Intense energy transfer and superharmonic resonance in a system of two coupled oscillators

Agnessa Kovaleva,[1,*] Leonid Manevitch,[2] and Elina Manevitch[3]
[1] *Space Research Institute, Russian Academy of Sciences, Moscow 117997, Russia*
[2] *Institute of Chemical Physics, Russian Academy of Sciences, Moscow 119991, Russia*
[3] *University of Illinois at Urbana-Champaign, Champaign, Illinois 61820, USA*
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The paper presents the analytic study of energy exchange in a system of coupled nonlinear oscillators subject to superharmonic resonance. The attention is given to complete irreversible energy transfer that occurs in a system with definite initial conditions corresponding to a so-called limiting phase trajectory (LPT). We show that the energy imparted in the system is partitioned among the principal and superharmonic modes but energy exchange can be due to superharmonic oscillations. Using the LPT concept, we construct approximate analytic solutions describing intense irreversible energy transfer in a harmonically excited Duffing oscillator and a system of two nonlinearly coupled oscillators. Numerical simulations confirm the accuracy of the analytic approximations.

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I. INTRODUCTION

In this paper, we study irreversible energy transfer arising due to superharmonic resonance in a system of weakly coupled nonlinear oscillators. Over the last decades, energy transfer in both conservative and dissipative systems has been a subject of growing interest in different fields of nonlinear dynamics. The theory has been developed to describe energy transfer in multibody systems [1–3], wave dynamics, primarily in fluids and plasmas [4–6], among other novel applications such as semiconductors [7–9].

Although most of the analytic and numerical results relate to energy transfer under the conditions of resonance 1:1, the superharmonic resonance can also play a significant role in the occurrence of energy exchange [4]. However, the problem seemed too complex to obtain an analytical solution and clarify the mechanism of the superharmonic energy transfer. The purpose of this paper is to give an analytic description of complete irreversible energy transfer in a system of coupled oscillators under the condition of superharmonic resonance.

The approach developed in this paper is based on the concept of *limiting phase trajectories* (LPTs) introduced in [10]. The LPT is defined as a trajectory corresponding to oscillations with the most intensive energy exchange between weakly coupled oscillators or an oscillator and a source of energy; the transition from energy exchange to energy localization at one of the oscillators is associated with the disappearance of the LPT. Recently, the LPT ideas have been used in the analysis of 1:1 resonance in single-degree-of-freedom (1DOF) [11–13] and two-degree-of-freedom (2DOF) [13] systems. This paper extends the LPT concept to the analysis of irreversible energy exchange under condition of superharmonic resonance.

The paper is organized as follows. The first part (Secs. I–IV) is concerned with the analysis of a harmonically excited Duffing oscillator. We construct the LPT (Sec. II) and obtain the relationships between the parameters guaranteeing the occurrence of strongly nonlinear oscillations of large amplitude in the nondissipative oscillator (Sec. III A). In Sec. III B we derive an explicit asymptotic representation of the LPT. An asymptotic approach to solving a dissipative system is developed in Sec. IV. In the remainder of the paper (Sec. V), we analyze superharmonic energy transfer in a 2DOF system, in which the initial impulse stands for an external excitation. We examine beating oscillations of an undamped system and consider transformations of beating to decaying oscillations in a system with dissipation.

II. EQUATIONS OF MOTION

The dimensionless equations of the system and the initial conditions are taken in the form

$$\frac{d^2y}{d\tau_0^2}+2\varepsilon\gamma\frac{dy}{d\tau_0}+y+8\varepsilon\alpha y^3=2F\sin\left(\frac{1}{3}+\varepsilon s\right)\tau_0,$$

$$\tau_0=0,\ \ y=0;\ \ \frac{dy}{d\tau_0}=0, \tag{2.1}$$

where γ, α, F, and s are positive parameters and $\varepsilon>0$ is a small parameter of the system. We recall that a maximum possible energy pumping from the source of excitation into the oscillator occurs if the oscillator is initially at rest. An orbit, satisfying the zero initial conditions is said to be *a limiting phase trajectory* [10].

For our purposes, it is convenient to separate the harmonic components in the solution. We present the solution of Eq. (2.1) in the form

$$y=y_0+u, \tag{2.2}$$

in which y_0 is a leading-order partial solution of the linear equation

$$\frac{d^2y_0}{d\tau_0^2}+y_0=2F\sin\omega_\varepsilon\tau_0,$$

$$y_0=\Lambda\sin\omega_\varepsilon\tau_0,\ \ \Lambda=9F/4,\ \ \omega_\varepsilon=\tfrac{1}{3}+\varepsilon s. \tag{2.3}$$

*a.kovaleva@ru.net

From Eqs. (2.1)–(2.3), we obtain

$$\frac{du}{d\tau_0} - v = 0,$$

$$\frac{dv}{d\tau_0} + u + 2\varepsilon\gamma\frac{du}{d\tau_0} + 8\varepsilon\alpha(u + y_0)^3 = 0,$$

$$\tau_0 = 0,\ \ u = 0;\ \ v = -\Lambda/3. \tag{2.4}$$

In Eq. (2.4), the principal harmonics of frequency 1/3 is of $O(\varepsilon)$; this implies that a generating solution of Eq. (2.4) includes only superharmonic components of frequency 1.

We analyze the nonstationary dynamics of Eq. (2.4) in terms of the complex-conjugate variables ψ and ψ^* [14]

$$\psi = v + iu,\ \ \psi^* = v - iu;\ \ u = -\frac{i}{2}(\psi - \psi^*),\ \ v = \frac{1}{2}(\psi + \psi^*). \tag{2.5}$$

Multiplying the first equation in Eq. (2.4) by i and adding to the second equation, we obtain the following (still exact) equation:

$$\frac{d\psi}{d\tau_0} - i\psi + \varepsilon\gamma(\psi + \psi^*) + \varepsilon i\alpha[(\psi - \psi^*) + \Lambda(e^{i\omega_\varepsilon\tau_0} - e^{-i\omega_\varepsilon\tau_0})]^3$$
$$= 0,\ \ \psi(0) = -\frac{\Lambda}{3}. \tag{2.6}$$

The solution of Eq. (2.6) is sought as a multiple scales expansion

$$\psi(t,\varepsilon) = \psi_0(\tau_0,\tau_1) + \varepsilon\psi_1(\tau_0,\tau_1) + \cdots,$$

$$\frac{d}{d\tau_0} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1}, \tag{2.7}$$

with the further selection of resonance terms [15]. In the leading-order approximation, we obtain

$$\frac{\partial\psi_0}{\partial\tau_0} - i\psi_0 = 0,\ \ \psi_0(\tau_0,\tau_1) = \varphi_0(\tau_1)e^{i\tau_0}, \tag{2.8}$$

where a slow function $\varphi_0(\tau_1)$ should be found at the next step of approximation. Equating the coefficients of order ε leads to the equation

$$\frac{d\psi_1}{d\tau_0} + \frac{\partial\varphi_0}{\partial\tau_1}e^{i\tau_0} - i\psi_1 + \gamma[\varphi_0(\tau_1)e^{i\tau_0} + \varphi_0^*(\tau_1)e^{-i\tau_0}]$$
$$+ i\alpha[(\varphi_0(\tau_1)e^{i\tau_0} - \varphi_0^*(\tau_1)e^{-i\tau_0}) + \Lambda(e^{i\omega_\varepsilon\tau_0} - e^{-i\omega_\varepsilon\tau_0})]^3 = 0. \tag{2.9}$$

In order to avoid the secular growth of $\psi_1(\tau_0,\tau_1)$ in τ_0, we eliminate the resonance terms from Eq. (2.9). First, we calculate the component of frequency 1 in the cubic function. We find

$$\{[\varphi_0(\tau_1)e^{i\tau_0} - \varphi_0^*(\tau_1)e^{-i\tau_0}] + \Lambda(e^{i\omega_\varepsilon\tau_0} - e^{-i\omega_\varepsilon\tau_0})\}^3$$
$$= -[3(|\varphi_0|^2\varphi_0 + 2\Lambda^2\varphi_0) - \Lambda^3 e^{3is\tau_1}]e^{i\tau_0} + N_r. \tag{2.10}$$

Here and below, N_r is a shorthand symbol for any nonresonance terms. Let us denote $\varphi = \Lambda\varphi_0$. Inserting Eq. (2.10) into Eq. (2.9) and summing with all other terms of frequency 1, we obtain the following equation for φ_0:

$$\frac{\partial\varphi_0}{\partial\tau_1} + \gamma\varphi_0 - i\alpha\Lambda^2[3(|\varphi_0|^2\varphi_0 + 2\varphi_0) - e^{3is\tau_1}] = 0,$$

$$\varphi_0(0) = -\frac{1}{3}. \tag{2.11}$$

We express Eq. (2.11) in real variables by letting

$$\varphi_0 = ae^{i\delta},\ \ \varphi_0^* = ae^{-i\delta},\ \ a > 0. \tag{2.12}$$

From Eqs. (2.5) and (2.12), we obtain the following asymptotic representation of the solution:

$$u(\tau_0,\varepsilon) = \Lambda a(\tau_1)\sin[\tau_0 + \delta(\tau_1)] + O(\varepsilon) \tag{2.13}$$

and, by Eq. (2.2),

$$y(\tau_0\varepsilon) = \Lambda\left\{a(\tau_1)\sin[\tau_0 + \delta(\tau_1)] + \sin\left(\tfrac{1}{3} + \varepsilon s\right)\tau_0\right\} + O(\varepsilon) \tag{2.14}$$

Substituting Eq. (2.12) into Eq. (2.11) and setting separately the real and imaginary parts of the resulting equations equal to zero leads to the averaged equations

$$\frac{da}{d\tau_1} = -\gamma a - \beta\sin\Delta,$$

$$a\frac{d\Delta}{d\tau_1} = a(-\zeta + 3\beta a^2) - \beta\cos\Delta,\ \ \Delta = \delta - 3s\tau_1 \tag{2.15}$$

with the initial conditions

$$a(0) = \tfrac{1}{3},\ \ \Delta(0) = -\pi. \tag{2.16}$$

Here, $\beta = \alpha\Lambda^2$ and $\zeta = 3(s - 2\alpha\Lambda^2)$. System (2.15) is similar to one derived in [12,13] and thus it can be analyzed in the same way.

