PHYSICAL REVIEW E **81**, 055201(R) (2010)

Synchronization transition in the Kuramoto model with colored noise

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(Received 3 February 2010; revised manuscript received 6 May 2010; published 27 May 2010)

We present a linear stability analysis of the incoherent state in a system of globally coupled identical phase oscillators subject to colored noise. In that we succeed to bridge the extreme time scales between the formerly studied and analytically solvable cases of white noise and quenched random frequencies.

DOI: 10.1103/PhysRevE.81.055201 PACS number(s): 05.45.Xt, 05.40.−a

Attractively coupled self-sustained oscillators tend to adjust their phases and frequencies in the process known as synchronization [1]. The transition to synchronization is mediated through the interplay between attractive coupling and heterogeneities. It is a fundamental mechanism of self-organization in nature, important for pattern formation, information processing, and transport among others.

It is known that inherent differences in the oscillators or external perturbations can inhibit synchronization to the point where the superposition of signals in a large system of oscillators is completely incoherent. By increasing the coupling strength this incoherent state can become unstable and global oscillations are observed. Values for the critical coupling strength are known analytically in some limiting cases of heterogeneities such as quenched unimodal random frequencies or white noise. However, natural systems are usually subject to a wide range of perturbations and the assumption of a separation of time scales is often not justified. The random drift of parameters, for instance, is a major grievance for experimentalists and the critical coupling strength can deviate substantially from the predicted value for quenched parameter heterogeneity with the same variance.

Here we present a framework to find the transition point to synchronization in the Kuramoto model for a system of globally coupled oscillators subject to colored noise. We find that the type of the random process is essential for the transition point to synchronization, as can be expected from the rich behavior of the Kuramoto model with different quenched frequency distributions [2–4]. Our method is demonstrated at the example of globally coupled van der Pol oscillators with parametric noise.

The phase reduction method of Kuramoto [4] is applicable for weakly coupled weakly nonidentical oscillators and the obtained phase equations are a powerful tool to make qualitative and quantitative predictions even in the presence of strong nonlinearities. In [4,5] Kuramoto considers the case of all-to-all coupling where each oscillator couples equally strongly to all other oscillators in the system. The Kuramoto phase equations for such a system are

$$\dot{\vartheta}_n = \sigma\eta_n + \frac{1}{N}\sum_{m=1}^{N} g(\vartheta_m - \vartheta_n), \tag{1}$$

where ϑ_n is the phase of the oscillator n, $\sigma\eta_n$ is an individual force which may be the natural frequency of the oscillator or a time-dependent perturbation, $g(\Delta\vartheta)$ is a periodic coupling function of a phase difference, and N is the total number of oscillators. Disorder is realized through a distribution of random forces $\sigma\eta_n$, where σ denotes the noise amplitude in units of coupling strength. When the forces are time independent the system models an ensemble of oscillators with nonidentical natural frequencies. For quenched random frequencies with unimodal distribution a continuous phase transition from an incoherent regime of evenly distributed phases to a regime of partial synchronization can be observed when σ is changed. If, on the other hand, the η_n change very rapidly, they may be approximated by white noise. Again, a continuous phase transition is predicted as the noise strength is changed [4]. These two analytically solvable cases mark the extremes of time scale separation between the dynamics of the oscillators and the fluctuations. It is of great interest to know how the critical coupling strength for the phase transition to partial synchronization for colored noise differs from the known values at quenched or white noise.

I. EVOLUTION OF PHASE AND FREQUENCY DISTRIBUTION

In the thermodynamical limit $N\to\infty$, system (1) can be described by a density $p(\vartheta,\eta,t)$ of phases ϑ and forces η. The evolution of this density is given by the nonlinear equation

$$\partial_t p = L_\vartheta[p]p + L_\eta p, \tag{2}$$

where we assume that the forces η_n are independent linear random processes described by an operator L_η, whereas the operator $L_\vartheta[p]\cdot=-\partial_\vartheta(\dot{\vartheta}\cdot)$ that acts on the phases depends on the mean field and is therefore a functional of the oscillator density p. Our strategy is to linearize Eq. (2) around the stationary incoherent state $p(\vartheta,\eta,t)=p(\eta)/2\pi$ and look for a critical condition of the stability of the eigenmodes of p.

