\int_0^{2\pi} d\phi_1 d\phi_2 \times \log[3+\tau^2 - \cos\phi_1 - \cos\phi_2 - \cos(\phi_1+\phi_2)], \tag{14}$$

where the superscripts (s) and (t) stand for the square and

TABLE I. Values of G_n, $\tilde{G}_n$, and Φ_n (higher coefficients available on request). Earlier results are also quoted.

	Triangular lattice CTM	Square lattice CTM	IFT [11]	References	
$\tilde{G}_1$	−1.357838341706595(2)	−1.3578383417066(1)	−1.35783835	−1.357838341706595496…	[5]
$\tilde{G}_2$	−0.048953289720(2)	−0.048953289720(1)	−0.0489589	−0.0489532897203…	[12,13,33]
$\tilde{G}_3$	0.0388639290(1)	0.038863932(3)	0.0388954	0.0387529 [36]; 0.03893	[28]
$\tilde{G}_4$	−0.068362121(1)	−0.068362119(2)	−0.0685060	−0.0685535 [36]; −0.0685(2)	[24]
$\tilde{G}_5$	0.18388371(1)	0.18388370(1)	0.18453		
$\tilde{G}_6$	−0.659170(1)	−0.6591714(1)	−0.66215		
G_2	−1.84522807823(1)	−1.8452280782328(2)	−1.8452283	−1.845228078232838…	[12,13,33]
G_4	8.3337117508(1)	8.333711750(5)	8.33410	8.33370(1)	[25]
G_6	−95.16897(3)	−95.16896(1)	−95.1884	−95.1689(4)	[25]
Φ_0	−1.197733383797993(1)	−1.197733383797993(1)	−1.1977320	−1.19773338379799339…	[15]
Φ_1	−0.3188101248906(1)	−0.318810124891(1)	−0.3188192	−0.31881012489061…	[29,34]
Φ_2	0.1108861966832(3)	0.110886196683(2)	0.1108915		
Φ_3	0.01642689465(1)	0.01642689465(2)	0.0164252		
Φ_4	$-2.6399783(1)\times 10^{-4}$	$-2.639978(1)\times 10^{-4}$	-2.64×10^{-4}		
Φ_5	$-5.140526(1)\times 10^{-4}$	$-5.140526(1)\times 10^{-4}$	-5.14×10^{-4}		
Φ_6	$2.08866(1)\times 10^{-4}$	$2.08865(1)\times 10^{-4}$	2.09×10^{-4}		
Φ_7	$-4.481969(2)\times 10^{-5}$	$-4.4819(1)\times 10^{-5}$	-4.48×10^{-5}		
Φ_8	$3.194(1)\times 10^{-7}$		3.16×10^{-7}		

triangular lattices, respectively. Write the nonlinear variables [Eq. (5)] in the form

$$m(\tau,H) = -C_\tau \tau a(\tau) + H^2 b(\tau) + O(H^4),$$

$$h(\tau,H) = C_h H[c(\tau) + H^2 d(\tau) + O(H^4)], \tag{15}$$

where $a(0)=c(0)=1$, $h(\tau,H)=-h(\tau,-H)$, and write the regular part in Eq. (13) as

$$F_{reg}(\tau,H) = A(\tau) + H^2 B(\tau) + O(H^4). \tag{16}$$

As shown in [33], the most singular subleading term, contributing to Eq. (13) is of the order of $\tau^{9/4}H^2 \sim m^6$ for the square lattice and $\tau^{13/4}H^2 \sim m^8$ for the triangular lattice.

Rewriting Eq. (14) in form (13) plus regular terms, one obtains

$$C_\tau^{(s)} = \sqrt{2}, \quad C_\tau^{(t)} = 3^{-1/4}\sqrt{2} \tag{17}$$

and

$$a^{(s)}(\tau) = 1 - \frac{3}{16}\tau^2 + \frac{137}{1536}\tau^4 + O(\tau^6),$$

$$a^{(t)}(\tau) = 1 - \frac{1}{24}\tau^2 + \frac{47}{10\,368}\tau^4 + O(\tau^6). \tag{18}$$