III. ANALYSIS OF THE UNDAMPED OSCILLATOR

In this section, we consider a nondissipative counterpart of Eq. (2.15) with $\gamma = 0$, namely,

$$\frac{da}{d\tau_1} = -\beta\sin\Delta,$$

$$a\frac{d\Delta}{d\tau_1} = a(-\zeta + 3\beta a^2) - \beta\cos\Delta. \tag{3.1}$$

A. Fixed points and dynamical transitions in Eq. (3.1)

First, we define the steady state of Eq. (3.1) from the equations

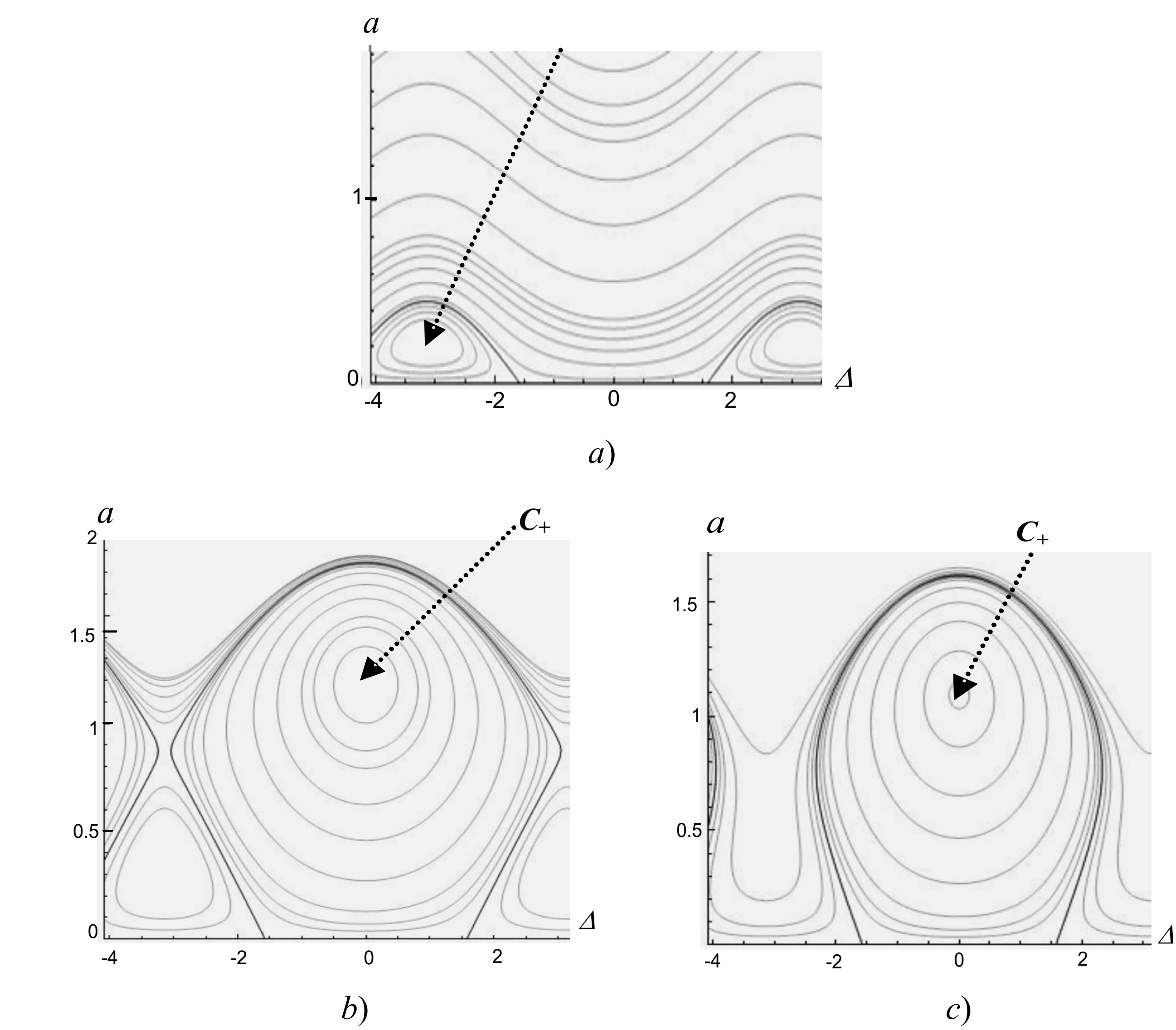


FIG. 1. Phase planes (a) (a,Δ): $\alpha<\alpha_1^*$; (b) $\alpha_1^*<\alpha<\alpha_2^*$; (c) $\alpha>\alpha_2^*$.

$$\frac{da}{d\tau_1}=0, \quad \frac{d\Delta}{d\tau_1}=0$$

or, by Eq. (3.1),

$$a(-\zeta+3\beta a)=\beta\cos\Delta_i,\ \ \Delta_1=0,\ \ \Delta_2=-\pi. \tag{3.2}$$

Depending on the parameters, Eq. (3.2) may have either three real solutions or a single real and two complex-valued solutions. In the first case, system (3.1) has two stable centers, C_-: $(-\pi,a_-)$, C_+: $(0,a_+)$, and an intermediate unstable hyperbolic point O: $(-\pi,a_0)$; in the second case, there exists a single stable center C_+: $(0,a_+)$. Using the same arguments as in [11–13], we obtain two critical relationships determining the centers of system (3.1) and the direction of the phase orbits

$$D_1=\beta^2-\frac{2\zeta^3}{81\beta}. \tag{3.3}$$

If $D_1<0$, then the LPT encircles the left stable center C_-: $(-\pi,a_-)$ (Fig. 1); otherwise, the LPT encircles the right stable center C_-: $(-\pi,a_-)$ (Fig. 2); the critical relationship is $D_1=0$. It follows from Eqs. (2.16) and (3.3) that the condition $D_1=0$ suggests the following result:

$$[3(s-2\beta)]^3=\frac{81}{2}\beta^3,\ \ s=\left(2+\sqrt[3]{\frac{3}{2}}\right)\beta.$$

Since $\beta=\alpha\Lambda^2$, $\Lambda=9F/4$, then $D_1=0$ if

$$\alpha=\alpha_1^*=0.063\frac{s}{F^2} \tag{3.4}$$

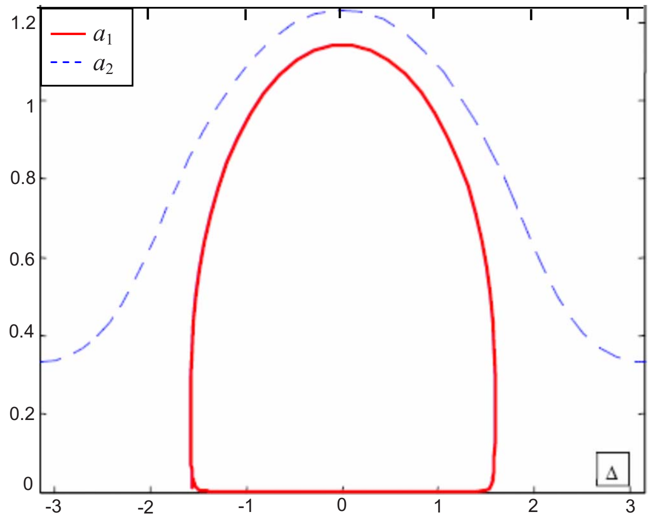


FIG. 2. (Color online) Phase portraits (a,Δ) for the initial conditions I (solid) and II (dashed).

$$D_2 = \left(\frac{\beta^2}{4} - \frac{\zeta^3}{81\beta}\right). \quad (3.5)$$

If $D_2<0$, then there exist two stable centers, C_-: $(-\pi, a_-)$, C_+: $(0, a_+)$, and an intermediate unstable hyperbolic point O: $(-\pi, a_0)$ (Figs. 1 and 2); if $D_2>0$, then there exists only a single stable center C_+: $(0, a_+)$ (Fig. 3); the second critical relationship is $D_2=0$. Arguing as above, we obtain that the equality $D_2=0$ yields $[3(s-2\beta)]^3=\frac{81}{4}\beta^3$, $s=(2+\sqrt[3]{\frac{3}{4}})\beta$, or

$$\alpha = \alpha_2^* = 0.068\frac{s}{F^2}. \quad (3.6)$$

Typical phase portraits of system (3.1) with different parameters α are depicted in Fig. 1.