Given the eigenvalues λ_n and eigenmodes $\varphi_n(\eta)$ of L_η with $L_\eta\varphi_n=\lambda_n\varphi_n$ and $0=\lambda_0>\mathrm{Re}\,\lambda_1\cdots$, we start by expanding $p(\vartheta,\eta)$ and $g(\Delta\vartheta)$ as

$$p(\vartheta,\eta) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} z_{kn}e^{ik\vartheta}\varphi_n(\eta),$$

$$g(\Delta\vartheta) = \sum_{k=-\infty}^{\infty} g_k e^{-ik\Delta\vartheta}. \tag{3}$$

The Kuramoto phase equation (1) gives

$$\dot{\vartheta} = \sigma\eta + \int_{-\pi}^{\pi} d\vartheta' \int_{-\infty}^{\infty} d\eta' g(\vartheta' - \vartheta) p(\vartheta', \eta')$$

$$= \sigma\eta + \sum_{k=-\infty}^{\infty} z_{k0} g_k e^{ik\vartheta}. \tag{4}$$

Inserting Eqs. (3) and (4) into Eq. (2), we find for the modes z_{kn} the nonlinear equation

$$\dot{z}_{kn} = -ik \sum_{l=-\infty}^{\infty} z_{l0} g_l z_{(k-l)n} - ik\sigma \sum_{m=0}^{\infty} z_{km} M_{mn} + \lambda_n z_{kn}, \tag{5}$$

with $\eta\varphi_n(\eta) = \sum_{m=0}^{\infty} M_{nm}\varphi_m(\eta)$. We remark that for this equation the celebrated ansatz of Ott and Antonsen [6] is in general only applicable when all eigenvalues λ_n are equal to zero, i.e., in the limit of quenched random frequencies. The order parameter $R = |\langle e^{i\vartheta}\rangle| = |z_{10}|$ is zero when the phases are distributed uniformly in the incoherent state. One can check that the incoherent distribution $p(\vartheta, \eta, t) = \varphi_0(\eta)/2\pi$ or $z_{kn} = \delta_{k0}\delta_{n0}$ is a stationary solution of the evolution equation (2). Thus, keeping only the terms linear in the small quantities z_{kn} we obtain linearized equations for the dynamics of the modes

$$\dot{z}_{k0} = -ik(g_0 + g_k) z_{k0} - ik\sigma \sum_m z_{km} M_{m0},$$

$$\dot{z}_{kn} = (\lambda_n - ikg_0) z_{kn} - ik\sigma \sum_m z_{km} M_{mn}. \tag{6}$$

The Fourier modes of the probability distribution, i.e., the z_{kn} for different k, do not interact linearly with one another and can be studied separately. The term $-ikg_0 = -ikg_0^*$ which appears as a self-interaction term for all eigenmodes can be neglected for the stability analysis since it is imaginary and only results in a bias to all frequencies. The incoherent state becomes unstable when the largest real part of the eigenvalues of the linear ODE (6) becomes positive.

II. RANDOM TELEGRAPH PROCESS

Consider the Kuramoto model with attracting sinusoidal coupling function $g(\Delta\vartheta) = \sin(\Delta\vartheta - \alpha)$, $\cos\alpha > 0$, and with independent random forces $\eta_n \in \{-1, 1\}$ which change sign as a dichotomous random Markov process with equal transition rate γ between both values (telegraph process, Fig. 1). The noise process has exactly two eigenmodes ($\lambda_0 = 0$, $\lambda_1 = -2\gamma$), and following the analysis in the previous section we find that the linear stability of the first Fourier mode at the incoherent state is determined by the eigenvalues of the matrix

$$J = \begin{pmatrix} \frac{1}{2} e^{i\alpha} & i\sigma \\ i\sigma & -2\gamma \end{pmatrix}. \tag{7}$$

Given α and the flipping rate γ, necessary and sufficient conditions for stability are