The contribution to the regular part reads as

$$A^{(s)}(\tau) = -\frac{2\mathcal{G}}{\pi} - \frac{\log 2}{2} + \frac{1}{2}\tau - \frac{(1+5\log 2)}{4\pi}\tau^2 - \frac{1}{12}\tau^3 + \frac{5(1+6\log 2)}{64\pi}\tau^4 + O(\tau^5),$$

$$A^{(t)}(\tau) = -\frac{5}{2\pi}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{1}{4}\log\frac{4}{3} + \frac{\tau}{3} - \left(\frac{2+3\log 12}{8\pi\sqrt{3}} - \frac{1}{36}\right)\tau^2 - \frac{7}{648}\tau^3 + \left(\frac{4+9\log 12}{288\pi\sqrt{3}} - \frac{1}{324}\right)\tau^4 + O(\tau^5). \tag{19}$$

Next, with definition (12) the zero-field spontaneous magnetization has the same expression for both lattices

$$M(\tau,0) = (1-k^2)^{1/8}, \quad \tau < 0. \tag{20}$$

Combining this with Eqs. (2), (6), (8), and (13) one obtains

$$C_h^{(s)} = -2^{3/16}/\tilde{G}_1, \quad C_h^{(t)} = -2^{5/16}3^{-3/32}/\tilde{G}_1 \tag{21}$$

and

$$c^{(s)}(\tau) = 1 + \frac{\tau}{4} + \frac{15\tau^2}{128} - \frac{9\tau^3}{512} - \frac{4333\tau^4}{98\,304} + O(\tau^5),$$

$$c^{(t)}(\tau) = 1 + \frac{\tau}{6} + \frac{5\tau^2}{96} + \frac{\tau^3}{576} - \frac{727\tau^4}{165\,888} + O(\tau^5). \tag{22}$$

Finally, consider the zero-field susceptibility. The second field derivative of Eq. (13) at $H=0$ gives

$$\chi(\tau) = -\frac{2GC_h^2 c(\tau)^2}{[\sqrt{2}|\tau|a(\tau)]^{7/4}} - \left.\frac{\partial^2 F_{sub}}{\partial H^2}\right|_{H=0} - 2B(\tau) + \frac{\tau a(\tau) b(\tau)}{\sqrt{2}\pi}\{1 + \log[2\tau^2 a(\tau)]\}, \quad (23)$$

where $G=G_2$ for $\tau>0$ and $G=\tilde{G}_2$ for $\tau<0$. No simple closed form expression for the zero-field susceptibility $\chi(\tau)$ is known. However, the authors of [33] obtained remarkable asymptotic expansions of $\chi(\tau)$ for the square lattice for small τ to within $O(\tau^{14})$ terms with high-precision numerical coefficients. Using their results in Eq. (23), one obtains

$$B^{(s)}(\tau) = 0.052\,066\,622\,546\,9 + 0.076\,912\,034\,189\,3\tau + 0.036\,020\,046\,230\,9\tau^2 + O(\tau^3) \quad (24)$$

and

$$b^{(s)}(\tau) = \mu_h^{(s)}\left(1 + \frac{\tau}{2} + O(\tau^2)\right), \quad \mu_h^{(s)} = 0.071\,868\,670\,814. \quad (25)$$

No similar expansion for $\tau\sim 0$ is available for the triangular lattice. We used our data for $\tau=0$ to estimate

$$B^{(t)}(\tau) = 0.024\,780\,558\,2(2) + O(\tau), \quad \mu_h = -0.010\,475(1) \quad (26)$$

and the coefficient $d(\tau)=e_h+O(\tau)$ in Eq. (15)

$$e_h^{(s)} = -0.007\,28(30), \quad e_h^{(t)} = +0.001\,29(1), \quad (27)$$

which is in agreement with $e_h^{(s)}=-0.007\,27(15)$ from [25].

The above expressions were used to analyze our extensive numerical data and extract the necessary information to obtain the universal scaling function. The results are summarized in Fig. 1 and Table I (not all higher order coefficients are presented). The numerical data is in perfect agreement with the quantum field theory results by Fonseca and Zamolodchikov [11]. We also report a remarkable agreement (11–14 digits) between our numerical values for $\tilde{G}_1$, G_2, and $\tilde{G}_2$ and the classic exact results of Barouch, McCoy, Tracy, and Wu [5,12,13] and a similar agreement between the values Φ_0 and Φ_1 and the exact predictions [15,34] of Zamolodchikov's integrable E_8 field theory [8]. Interestingly, this E_8 symmetry has now been observed in experiments on the transverse Ising chain [37].

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