B. Analysis of nonlinear oscillations

In what follows, we examine strongly nonlinear oscillations ($\alpha>\alpha_2^*$) associated with intense energy pumping from the source of energy into the object. In order to evaluate an intensity of energy transfer due to superharmonic oscillations, we compare solutions of Eq. (3.1) obeying the initial conditions

$$(\mathrm{I})a(0)=0, \quad \Delta(0)=-\pi/2; \quad (\mathrm{II})a(0)=\tfrac{1}{3}, \quad \Delta(0)=-\pi. \quad (3.7)$$

The initial conditions (I) define the LPT of system (2.4) and thus correspond to motion with a maximum possible transfer of energy into superharmonic oscillations; the conditions (II) coincide with Eq. (2.16). For brevity, we term respective solutions as solution *I* and solution *II*.

Throughout the paper, we take the following parameters of numerical simulation:

$$\beta=1.35, \quad \zeta=0.3, \quad F=1; \quad \varepsilon=0.007; \quad \gamma=0 \text{ or } \gamma=0.2. \quad (3.8)$$

This yields $\alpha=0.27$, $s=2.8$, and $\alpha_2^*=0.187$; $\alpha>\alpha_2^*$.

As seen in Fig. 2, the phase orbits $a(\Delta)$ are close and similar to that in Fig. 1(c) but Fig. 3 highlights the qualitative difference between the solutions. In the solution II, the phase $\Delta_2(\tau_1)$ is changed by a quasilinear law and the amplitude $a_2(\tau_1)$ contains a dominant harmonic component.

To analyze $a_2(\tau_1)$ and $\Delta_2(\tau_1)$, we recall that system (3.1) conserves the integral of motion

$$h=-a(\zeta a+3\beta a^3/2-2\beta\cos\Delta), \quad (3.9)$$

identifying the phase trajectories in the plane (a,Δ). The LPT corresponds to the contour $h=0$, as only in this case a trajectory passes the point $a=0$; if a trajectory begins at Eq. (3.2), then

$$h=\frac{1}{3}\left(-\frac{\zeta}{3}+\frac{37\beta}{18}\right). \quad (3.10)$$

We employ Eqs. (3.9) and (3.10) to exclude Δ and reduce Eq. (2.15) to a single second-order equation. From Eqs. (3.1) and (3.9), we obtain

$$\cos(\Delta(a)) = \frac{1}{2\beta}\left(\frac{3\beta a^3}{2} - \zeta a - \frac{h}{a}\right), \quad \frac{d\Delta}{d\tau_1} = \Omega(a),$$

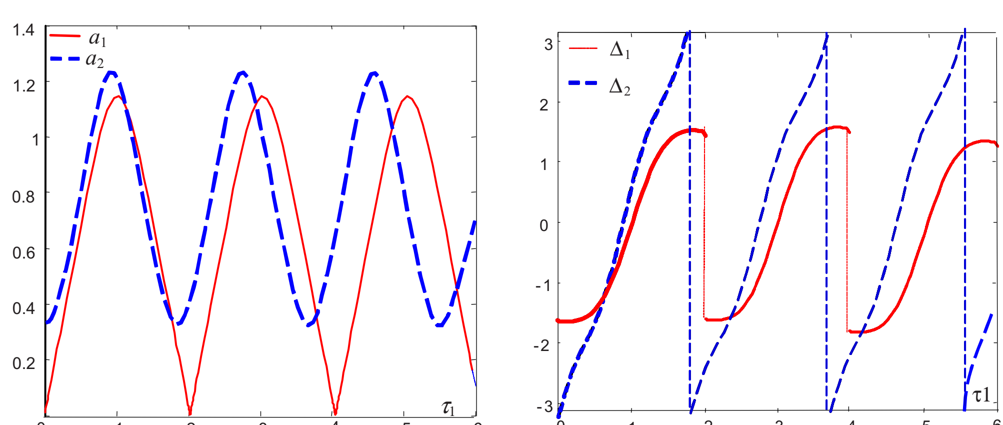


FIG. 3. (Color online) (a) Temporal behavior of $a(\tau_1)$ and (b) principal value of $\Delta(\tau_1)$ in the range $(-\pi,\pi]$ for the initial conditions I (solid) and II (dashed).

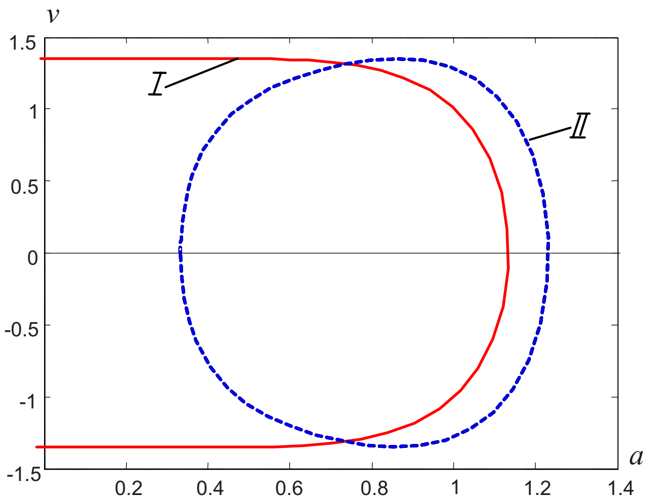


FIG. 4. (Color online) Phase portraits of Eq. (3.12) for the initial conditions i (solid curve I) and ii (dashed curve II)

$$\Omega(a) = \frac{1}{2}\left(\frac{9\beta a^2}{2} - \zeta\right) + \frac{h}{2a^2} \tag{3.11}$$

and, therefore,

$$\frac{d^2 a}{d\tau_1^2} + f(a) = 0, \quad \frac{da}{d\tau_1} = v,$$

$$f(a) = \Omega(a)\beta\cos[\Delta(a)] = \frac{a}{4}\left[\left(\frac{3}{2}\beta a^2 - \zeta\right)\left(\frac{9}{2}\beta a^2 - \zeta\right) + 3\beta h - \frac{h^2}{a^4}\right],$$

$$\tau_1 = 0{:}(i)a = 0, \ \ v = \beta; \quad (ii)a = \tfrac{1}{3}, \ \ v = 0. \tag{3.12}$$

The phase portrait of the oscillator (3.11) is given in Fig. 4.

Note that Eq. (3.12) can be interpreted as the equation of a conservative oscillator with the potential

$$\Psi(a) = \int_0^a f(x)dx = \frac{a^2}{8}\left(\frac{3}{2}\beta a^2 - \zeta\right)^2 + \frac{3}{8}\beta h a^2 - \frac{h}{8a^2}, \tag{3.13}$$

yielding the integral of energy $E = v^2/2 + \Psi(a) = E_0$, the initial energy E_0 is determined by the conditions (ii). The amplitude of oscillations $A = [\max a(\tau_1);\ v = 0]$ can be found by formula $\Psi(A) = E_0$; under given A, the period of oscillation $T(A)$ is calculated as [16]

$$T(A) = 2\int_0^A \frac{da}{\sqrt{E_0 - 2\Psi(a)}}. \tag{3.14}$$

Equalities (3.13) and (3.14) hold for an exact solution of Eq. (3.1). However, for our purposes, an approximate consideration suffices. We calculate $a(\tau_1)$ and $\Delta(\tau_1)$ with help of the following iterative procedure:

$$\frac{da_{i+1}}{d\tau_1} = -\beta\sin\Delta_i, \quad \frac{d\Delta_i}{d\tau_1} = \Omega(a_i), \quad i = 0, 1, \ldots ,$$

$$a_i(0) = \tfrac{1}{3}; \quad \Delta_i(0) = -\pi, \tag{3.15}$$

where $a_0 = a(0) = \frac{1}{3}$, $\Omega_0 = \Omega(a_0)$, and

$$\frac{da_1}{d\tau_1} = -\beta\sin\Delta_0, \quad \frac{d\Delta_0}{d\tau_1} = \Omega_0,$$

$$a_1(0) = \tfrac{1}{3}; \quad \Delta_0(0) = -\pi, \tag{3.16}$$

where $\Omega(a)$ is defined by Eq. (3.12). This yields the following leading-order approximation:

$$a_0(\tau_1) = 1/3 + (\beta/\Omega_0)[1 - \cos(\Omega_0\tau_1)], \quad \Delta_0(\tau_1) = -\pi + \Omega_0\tau_1. \tag{3.17}$$

It follows from Eqs. (3.8) and (3.11) that $\Omega_0 \approx 3.49$; this yields $T_0 = 2\pi/\Omega_0 \approx 1.8$; the amplitude of oscillations $A_0 = a_0(T_0/2) \approx 1.1$. As seen in Fig. 5, a discrepancy between the numerical solution of Eq. (3.1) and approximation (3.17) is about 10%.

Figures 6 and 7 compare the numerical solution $y(\tau_0)$ of the Duffing Eq. (2.1) to its analytic approximation (2.14); in this latter case, a and Δ are calculated by Eq. (3.17). The Duffing equation has the parameters (3.8); in addition, we take $\gamma = 0$, $\varepsilon = 0.007$. Figures 6 and 7 demonstrate a fairly good agreement of the numerical and analytical solutions despite slight irregularity of high-frequency components in Fig. 7.

IV. TRANSIENT SUPERHARMONIC OSCILLATIONS OF THE DISSIPATIVE OSCILLATOR

In this section, we consider the dynamics of dissipative system (2.1). Figure 8 depicts a typical behavior of a strongly nonlinear oscillator with weak dissipation ($\gamma = 0.2$, $\varepsilon = 0.007$). The plots in Figs. 6 and 8 agree with the basic assumption [13]: motion of the damped system is similar to motion of the undamped system up to an instant of the first maximum of the envelope of oscillations, then the damped motion develops into stationary oscillations generated by external forcing and independent of the initial conditions. This assumption underlies the approximation procedure.

