Soliton tunneling in the nonlinear Schrödinger equation with variable coefficients and an external harmonic potential

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We report on the nonlinear tunneling effects of spatial solitons of the generalized nonlinear Schrödinger equation with distributed coefficients in an external harmonic potential. By using the homogeneous balance principle and the F-expansion technique we find the spatial bright and dark soliton solutions. We then display tunneling effects of such solutions occurring under special conditions; specifically when the spatial solitons pass unchanged through the potential barriers and wells affected by special choices of the diffraction and/or the nonlinearity coefficients. Our results show that the solitons display tunneling effects not only when passing through the nonlinear potential barriers or wells but also when passing through the diffractive barriers or wells. During tunneling the solitons may also undergo a controllable compression.

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I. INTRODUCTION

Spatial solitary waves have been the subject of intense theoretical and experimental studies in recent years [1,2]. Spatial solitons—localized pulses or bounded self-guided beams in space—evolve from a nonlinear (NL) change in the refractive index of material induced by the distribution of light intensity. When the combined effects of refractive nonlinearity and beam diffraction exactly compensate each other, the beam propagates without change in its shape and is said to be self-trapped. NL effects responsible for the formation of a spatial soliton are, in general, Kerr-like; they induce a local self-focusing index change directly proportional to the light intensity. In this case the paraxial wave equation governing the pulse evolution becomes the cubic NL Schrödinger equation (NLSE) for the complex envelope of the electric field. In $(1+1)$ dimensions $[(1+1)D]$ two distinct types of stable localized solutions are supported by the NLSE, the bright, and the dark solitons.

Recently various forms of NLSE have been studied in Refs. [3–5]. The NL compression of chirped solitary waves and the quasi solitonic phase modulation or the gain or loss terms have been discussed in detail [3]. Soliton solutions [4], a broad class of self-similar solutions describing both bright and dark solitary waves [5] and periodic solutions [6], were obtained. Generally speaking spatial solitons depicted by the NLSE in $(1+1)D$ with constant coefficients can propagate without change. An important question is how the shape and the width of solitons change when the propagation is in a Kerr-like medium with varying coefficients of diffraction, nonlinearity, gain or loss, and an external harmonic potential. Varied behavior is observed including instabilities and beam collapse in more than one transverse dimension. Also various NLSE models can be used to design novel dispersion-managed transmission systems [7,8].

The tunneling effects of spatial solitons governed by the NLSE have been less investigated until now. In 1978 Newell predicted the tunneling effect which exists in NL media [9]. The concept of the NL tunneling effect originates from the wave equations that stem from the NL dispersion relation. Research has shown that the soliton can pass lossless through the barrier under special conditions which depend on the ratio between the solitonic amplitude and the height of the barrier [10–12]. Solitonic tunneling may create a new field of applications of spatial solitons as the tunneling effect makes it possible to design new kinds of all-optical switches and logic circuits.

In this paper closed-form solutions to the NLSE with variable coefficients in an external harmonic potential are obtained by employing the homogeneous balance principle and the F-expansion technique. We demonstrate that the propagation of spatial solitons with varying diffraction, nonlinearity and the external potential can simply be interpreted as a tunneling through the barriers and wells produced by the changing coefficients. After passing through a barrier the soliton reverts to its original shape regardless of the size of the disturbance caused by the barrier. For the barriers considered no radiation emanating from the barrier could be detected.

The paper is arranged as follows. In Sec. II we briefly describe and apply the homogeneous balance principle and the F-expansion technique to the NLSE with variable coefficients in a harmonic potential to obtain solitonic solutions. In Sec. III several NL tunneling examples are considered. In particular some special conditions on the potential barriers and wells are discussed. Finally a short summary is presented.

II. SOLITARY SOLUTIONS

The generalized NLSE in $(1+1)D$ with variable coefficients and an external harmonic potential is given by [6,13]

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\beta(t)\frac{\partial^2 u}{\partial x^2}+\chi(t)u|u|^2+\frac{1}{2}V(t)x^2u-i\Gamma(t)u=0, \quad (1)$$

where $u(t,x)$ is the complex envelope of the electrical field in the moving frame, t is the normalized distance of propaga-

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tion, and x is the normalized coordinate in the transverse direction. The function $\beta(t)$ represents the diffraction coefficient, $\chi(t)$ is the coefficient of nonlinearity, $\Gamma(t)$ is the gain or loss coefficient, and $V(t)$ is the coefficient of the harmonic potential, all of which can be controlled and manipulated by the choice of media. It is worth mentioning that the same equation, but with the t coordinate interpreted as time and with $\beta=1$, is known as the $(1+1)D$ Gross-Pitaevskii equation. It describes the evolution of the wave function of the Bose-Einstein condensate and can also develop solitary-wave solutions [14].