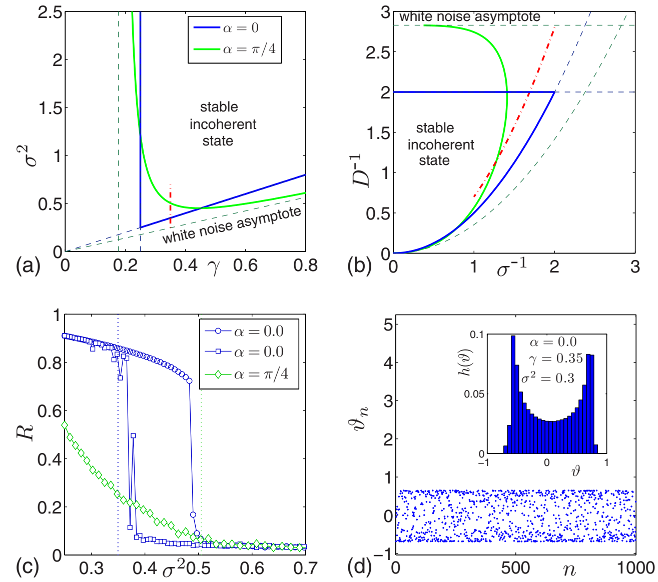

FIG. 1. (Color online) Case of the random telegraph process. Subfigures (a) and (b) show the parameter regions of linear stability of the incoherent state for (a) switching rate γ and amplitude σ^2 or (b) inverse amplitude σ^{-1} and noise strength D^{-1}. Shown are the two cases $\alpha=0$ (dark blue, solid line) and $\alpha=\pi/4$ (light green, solid line). Subfigure (b) can directly be compared to the case of Ornstein-Uhlenbeck type noise in Fig. 2. The dashed lines are asymptotes obtained from Eq. (8). Subfigure (c) shows the time-averaged order parameter. The theoretical predictions are verified by direct numerical simulation in systems of size $N=5000$ at $\gamma=0.35$ using σ^2 as bifurcation parameter. The corresponding one dimensional curve is shown as dashed and dotted (red) line in (a) and (b). The vertical dotted lines mark the theoretical transition points. Hysteresis is observed for $\alpha=0.0$ (blue circles and squares). Subfigure (d) shows the phases of 1000 oscillators in the partially synchronized state for $\alpha=0.0$, $\gamma=0.35$, and $\sigma^2=0.3$ with a phase histogram in the inset.

$$\sigma^2 > \gamma\cos\alpha\left(1 + \frac{\sin^2\alpha}{(4\gamma - \cos\alpha)^2}\right), \quad \cos\alpha - 4\gamma < 0. \tag{8}$$

The frequency Ω of the mean field at the bifurcation is $\Omega = 2\gamma\sin\alpha/(4\gamma - \cos\alpha)$. The white-noise limit with $D = 2\sigma^2/\gamma$ is recovered from Eq. (8) with $D_{\mathrm{cr}} = 0.5\cos\alpha$ and $\Omega = 0.5\sin\alpha$. From the linear stability analysis we are not able to predict whether the transition is supercritical or subcritical. The onset of hysteresis in a very similar model with dichotomous Markov noise has been investigated in detail in [7].

III. ORNSTEIN-UHLENBECK PROCESS

Let us now consider i.i.d. random forces diffusing in the fashion of an Ornstein-Uhlenbeck (OU) process according to the Langevin equation $\dot{\eta} = -\gamma\eta + \sqrt{2\gamma}\xi(t)$ with white noise $\langle\xi(t)\xi(t')\rangle = \delta(t - t')$. The rate γ determines the time scale of the diffusion. To visualize both the white noise and the quenched noise limit, it is of advantage to use the parameter

$D=\sigma^2/\gamma$. Then the white-noise limit with finite noise strength D is reached as $\sigma^{-1}\to 0$ and quenched noise corresponds to $D^{-1}\to 0$. The eigenvalues and eigenfunctions of the Fokker-Planck operator L_η for an OU process are intimately related to those of the quantum harmonic oscillator [8]. One finds $\lambda_n=-\gamma n$ and $M_{mn}=\delta_{mn-1}\sqrt{n}+\delta_{mn+1}\sqrt{n+1}$. This turns the eigenvalue problem of Eq. (6) into an infinite system of second order difference equations.