We denote by $a_\gamma(\tau_1)$ a solution of Eq. (2.15) satisfying the initial conditions (2.16): by $a_\gamma(\tau_1^*)$, its first maximum at $\tau_1 = \tau_1^*$ (Fig. 9); by $a(\tau_1)$, a similar solution of Eq. (3.1) in the absence of damping ($\gamma = 0$). Using the above assumption, $a_\gamma(\tau_1)$ is partitioned into two segments: on the interval $[0, \tau_1^*]$, $a_\gamma(\tau_1)$ is considered as being close to the solution $a(\tau_1)$ of the undamped system (3.1); on the interval $\tau_1 \geq \tau_1^*$, the solution $a_\gamma(\tau_1)$ is similar to smooth decaying oscillations of the dissipative system.

If $\gamma \neq 0$, the steady state O: $(a_\gamma^0, \Delta_\gamma^0)$ is determined by the equality

$$a^2[(\zeta - 3\beta a^2)^2 + \gamma^2] = \beta^2 \tag{4.1}$$

or, for sufficiently small γ,

$$\gamma a_\gamma^0 = -\beta\sin\Delta_\gamma^0, \quad \zeta a_\gamma^0 - 3\beta(a_\gamma^0)^3 = -\beta\cos\Delta_\gamma^0,$$

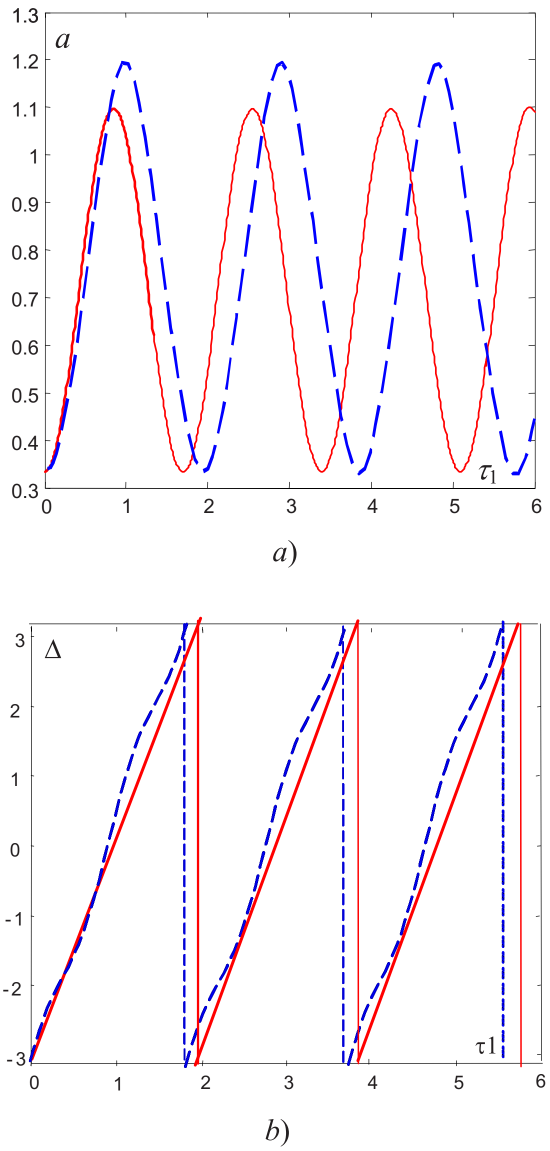


FIG. 5. (Color online) Numerical solution of Eq. (3.1) (dashed) and its analytic approximation (3.17) (solid): (a) temporal behavior of $a(\tau_1)$; (b) principal value of $\Delta(\tau_i)$ in the range $(-\pi,\pi]$.

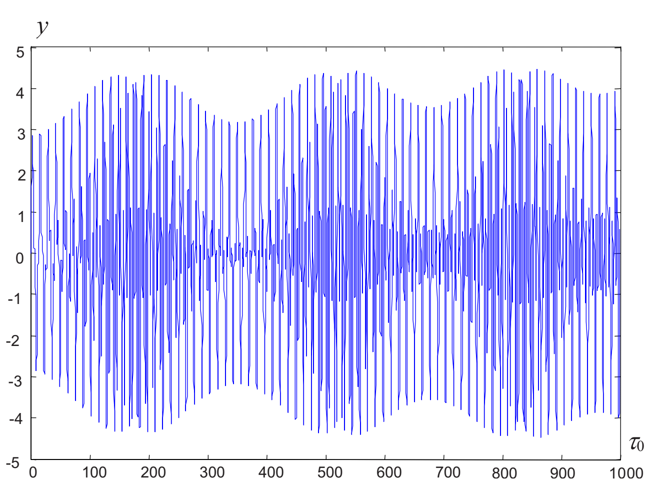


FIG. 6. (Color online) Numerical solution of Eq. (2.1); $\gamma=0$.

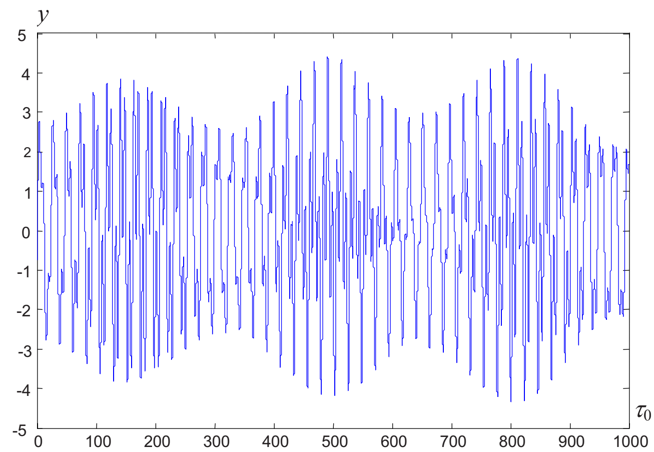


FIG. 7. (Color online) Analytic approximation (2.14) based on Eq. (3.17); $\gamma=0$.

$$\Delta^0 \approx -\gamma a_\gamma^0/\beta + O(\gamma^3), \quad a_\gamma^0[\zeta - 3\beta(a_\gamma^0)^2] = -\beta + O(\gamma^2). \tag{4.2}$$

Using Eq. (3.8), we find $a_\gamma^0 \approx 0.73$ and $\Delta_\gamma^0 \approx -0.54$. In addition, we note that the contribution of nonlinear force in oscillations near O is relatively small. Under this assumption, on the interval $\tau_1 \geq \tau_1^*$, one can consider the system, linearized near the steady state a_γ^0, Δ_γ^0,

$$\frac{d\xi}{d\tau_1} + \beta\eta = -\gamma\xi, \quad \frac{d\eta}{d\tau_1} - \frac{k_1}{a_\gamma^0}\xi = -\gamma\eta, \tag{4.3}$$

where $\xi = a_\gamma - a_\gamma^0$, $\eta = \Delta_\gamma - \Delta_\gamma^0$, and $k_1 = 9\beta(a_\gamma^0)^2 - \zeta$. The matching condition at $\tau_1 = \tau_1^*$ is

$$a_\gamma^0 + \xi = a(\tau_1^*) = A_0, \quad \frac{d\xi}{d\tau_1} = 0. \tag{4.4}$$

If $k_1 > 0$, then we obtain from Eq. (4.3)

$$\xi(\tau_1) = c_0 e^{-\gamma(\tau_1 - \tau_1^*)} \cos \kappa(\tau_1 - \tau_1^*),$$

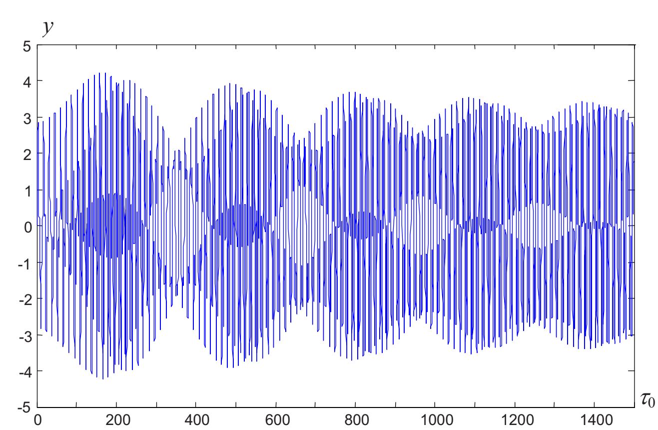


FIG. 8. (Color online) Numerical solution of Eq. (2.1); $\gamma=0.2$.