In order to solve the NL partial differential Eq. (1) and to obtain solitary solutions, we look for the solution in the form [15]

$$u(t,x)=A(t,x)e^{iB(t,x)}, \tag{2}$$

where the amplitude $A=f_1(t)F(\theta)+f_{-1}(t)F^{-1}(\theta)$ and the phase $B=a(t)x^2+b(t)x+e(t)$ are real functions of their arguments. $F(\theta)$ is the Jacobi elliptic function (JEF), which is the solution of an ordinary NL differential equation $(dF/d\theta)^2=c_0+c_2F^2+c_4F^4$ (here c_0, c_2, and c_4 are real constants related to the elliptic modulus of JEFs, see Table I of Refs. [6,16]), and the traveling variable is of the form $\theta=k(t)x+\omega(t)+\theta_0$. Using the same method as in Refs. [6,16] and the same procedure of solution, we obtain the traveling-wave solution of Eq. (1):

$$u=f_{10}\left[F(\theta)+\varepsilon\sqrt{\frac{c_0}{c_4}}F^{-1}(\theta)\right]\exp\left[i(ax^2+bx+e)+\int_0^t(\Gamma-a\beta)dt\right], \tag{3}$$

where $\theta=k_0\exp(-2\int_0^t a\beta dt)x-b_0k_0\int_0^t\beta\exp(-4\int_0^t a\beta dt)dt+\theta_0$, $\varepsilon=0,\pm1$, $e=\frac{1}{2}(c_2k_0^2-b_0^2)\int_0^t\beta\exp(-4\int_0^t a\beta dt)dt+e_0$, $b=b_0\exp(-2\int_0^t a\beta dt)$, and $a(t)$ is the chirp function. Here all symbols with subscript 0 are the initial values of the corresponding parameters. The coefficient of nonlinearity, $\chi(t)$, is expressed in terms of the other coefficients $\beta(t)$, $\Gamma(t)$, and $a(t)$: $\chi=-(k_0^2c_4/f_{10}^2)\beta\exp[-2\int_0^t(\Gamma+a\beta)dt]$. This equation can be conveniently understood as an integrability condition on Eq. (1):

$$\frac{1}{\beta}\frac{d\beta}{dt}-\frac{1}{\chi}\frac{d\chi}{dt}=2(\Gamma+a\beta). \tag{4}$$

Likewise the chirp parameter $a(t)$, the diffraction coefficient $\beta(t)$, and the coefficient of the harmonic potential $V(t)$ cannot be arbitrary simultaneously. In fact $a(t)$ is determined from the following Riccati equation:

$$\frac{da}{dt}=-2\beta a^2+\frac{1}{2}V. \tag{5}$$

In general Eq. (1) is a nonintegrable system. It is noted that, if one chooses $a(t)=0$, from Eq. (5) one obtains $V(t)=0$. Equation (1) then becomes the standard NLSE with variable coefficients; hence $a(t)\neq0$ in this paper.

The solutions to Eq. (1) can exist only under the conditions specified in Eqs. (4) and (5); the coefficients $\beta(t)$, $\chi(t)$, $V(t)$, and $\Gamma(t)$ cannot be all chosen independently. In accordance with the Table I of Refs. [6,16], by choosing $\varepsilon=0$ and $m=1$, and under the conditions specified in Eqs. (4) and (5), we obtain interesting single bright soliton (BS) and dark soliton (DS) solutions of the form:

$$u_1^{BS}=f_{10}\,\mathrm{sech}(\theta)\exp\left\{i\left[ax^2+b_0\exp\left(-2\int_0^t a\beta dt\right)x+\frac{1}{2}(k_0^2-b_0^2)\int_0^t\beta\exp\left(-4\int_0^t a\beta dt\right)dt+e_0\right]+\int_0^t(\Gamma-a\beta)dt\right\}, \tag{6}$$