A complete solution of the eigenvalue problem for the ODE (6) depending on D, σ, and α is not necessary to determine the critical parameters at the synchronization transition. Instead we notice that at criticality there is an imaginary eigenvalue $i\Omega$, which gives us an implicit condition for the bifurcation line of the first Fourier mode

$$i\Omega z_{10} = -ig_1 z_{10} - i\sigma z_{11},$$

$$i\Omega z_{1n} = -\sigma^2 D^{-1} n z_{1n} - i\sigma\sqrt{n} z_{1n-1} - i\sigma\sqrt{n+1} z_{1n+1}. \quad (9)$$

At the transition Ω is the frequency of the emerging mean field. Defining $\mu_n=-i\sqrt{n+1}z_{1n+1}/z_{1n}=f_{n+1}/f_n$ we obtain the equations

$$-\mu_{n-1} = \frac{n}{(nx+iy)-\mu_n}, \quad \mu_0 = ig_1\sigma^{-1}+iy,$$

$$0 = nf_{n-1} + (nx+iy)f_n - f_{n+1} \quad (10)$$

with the dimensionless quantities $x=\gamma\sigma^{-1}=D^{-1}\sigma$ and $y=\Omega\sigma^{-1}$ relating the time scales in the system. The first equation defines the value of μ_0 as a continued fraction in terms of g_1/σ, x, and y. The second equation can be used to obtain an analytic expression for this continued fraction. Using a technique of Euler [9] or recurrence relations of special functions, we find $\mu_0(x,y)$ in terms of Kummer's function $M(a,b,z)$ [10]

$$\mu_0(x,y) = ig_1\sigma^{-1} + iy = -\frac{1}{(\beta+1)x}\frac{M(2,\beta+2,x^{-2})}{M(1,\beta+1,x^{-2})}, \quad (11)$$

with $\beta=x^{-1}(x^{-1}+iy)$, and finally at criticality the relations

$$\sigma = \frac{|g_1|}{|\mu_0(x,y)-iy|}, \quad \mathrm{Im}\, g_1 = -\sigma\, \mathrm{Re}\, \mu_0(x,y). \quad (12)$$

Equations (11) and (12) are one of the main results of this Rapid Communication. They enable us to determine the critical conditions without the need of extensive simulations.

Let us consider the stability of the first Fourier mode in the Kuramoto model with sinusoidal coupling function $g(\Delta\vartheta)=\sin(\Delta\vartheta-\alpha)$, i.e., $g_1=i/2\exp(i\alpha)$. The critical lines for that case (Fig. 2) in the (σ^{-1},D^{-1}) parameter plane are parametrized by the time scale ratio $x=0\ldots\infty$. For fixed nonzero α the time scale ratio $y(x)$ has to be determined numerically from Eq. (12). For vanishing α, μ_0 is real, i.e., $y=0$, and the critical line can directly be computed. The white-noise limit $D_{\mathrm{cr}}=0.5\cos\alpha$ and $\Omega=0.5\sin\alpha$ are obtained from Eq. (10) letting $\sigma\to\infty$. The quenched noise limit $x\to 0$ is not trivial. For $\alpha=0$ it must be compared to $\sigma_{\mathrm{cr}}^{-1}=2/\pi\varphi_0(0)=\sqrt{8/\pi}$ [4]. No simple expression exists for $\alpha \neq 0$ [11]. To test our analytic results Monte Carlo simulations of the Kuramoto model Eq. (1) with OU random forces have been carried out with finite step size dt. The displacements of phase ϑ and force η can be drawn directly from the transition probability $p(\vartheta+d\vartheta,\eta+d\eta,t+dt|\vartheta,\eta,t)$ under the assumption of a slowly changing mean field force $f(\vartheta)$ which is assumed constant during an integration step. The two random variables