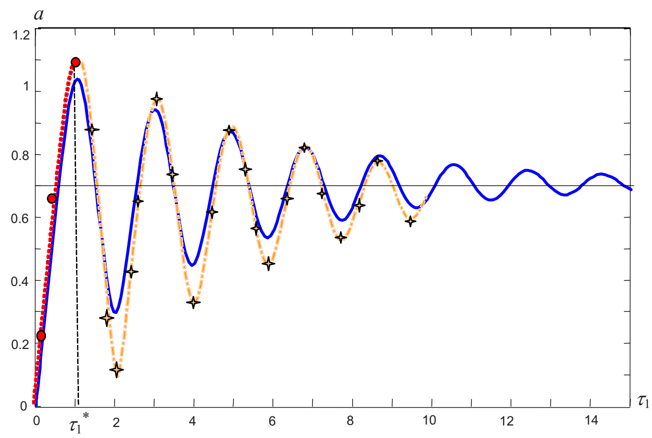


FIG. 9. (Color online) Numerical and analytic solutions; —: numerical solution of (2.15); analytic approximation; ---: segment (3.17); -·-: segment (4.5); γ=0.2

$$\eta(\tau_1) = rc_0 e^{-\gamma(\tau_1-\tau_1^*)} \sin \kappa(\tau_1-\tau_1^*), \quad \tau_1-\tau_1^*>0, \tag{4.5}$$

where $c_0=A_0-a_\gamma^0$, $\kappa^2=\Phi k_1/a_\gamma^0>0$, and $r=\kappa/F$. Using Eqs. (3.14) and (4.4), we obtain $\tau_1^*\approx 1$, $A_0=1.1$, $a_\gamma^0=0.73$, and $k_1=6$, and, therefore, $c_0=0.48$ and $\kappa=3.35$.

Figure 9 demonstrates a good agreement between the numerical solution of Eq. (2.15) (solid line) and the approximation found by matching the segment (3.17) (dot line) with the solution (4.5) of the linearized systems (dot-dashed line) at the point τ_1^*. Despite a certain discrepancy in the initial interval of motion, the numerical and analytic solutions approach closely to the steady state a_γ^* as τ_1 increases. This implies that a simplified model (3.17), being matched with the solution (4.5), suffices to describe the complicated resonance dynamics. Once the approximate solution of Eq. (2.15) is constructed, the overall response $y(\tau_0)$ is calculated by formula (2.14).

V. DYNAMICS OF A 2DOF SYSTEM

In this section, we investigate superharmonic energy transfer in a 2DOF system. The system is designed as a linear oscillator of mass M with an attached mass m; the attachment is coupled with the base by a cubic spring (Fig. 10). We denote by u_1 and u_2 the displacements of the masses M and m, respectively; by $k_{1,2,3}$, stiffness of the linear oscillator ($k_1>0$), linear coupling between the masses ($k_2>0$), and nonlinear coupling between the attachment and the base ($k_3>0$); the parameters $h_1>0$ and $h_2>0$ characterize dissipation in the linear oscillator and coupling, respectively (Fig. 10). An initial impulse applied to the mass M is treated as an external excitation; the attachment stands for an energy sink. We suppose that the energy imparted in the system at the initial time $t=0$ is partitioned among the principal and superharmonic modes but the energy exchange is due to superharmonic oscillations.

Under given assumptions, the equations of motion and the initial conditions have the following form:

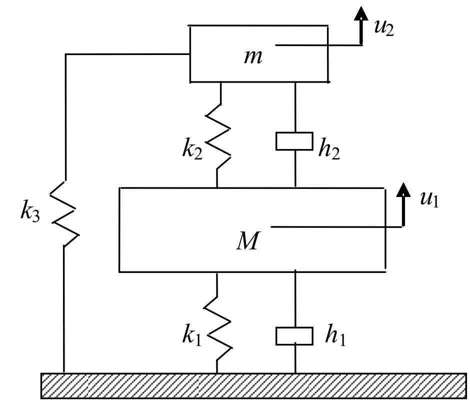


FIG. 10. Scheme of a 2DOF system.

$$M\frac{d^2u_1}{dt^2} + h_1\frac{du_1}{dt} + k_1u_1 + k_2(u_1-u_2) + h_2\left(\frac{du_1}{dt}-\frac{du_2}{dt}\right) = 0,$$

$$m\frac{d^2u_2}{dt^2} - k_2(u_1-u_2) + k_3u_2^3 - h_2\left(\frac{du_1}{dt}-\frac{du_2}{dt}\right) = 0,$$

$$t=0: x_1 = x_2 = 0; \quad \frac{du_1}{dt} = v_0 > 0, \quad \frac{du_2}{dt} = 0. \tag{5.1}$$

In this section, we consider the dynamics of the undamped system in which $h_1=h_2=0$. We assume that $m/M=\varepsilon\ll 1$, $k_2/k_1=\varepsilon c_0$, $k_3/k_1=\varepsilon k_0$. Introducing the dimensionless variable $t_0=\omega_1 t$, $\omega_1=\sqrt{k_1/M}$, and taking into account the relationships between the parameters, system (5.1) becomes

$$\frac{d^2u_1}{dt_0^2} + (1+\varepsilon c_0)u_1 = \varepsilon c_0 u_2,$$

$$\frac{d^2u_2}{dt_0^2} + k_0u_2^3 + c_0(u_2-u_1) = 0,$$

$$t_0=0: u_1=u_2=0, \quad \frac{du_1}{dt_0} = \frac{v_0}{\omega_1} = V_0, \quad \frac{du_2}{dt_0} = 0. \tag{5.2}$$

A. Reduction of the 2DOF system to a single oscillator

We introduce the independent variable $\tau_0=3t_0$ and rewrite (5.2) as

$$\frac{d^2u_1}{d\tau_0^2} + \omega_\varepsilon^2 u_1 = \varepsilon c u_2,$$

$$\frac{d^2u_2}{d\tau_0^2} + 8\alpha u_2^3 + c(u_2-u_1) = 0,$$

$$t_0=0: u_1=u_2=0, \quad \frac{du_1}{d\tau_0} = V, \quad \frac{du_2}{d\tau_0} = 0, \tag{5.3}$$

where $c=c_0/9$, $\alpha=k_0/72$, $V=V_0/3$; $\omega_\varepsilon^2=\frac{1}{9}+\varepsilon c$, and $\omega_\varepsilon=\frac{1}{3}+(\frac{3}{2})c$. Then, as in [13], the 2DOF system is reduced to a single oscillator. The first equation of Eq. (5.3) allows us to express u_1 as a solution of the linear equation with the input u_2,

$$u_1(\tau_0) = \varepsilon c J_{u_2}(\tau_0) + \omega_\varepsilon^{-1} V \sin(\omega_\varepsilon \tau_0), \tag{5.4}$$

where $J_{u_2}(\tau_0)\int_0^{\tau_0}\sin[\omega_\varepsilon(\tau_0-r)]u_2(r)dr$.

Here and below, the subscript denotes the function to be integrated. Substituting Eq. (5.4) into Eq. (5.3) and introducing the parameter μ to underline the superharmonic dynamics, we reduce Eq. (5.4) to the following form:

$$\frac{d^2u_2}{d\tau_0^2} + u_2 + \varepsilon\mu[(c-1)u_2 + 8\alpha u_2^3] - \varepsilon c^2 J_{u_2}(\tau_0) = c\omega_\varepsilon^{-1} V \sin(\omega_\varepsilon \tau_0),$$

$$\tau_0 = 0{:}u_2 = 0, \quad v_2 = \frac{du_2}{d\tau_0} = 0, \tag{5.5}$$

in which $\mu = 1/\varepsilon$; the parameter $\varepsilon\mu$ suggests that the sum in the parenthesis is small compared to all other terms of order 1.

By analogy with Eq. (2.2), the solution of Eq. (5.5) is represented as

$$u_2 = u + y_0, \tag{5.6}$$

where the low-frequency component y_0 is defined as a leading-order solution of the equation

$$\frac{d^2y_0}{d\tau_0^2} + y_0 - \varepsilon c^2 J_{y_0}(\tau_0) = c\omega_\varepsilon^{-1} V \sin(\omega_\varepsilon \tau_0), \tag{5.7}$$

where $J_{y_0}(\tau_0)$ is obtained from Eq. (5.5) by substituting y_0 for u_2. In order to separate the resonance harmonics, we find the eigenfrequencies of Eq. (5.7) from the characteristic equation

$$(s^2+1)(s^2+\omega_\varepsilon^2) - \varepsilon\omega_\varepsilon c^2 = 0. \tag{5.8}$$

This gives $(s_{1,2})^2 = -(\omega_{1,2})^2$, where

$$\omega_1 = 1 + \varepsilon\sigma_1, \quad \omega_2 = \frac{1}{3} + \varepsilon\sigma_2; \quad \sigma_1 = \frac{3}{16}c^2; \quad \sigma_2 = \frac{3c}{2}\left(1 - \frac{3}{8}c\right).$$

It follows from Eq. (5.7) that the sought solution of frequency ω_2 is $y_0(\tau_0) + O(\varepsilon)$, where

$$y_0(\tau_0) = \Lambda_1 \sin(\omega_2\tau_0), \quad \Lambda_1 = \frac{27cV}{8}. \tag{5.9}$$

We will show that transformations (5.6) and (5.9) eliminate harmonic components of frequency $\omega = \frac{1}{3} + O(\varepsilon)$ from the leading-order equation for u and thus prevents from the resonance divergence of the successive approximations. Note that $\omega_2 = 1/3 + \varepsilon\sigma_2$ is the eigenfrequency of the 2DOF linear subsystem of Eq. (5.3), whereas $\omega_\varepsilon = \frac{1}{3} + (\frac{3}{2})\varepsilon c$ is the partial frequency of the linear oscillator.