$$u_2^{DS}=f_{10}\tanh(\theta)\exp\left\{i\left[ax^2+b_0\exp\left(-2\int_0^t a\beta dt\right)x-\frac{1}{2}(2k_0^2+b_0^2)\int_0^t\beta\exp\left(-4\int_0^t a\beta dt\right)dt+e_0\right]+\int_0^t(\Gamma-a\beta)dt\right\}, \tag{7}$$

where $\theta=k_0[\exp(-2\int_0^t a\beta dt)x-b_0\int_0^t\beta\exp(-4\int_0^t a\beta dt)dt-x_0]$ and $\theta_0=-k_0x_0$.

III. GENERAL NL TUNNELING EFFECTS

Recently, based upon the NLSE (1) with variable coefficients in a gainless medium and without an external potential, the tunneling effect of BSs has been analyzed by Serkin *et al.* [10]. Their results display the process of NL tunneling of a BS through a strong NL thin film barrier, which exhibits jumplike nonadiabatic evolution and eventually leads to the soliton "fission reaction." Yang confirmed, using Eq. (1) without an external potential, that an optical pulse can be compressed by NL barriers [11]. Also Wang discussed the NL tunneling effect in Eq. (1) under the assumption $\beta(t)=1$ and showed that BS and DS similarly can pass through the NL barriers or wells [12].

We explore the general NL tunneling effect using the complete Eq. (1) including diffractive (or dispersive) potential barriers and wells. As mentioned above, the solutions to the generalized NLSE can be categorized in terms of arbitrary functions, $\beta(t)$, $\chi(t)$, $V(t)$, and $\Gamma(t)$. We have freedom in selecting three of these functions independently in accordance with some actual physical requirements. The idea is to

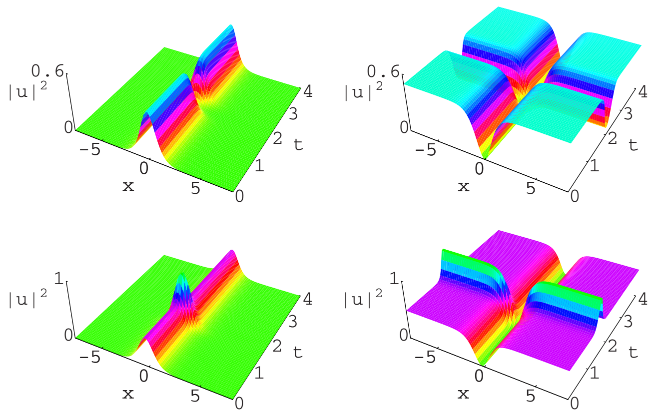

FIG. 1. (Color online) Comparison of tunneling effects for the propagation of BSs and DSs through potential barriers (top row) with the propagation through potential wells (bottom row). Here x is the transverse variable and t is the propagation variable. The barriers (wells) are formed using the diffraction coefficient $\beta(t)$. The system parameters are $\lambda=5$, $\eta=8$, $a_0=0.02$, and $t_0=2$. BS is at the left and DS is at the right. Other parameters are $f_{10}=k_0=1$, $b_0=0$, and $x_0=0$.

make such choices that display tunneling effects of spatial solitons. By tunneling we mean the propagation of solitons across regions in which deliberate changes in the diffraction coefficient and the coefficient of nonlinearity are introduced such that the solitons emerge without any change.

In order to understand further the dynamical behavior of NL tunneling in Eqs. (6) and (7) we consider the following two examples, namely, two possibilities for the system coefficients: (1) the system has a diffraction potential barrier (well); i.e., the diffraction coefficient $\beta(t)$ is chosen as $\beta(t)=\beta_0 e^{-rt}\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]$, while $\chi(t)=1$. (2) The system has a nonlinearity potential barrier (well); i.e., the coefficient of nonlinearity is chosen as $\chi(t)=\chi_0 e^{-rt}\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]$, while $\beta(t)=1$. Here the positive or the negative sign denotes the barrier or the well. Also λ represents the barrier's amplitude, η is the parameter relating to the barrier's width, t_0 marks the position of the barrier, and r is a decay parameter. The coefficients are set such that the resulting tunneling structures are stable and do not radiate, while still allowing for the analytical treatment of equations.