Inserting Eq. (5.6) in Eq. (5.5) and ignoring insubstantial terms, we obtain the following equation:

$$\frac{d^2u}{d\tau_0^2} + u + \varepsilon\mu f(u+y_0) - \varepsilon c^2 J_u(\tau_0) = 0,$$

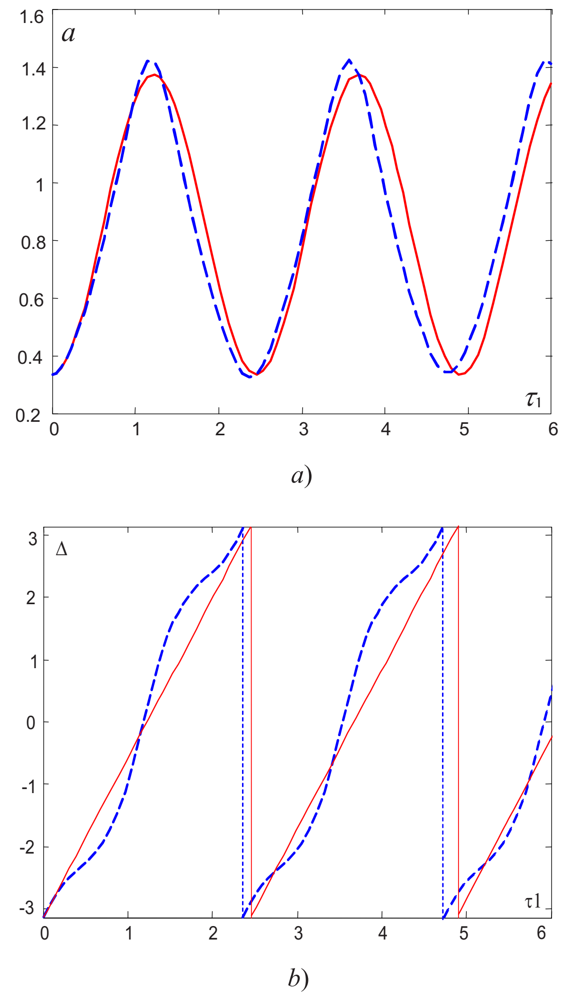


FIG. 11. (Color online) Numerical solution of (5.19) (dashed) and its analytic approximation (5.21) (solid); (a) temporal behavior of $a(\tau_1)$; (b) principal value of $\Delta(\tau_i)$ in the range $(-\pi, \pi]$.

$$\tau_0 = 0{:}u = 0, \quad v = \frac{du}{d\tau_0} = -\frac{\Lambda_1}{3}, \tag{5.10}$$

where $f(u) = (c-1)u + 8\alpha u^3$; $J_u(\tau_0)$ is obtained from Eq. (5.5) by substituting u for u_2. It is obvious that the generating subsystem of Eq. (5.10) ($\varepsilon = 0$) yields the solution of superharmonic frequency 1 and thus representation (5.6), (5.9) leads to the partition of spectral constituents.

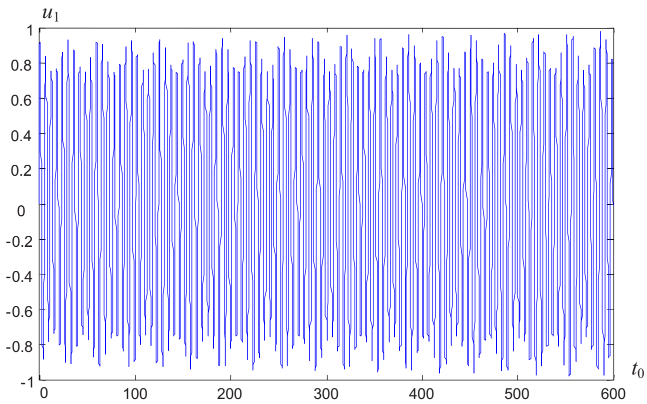
We analyze (5.10) in terms of the complex-conjugate variables ψ and ψ^* (2.5)

$$\psi = v + iu, \quad \psi^* = v - iu. \tag{5.11}$$

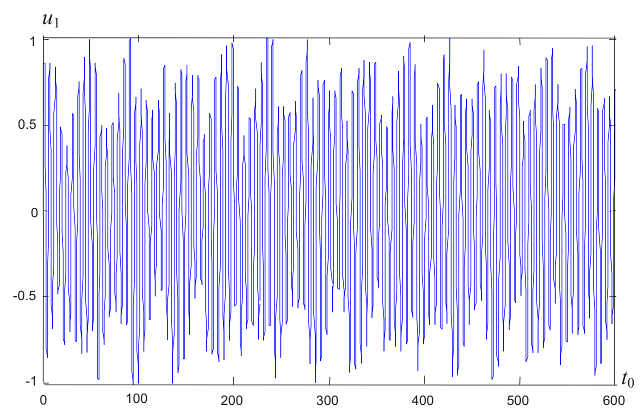
Substituting Eq. (5.11) into Eq. (5.10), we obtain the equation similar to Eq. (2.6)

$$\frac{d\psi}{d\tau_0} - i\psi + \varepsilon\mu f\left[-\frac{i}{2}(\psi - \psi^*) + y_0\right] - \varepsilon c^2 I_\varepsilon = 0,$$

$$\psi(0) = -\frac{\Lambda_1}{3},$$



a)



b)

FIG. 12. (Color online) Displacement u_1: (a) numerics, (b) theory; $\varepsilon=0.0675$, $c_0=6$, $k_0=4$, $V_0=1$.

$$I_\varepsilon = -\frac{i}{2}\int_0^{\tau_0} \sin[\omega_\varepsilon(\tau_0 - r)][\psi(r) - \psi^*(r)]dr. \quad (5.12)$$

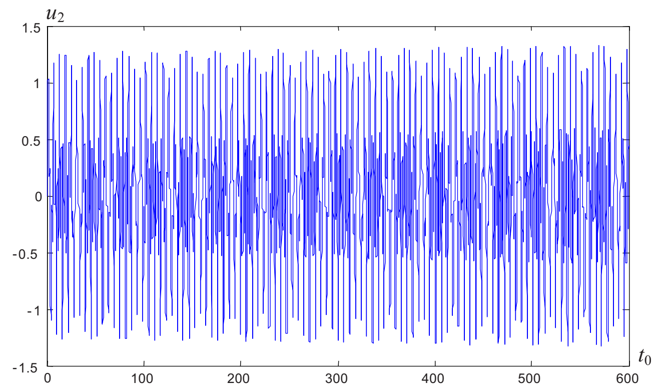
The solution of Eq. (5.12) is sought in the form (2.7). Using the previous arguments, we obtain ψ_0 satisfying the equation

$$\frac{\partial \psi_0}{\partial \tau_0} - i\psi_0 = 0, \quad \psi_0(\tau_0,\tau_1) = \varphi(\tau_1)e^{i\tau_0}, \quad \varphi(0) = -\frac{\Lambda_1}{3}. \quad (5.13)$$

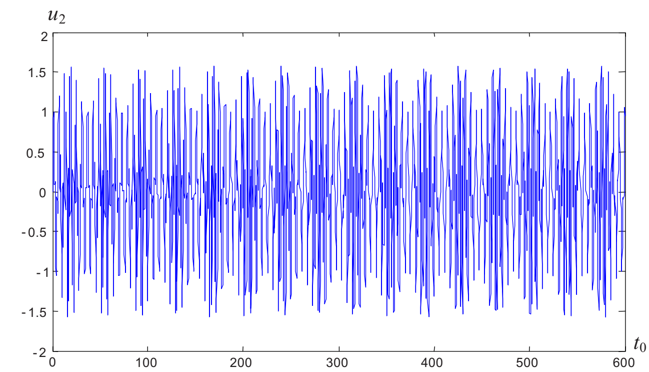
The first-order equation collecting the ε-order terms is

$$\frac{d\psi_1}{d\tau_0} - i\psi_1 + \frac{d\varphi}{d\tau_1}e^{i\tau_0} - c^2 I_{0\varepsilon} + i\mu\kappa(\psi_0 - \psi_0^* + \zeta_0) + \mu\alpha[(\psi_0 - \psi_0^*) + \zeta_0]^3 = 0, \quad (5.14)$$

where $\kappa=(1-c)/2$, $\zeta_0=\Lambda_1[(e^{i(\tau_0/3+\sigma_2\tau_1)}-e^{-i(\tau_0/3+\sigma_2\tau_1)})]$, $I_{0\varepsilon}$ is obtained from I_ε by substituting ψ_0 for ψ and τ_1 in the integrand is considered as a fixed parameter. In order to avoid the secular growth of $\psi_1(\tau_0,\tau_1)$ in τ_0, we eliminate the terms proportional to $e^{i\tau_0}$ from Eq. (5.14). Note that $I_{0\varepsilon}$ does not involve resonance terms, as the kernel $\sin\omega_\varepsilon(\tau_0-r)$ has the frequency $\omega_\varepsilon=\frac{1}{3}+O(\varepsilon)$ but the functions ψ_0 and ψ_0^* are of frequency 1. The calculation of the component $e^{i\tau_0}$ in the functions $\Psi=(\psi_0-\psi_0^*+\zeta_0)$ and Ψ^3 gives



a)



b)

FIG. 13. (Color online) Displacement u_2: (a): numerics, (b): theory; $\varepsilon=0.0675$, $c_0=6$, $k_0=4$, $V_0=1$.

$$\Psi = \varphi(\tau_1)e^{i\tau_0} - \varphi^*(\tau_1)e^{-i\tau_0} + \Lambda_1(e^{i(\tau_0/3+\sigma_2\tau_1)} - e^{-i(\tau_0/3+\sigma_2\tau_1)}) = \varphi(\tau_1)e^{i\tau_0} + N_r,$$

$$\Psi^3 = [\varphi(\tau_1)e^{i\tau_0} - \varphi^*(\tau_1)e^{-i\tau_0} + \Lambda_1(e^{i(\tau_0/3+\sigma_2\tau_1)} - e^{-i(\tau_0/3+\sigma_2\tau_1)})]^3 = -[3(|\varphi|^2\varphi + 2\Lambda_1^2\varphi) - \Lambda_1^3 e^{3i\sigma_2\tau_1})e^{i\tau_0} + N_r. \quad (5.15)$$

Let us denote $\varphi=\Lambda_1\varphi_0$. Inserting Eq. (5.15) into Eq. (5.14) and summing with all other terms of frequency 1, we obtain the following equation for $\varphi_0(\tau_1)$:

$$\frac{d\varphi_0}{d\tau_1} - i\mu\alpha\Lambda_1^2(3|\varphi_0|^2\varphi_0 - e^{3i\sigma_2\tau_1}) + i\mu\kappa_1\varphi_0(\tau_1) = 0,$$

$$\varphi_0(0) = -\frac{1}{3}, \quad (5.16)$$

where $\kappa_1=\kappa-6\alpha\Lambda_1^2$. The change of variables $\varphi_0=ae^{i\delta}$, $\varphi_0^*=ae^{-i\delta}$ reduces Eq. (5.17) to the form

$$\frac{da}{d\tau_1} = -\mu\alpha\Lambda_1^2 \sin\Delta,$$

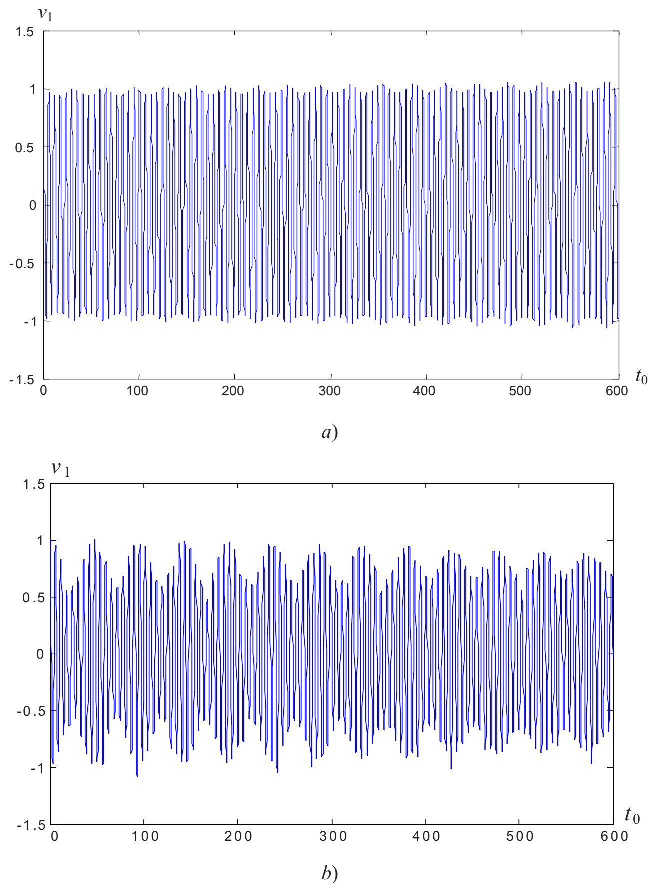


FIG. 14. (Color online) Velocity $v_1 = du_1/dt_0$: (a) numerics, (b) theory; $\varepsilon = 0.0675$, $c_0 = 6$, $k_0 = 4$, $V_0 = 1$.

$$a\frac{d\Delta}{d\tau_1} = -\zeta a + \mu\alpha\Lambda_1^2(3a^3 - \cos\Delta), \tag{5.17}$$

where $\Delta = \delta - 3\sigma_2\tau_1$ and $\zeta = \mu\kappa_1 + 3\sigma_2$. We now rewrite Eq. (5.17) in the form similar to Eq. (3.1)

$$\frac{da}{d\tau_1} = -\beta\sin\Delta,$$

$$a\frac{d\Delta}{d\tau_1} = a(-\zeta + 3\beta a^2) - \beta\cos\Delta,$$

$$a(0) = \tfrac{1}{3}, \quad \Delta(0) = \delta(0) = -\pi, \tag{5.18}$$

where $\beta = \mu\alpha\Lambda_1^2$. With these notations, system (5.18) can be investigated in the same way as in the previous sections. In particular, we obtain the following leading-order approximations of the solution of Eq. (5.5) [cf. Eq. (2.16)]

$$u_2(\tau_0,\tau_1) = \Lambda_1\left\{a(\tau_1)\sin[\tau_0 + \delta(\tau_1)] + \sin\left(\tfrac{1}{3}\tau_0 + \sigma_2\tau_1\right)\right\},$$

$$v_2(\tau_0,\varepsilon) = \Lambda_1\{a(\tau_1)\cos[1 + \delta(\tau_1)] + \tfrac{1}{3}\cos\left(\tfrac{1}{3}\tau_0 + \sigma_2\tau_1\right)\right]. \tag{5.19}$$

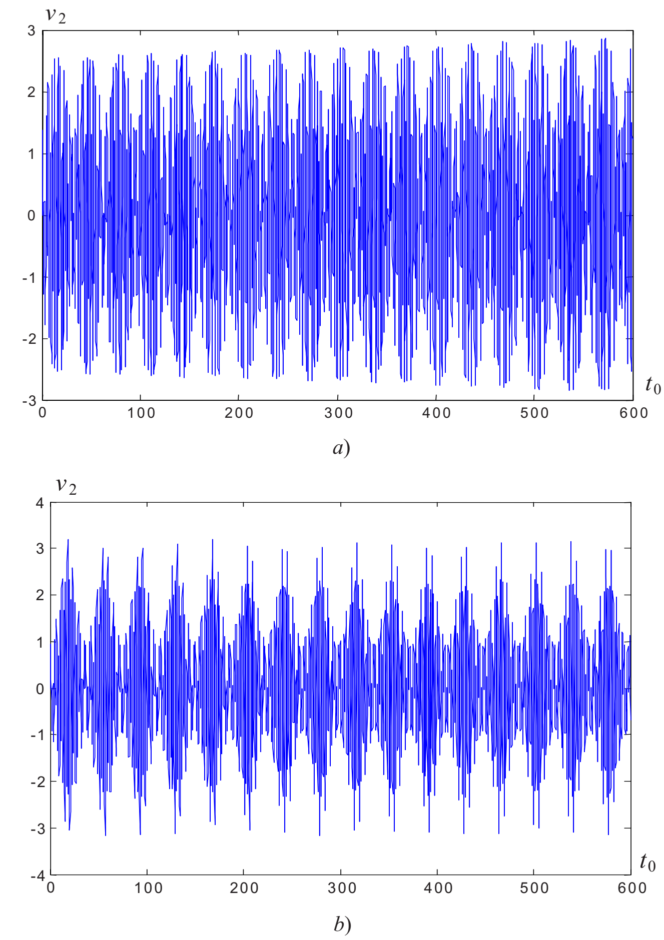


FIG. 15. (Color online) Velocity $v_2 = du_2/dt_0$: (a) numerics, (b) theory; $\varepsilon = 0.0675$, $c_0 = 6$, $k_0 = 4$, $V_0 = 1$.

After the solution u_2 is found, the response u_1 can be calculated by Eq. (5.4). Hence, the solution of the averaged system (5.18) suffices to determine the overall response of system (5.1).

B. Numerical analysis of motion

We present computational results for the system with the parameters $\varepsilon = 0.0675$, $c_0 = 6$, $k_0 = 4$, and $V_0 = 1$. To justify the choice of the parameters, we evaluate the right-hand side of Eq. (5.17). By definition, the right-hand side must be $O(1)$; the above parameters give $\beta = 1.39$ and $\zeta = 1.92$, and thus the required transformations are not contradictory. Note that the chosen parameters satisfy the second critical condition (3.5), that is the system exposes large nonlinear oscillations.

Reproducing the transformations of Sec. III, we obtain that the leading-order approximation to the solution of Eq. (5.18) is similar to Eq. (3.17)

$$a_0(\tau_1) = 1/3 + (\beta/\Omega_0)[1 - \cos(\Omega_0\tau_1)], \quad \Delta_0(\tau_1) = -\pi + \Omega_0\tau_1, \tag{5.20}$$

in which $\Omega_0 = 2.56$ and $\beta/\Omega_0 = 0.52$. The analytic approximation (5.20) is shown by solid curves in Fig. 11; dashed curves in Fig. 11 depict the numerical solution of Eq. (5.18). Note