A. Specific NL tunneling effects

Now we consider the tunneling effects of a solitary wave with the parameters chosen as follows: the constant chirp $a(t)=a_0$ $(a_0>0)$, $\beta(t)=1\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]$ (namely, $r=0$ and $\beta_0=1$) and $\chi(t)=1$. Then from the constraint conditions of Eqs. (4) and (5) we find the coefficient of the external harmonic potential $V(t)$ and the gain coefficient $\Gamma(t)$: $V(t)=4a_0^2\{1\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]\}$ and

$$\Gamma(t)=-a_0(1\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)])\pm\lambda\eta\,\mathrm{sech}^2[\eta(t-t_0)]\times\tanh[\eta(t-t_0)]/\{1\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]\}.$$

In Fig. 1 a comparison is presented of the propagation of BS and DS through a potential barrier with propagation through a potential well for the case of the diffraction barrier.

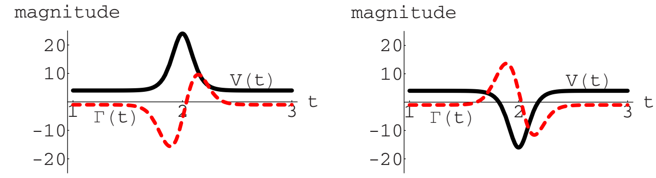

FIG. 2. (Color online) The change in the external harmonic potential coefficient $V(t)$ and the gain coefficient $\Gamma(t)$ along distance t. The system parameters are $\lambda=5$, $\eta=8$, $a_0=1$, and $t_0=2$. The potential barrier is at the left and the potential well at the right.

When the BS passes through the diffraction barrier, it vanishes and a valley is formed, while the DS forms a channel in the background near $t=t_0$; after the tunneling the solitons are restored to their original shapes. Conversely, when the bright or the dark soliton crosses the diffraction well, the intensity of the soliton grows and forms a peak; afterwards it again recovers its original shape. Please note that, when the soliton passes through the diffraction potential barrier (well), the amplitude decreases slightly.

Figure 2 illustrates the profiles of the coefficient of the external harmonic potential $V(t)$ and the gain coefficient $\Gamma(t)$ along the propagation distance t. From Fig. 2 it is noted that the harmonic coefficient $V(t)$ forms a potential barrier or a well at $t=t_0$, which results in the corresponding change in the gain coefficient $\Gamma(t)$ near $t=t_0$. For the barrier the gain coefficient $\Gamma(t)$ forms a concave part in the vicinity of $t=t_0$ and then forms a convex part (see the left part of Fig. 2). For the potential well the reverse occurs (see the right part of Fig. 2).

We consider next solitary waves passing through a potential barrier (well) formed by the nonlinearity when $a(t)=a_0\tanh(2a_0 t)$ $(a_0>0)$, $\chi(t)=1\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]$ (namely, $r=0$ and $\chi_0=1$), and $\beta(t)=1$. From Eqs. (4) and (5) we obtain the other coefficients:

$$\Gamma(t)=-a_0\tanh(2a_0 t)\pm\lambda\eta\,\mathrm{sech}^2[\eta(t-t_0)]\times\tanh[\eta(t-t_0)]/\{1\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]\}$$

and $V(t)=4a_0^2$. Figure 3 depicts the effects of tunneling. When the soliton crosses the barrier, the intensity of the pulse decays and becomes a gully with a bank; afterwards the soliton is restored to its original shape. When the soliton crosses a well, at first it produces a peak, then plunges into a ravine and in the end recovers a peak; finally the soliton propagates with its original shape. It is noted that the pulse forms two adjacent banks or peaks.

An analysis of Figs. 1 and 3 shows that, when the NLSE solitons pass through the two types of potential barriers, the pulse undergoes tunneling phenomenon; the intensity of the soliton grows or declines within the barrier, but in the end the solitons pass through unchanged. We could not detect any radiation emanating from the barrier region. Also this tunneling effect has nothing to do with the loss or gain in media. We have reported similar phenomena in [6], where we demonstrated lossless scattering off a diffractive wall as well as a snakelike propagation through combined t-dependent diffraction and nonlinearity coefficients.