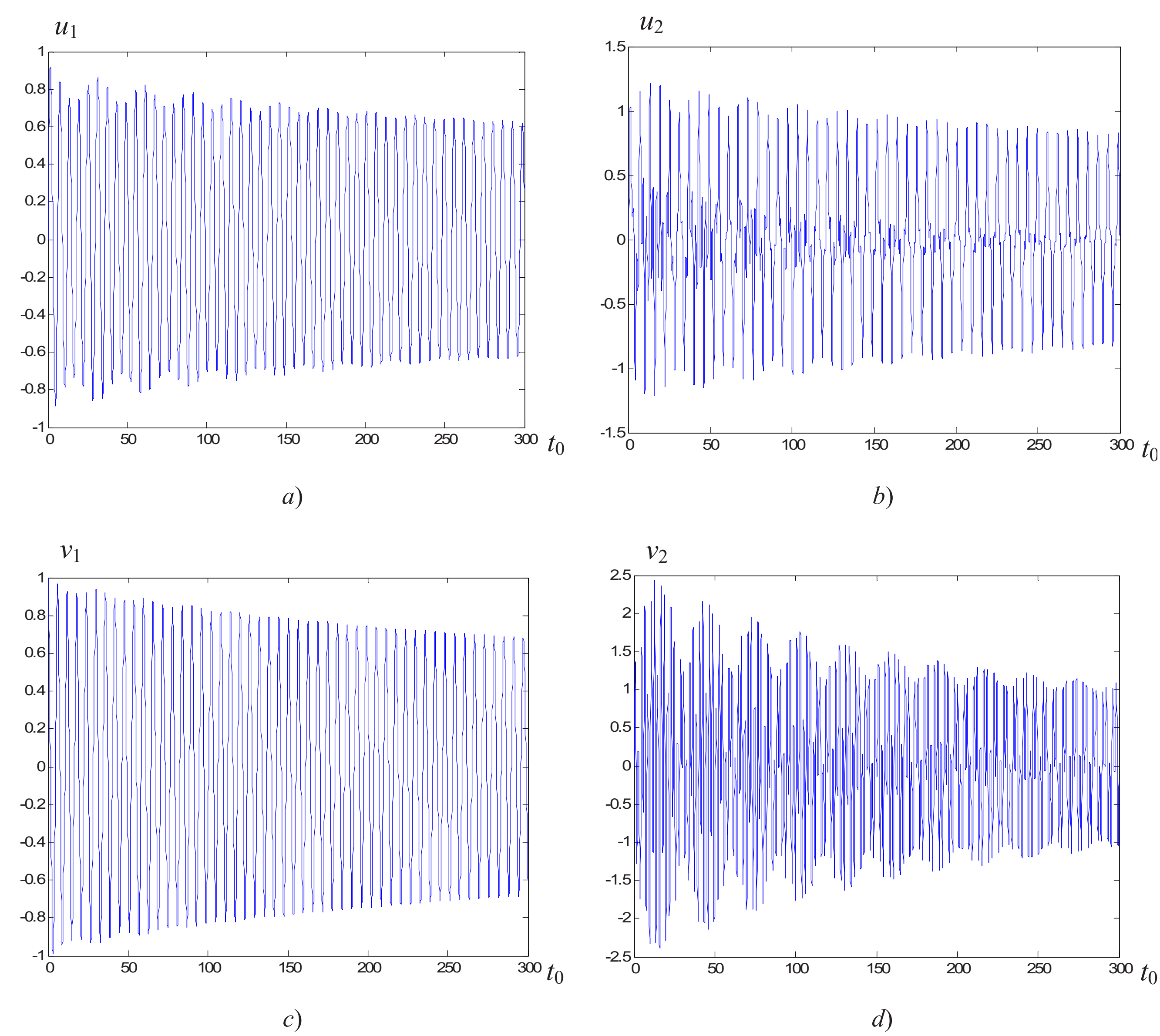


FIG. 16. (Color online) Temporal evolution of displacements and velocities in the dissipative system: (a) u_1, (b) u_2, (c) v_1, (d) v_2.

that a discrepancy between numerics and the analytic approximation is about 3%.

Figures 12–15 compare the results of numerical simulations of the original system (5.2) to the theoretical results. The solution u_2 is calculated by Eqs. (5.19) and (5.20); the solution u_1 is found from Eq. (5.4), in which u_2 is taken from Eqs. (5.19) and (5.20). For the computational purposes, we use the time scale $t_0=\tau_0/3$ introduced in Eq. (5.2). As seen in Figs. 12 and 13, the number and the distribution of the peaks of the envelope, the periods, and the magnitudes of the numerical and analytic solutions are very close, despite a slight irregularity of the analytic approximation u_1; moreover, the time positions of local minima of u_1 and v_1 corresponds to the positions of local maxima of u_2 and v_2 both for numerical and theoretical results. This confirms the existence of energy exchange between the oscillators and the sink.

C. Transient dynamics of a 2DOF system

Here we give a brief description of the resonance dynamics in a weakly dissipated system (5.1). We recall that, if $t\to\infty$, the transient process in Eq. (5.1) does not turn into stationary oscillations; it vanishes at rest O_1: $(u_1=u_2=0,\ v_1=v_2=0)$. To describe transient oscillations, one can use the same arguments as in Sec. IV. The trajectory is separated into two parts; motion is assumed to be close to strongly nonlinear undamped oscillations over the interval $[0,\tau_1^*]$, then the orbit approaches the rest state O_1 with an exponentially decreasing amplitude of oscillations.

For simplicity, we let $h_1=0$ in Eq. (5.1). Thus, the system under consideration is

$$M\frac{d^2u_1}{dt^2}+k_1u_1+k_2(u_1-u_2)+h_2\left(\frac{du_1}{dt}-\frac{du_2}{dt}\right)=0,$$

$$m\frac{d^2u_2}{dt^2} - k_2(u_1-u_2) + k_3u_2^3 - h_2\left(\frac{du_1}{dt} - \frac{du_2}{dt}\right) = 0,$$

$$t=0{:}x_1=x_2=0;\quad \frac{du_1}{dt}=v_0>0,\quad \frac{du_2}{dt}=0. \tag{5.21}$$

After introducing the variable $t_0=\omega_0 t$, $\omega_0=\sqrt{k_1/M}$, system (5.21) becomes

$$\frac{d^2u_1}{dt_0^2} + (1+\varepsilon c_0)u_1 + \varepsilon^2\eta_0\left(\frac{du_1}{dt_0} - \frac{du_2}{dt_0}\right) = \varepsilon c_0 u_2,$$

$$\frac{d^2u_2}{dt_0^2} + k_0u_2^3 + c_0(u_2-u_1) + \varepsilon\eta_0\left(\frac{du_2}{dt_0} - \frac{du_1}{dt_0}\right) = 0,$$

$$t_0=0{:}u_1=u_2=0,\quad \frac{du_1}{dt_0}=V_0,\quad \frac{du_2}{dt_0}=0, \tag{5.22}$$

where $\varepsilon^2\eta_0=h_2/\sqrt{k_1M}$. Since the contribution of nonlinear force in oscillations near O_1 is negligibly small, then the resonance construction of Sec. V is incorrect if τ_1 is large enough. In order to describe motion near rest O_1: $(x_1=x_2=0,\ v_1=v_2=0)$, one can consider system (5.22) linearized near O_1.

Let ξ_1 and ξ_2 denote small deviations of x_1, x_2 from O_1. A corresponding linearized system takes the form

$$\frac{d^2\xi_1}{dt_0^2} + (1+\varepsilon c_0)\xi_1 + \varepsilon^2\eta_0\left(\frac{d\xi_1}{dt_0} - \frac{d\xi_2}{dt_0}\right) = 0,$$

$$\frac{d^2\xi_2}{dt_0^2} + c_0(\xi_2-\xi_1) + \varepsilon\eta_0\left(\frac{d\xi_2}{dt_0} - \frac{d\xi_1}{dt_0}\right) = 0. \tag{5.23}$$

Formally, one can reduce Eq. (5.23) to a single integrodifferential equation. However, a system of low dimensionality enables a straightforward investigation.

The solution near O_1 can be written as

$$\xi_1(t_0,\varepsilon) \sim C_{11}e^{-\varepsilon^2\eta(t_0-t_0^*)/2}\sin\omega_\varepsilon(t_0-t_0^*) + \varepsilon^2\ldots,$$

$$\xi_2(t_0,\varepsilon) \sim C_{21}e^{-\varepsilon\eta(t_0-t_0^*)/2}\sin\lambda_0(t_0-t_0^*) + C_{22}e^{-\varepsilon^2\eta(t_0-t_0^*)/2}\sin\omega_\varepsilon(t_0-t_0^*) + \varepsilon, \tag{5.24}$$

where $\lambda_0=c_0^{1/2}$ is the partial frequency of the nonlinear sink; t_0^* is a matching point defined as in Sec. IV. It follows from Eq. (5.24) that the rate of decay of the harmonic component with frequency λ_0 vastly exceeds decrement of oscillations with frequency ω_ε and thus the convergence to rest obeys the low $\xi_{1,2}(t_0,\varepsilon)\sim e^{-\varepsilon^2\eta(t_0-t_0^*)/2}\sin\omega_\varepsilon(t_0-t_0^*)$ for both components as $t_0\gg t_0^*$. The results of numerical simulations (Fig. 16) demonstrate the disappearance of beating and the fast decay of superharmonic oscillations, transforming into slowly decaying resonance 1:1. The parameters of simulations are $\varepsilon=0.0675$, $c_0=6$, $k_0=4$, $V_0=1$, and $\eta_0=1/3$.

VI. CONCLUSION

In this paper, we obtained an analytic description of the superharmonic energy exchange. First, a periodically excited Duffing oscillator was considered. We proved that for certain values of the parameters, the incoming energy is partitioned among the principal and superharmonic modes but the energy exchange is due to intense superharmonic oscillations. Using the LPT concept, we constructed an explicit asymptotic solution describing the energy transfer from the source of energy into the oscillator. The LPT concept was extended to a system consisting of a linear oscillator weakly coupled with a nonlinear energy sink. We developed a procedure of reducing the system to a single oscillator and then analyzed superharmonic energy transfer in the reduced oscillator.

ACKNOWLEDGMENTS

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