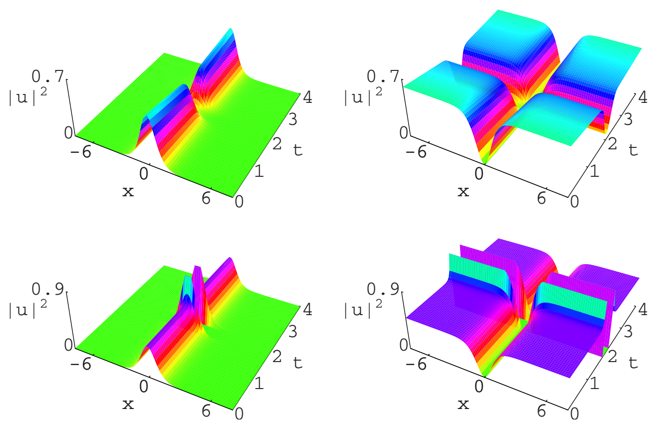

FIG. 3. (Color online) Intensity evolution during the tunneling effect through the barrier (upper row) or the well (lower row) formed by the nonlinearity coefficient. The system parameters have the values $\lambda=5$, $\eta=8$, $a_0=0.02$, and $t_0=2$. BS is at the left and DS at the right. Other parameter are $f_{10}=k_0=1$, $b_0=0$, and $x_0=0$.

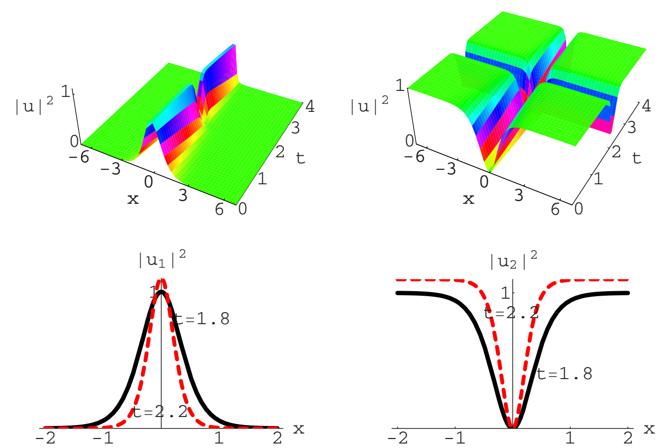

FIG. 4. (Color online) Evolution of a pulse while tunneling (top row) and the pulse compression (bottom row) at distances $t=1.8,2.2$, respectively. The system parameters have values $\lambda=2$, $\eta=4$, $r=0.02$, and $t_0=2$. BS is at the left and DS at the right. Other parameters are $f_{10}=k_0=1$, $b_0=0$, and $x_0=0$.

B. NL tunneling compression

Finally we find that the pulse can be compressed when the soliton passes through the barriers of diffraction or nonlinearity of special form. This presents a potentially important method for an efficient compression of solitons. We consider a system with decaying nonlinearity and with a diffraction barrier riding on a decaying exponential of the form: $\beta(t)=\beta_0 e^{-rt}\pm\lambda\,\mathrm{sech}^2[\eta(t-t_0)]$, $\Gamma(t)=0$ and $\chi(t)=\chi_0 e^{-rt}$, where r (>0), β_0 (>0), and χ_0 (>0) are the constant parameters of the system. Figure 4 presents the evolution of the soliton pulse at different distances with the diffraction barrier on an exponential. From Fig. 4 it can be seen that after passing the barrier the soliton is compressed about $t=t_0$ and then it propagates unchanged. This means that, when the pulse passes through the dispersion barrier (well) with an exponential decay of coefficients, it can be compressed to a desired extent by the choice of the barrier (well) parameters.

IV. CONCLUSIONS

In conclusion we have analyzed the tunneling effects of spatial solitons passing through the potential barriers and wells formed by the coefficients of diffraction and nonlinearity. By using the homogeneous balance principle and the F-expansion technique we introduced the spatial bright and dark soliton solutions that are then tunneled through the barriers and wells of diffraction and nonlinearity. Our results show that the solitons peak or form valleys as they pass through these potential barriers; however, there is no relation of these phenomena to the loss or gain in media. No radiation is detected after solitons tunnel through the barriers. It is significant that the spatial solitons propagate undeformed in media with such barriers. It is also shown that the compression of solitons to an extent desired can be achieved by passing them through appropriately chosen barriers and wells placed atop diffraction and nonlinearity coefficients of exponential decay.

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