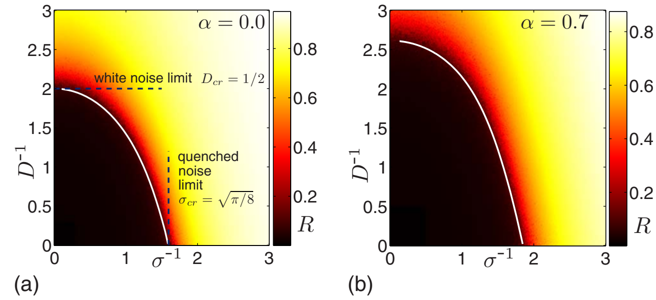

FIG. 2. (Color online) Synchronization transition of the Kuramoto model Eq. (1) with $g(\Delta\vartheta)=\sin(\Delta\vartheta-\alpha)$ subject to random forces of Ornstein-Uhlenbeck type with variance σ^2 and dissipation rate γ. Shown is the time averaged order parameter R as a function of σ^{-1} and $D^{-1}=\gamma/\sigma^2$ for (a) $\alpha=0.0$ and (b) $\alpha=0.7$ in a system of size $N=5000$ averaged over 150 time units after a transient of 50 units. The solid white line marks the critical conditions Eq. (11) obtained by changing $x=D^{-1}\sigma$ from 0.01 to 20. The white-noise limit is located on the ordinate axis for $\sigma^{-1}\to 0$ and the quenched noise limit on the abscissa axis for $D^{-1}\to 0$.

$$r_1 = d\vartheta - f(\vartheta)dt + D\sigma^{-1}(1-e^{-\sigma^2 D^{-1}dt})\eta(t),$$

$$r_2 = \eta(t+dt) - e^{-\sigma^2 D^{-1}dt}\eta(t) \quad (13)$$

are Gaussian [8] with correlation matrix

$$\Sigma_{11} = D^2\sigma^{-2}(2\sigma^2 D^{-1}dt - 3 + 4e^{-\sigma^2 D^{-1}dt} - e^{-2\sigma^2 D^{-1}dt}),$$

$$\Sigma_{12} = \Sigma_{21} = D\sigma^{-1}(1-e^{-2\sigma^2 D^{-1}dt})^2,$$

$$\Sigma_{22} = 1 - e^{-2\sigma^2 D^{-1}dt}. \quad (14)$$

Both values in Eq. (13) can be sampled at all time scales and in particular also for $\sigma^{-1}\to 0$, as well as $D^{-1}\to 0$. The critical line obtained from Eq. (11) and the numerical simulations are in excellent agreement (Fig. 2).

IV. EXAMPLE OF GLOBALLY COUPLED VAN DER POL OSCILLATORS

Our result can be used to obtain an accurate value of the critical coupling strength in large systems of nonlinear self-sustained oscillators with mean field coupling whenever the Kuramoto phase reduction method is applicable. Here we demonstrate our method at the example of weakly coupled weakly nonidentical van der Pol oscillators. The weak heterogeneity is necessary to apply the averaging and to obtain the coupling function $g(\Delta\vartheta)$ that only depends on a phase difference. Thus we consider perturbations at a time scale

$\gamma \ll \omega_0 = 2\pi/T$ and amplitude σ_{eff} of comparable order, i.e., $x = \gamma/\sigma_{\text{eff}} \approx 1$. Let the dynamics of the system be given by

$$\ddot{u}_n - \epsilon_n(1 - u_n^2)\dot{u}_n + u = \frac{K}{N}\sum_{m=1}^{N}(\dot{u}_m - \dot{u}_n), \tag{15}$$

with coupling strength K, time dependent parameter $\epsilon_n = 2 + \sigma_0 \eta_n(t)$, and normalized OU-noise $\eta_n(t)$ with relaxation rate $\gamma = 10^{-2}$. The phase response curve $Z(\vartheta)$ necessary for the phase reduction to Eq. (1) [4] has been calculated using the tool XPPAUT [12] [Fig. 3(a)]. Depending on the phase, a perturbation in ϵ will advance or delay an individual oscillator. Since the values of ϵ_n change slowly we can average the effect over one period to obtain an effective frequency heterogeneity of amplitude $\sigma_{\text{eff}} = \sigma_0 |h|$ with $h = -12.2 \times 10^{-2}$. Kuramoto's phase equations for the system of globally coupled van der Pol oscillators Eq. (15) are

$$\dot{\vartheta}_n = \omega_0 + h\sigma_0\eta_n + \frac{K}{N}\sum_{m=1}^{N} g(\vartheta_m - \vartheta_n). \tag{16}$$

with period $T = 2\pi/\omega_0 = 7.650$, $g_1 = 22.7 \times 10^{-2} \exp(i\alpha)$, and $\alpha = 16.5 \times 10^{-2}$. Using a perturbation of amplitude $\sigma_0 = 8.1 \times 10^{-2}$ we have $\sigma_{\text{eff}} = 10^{-2}$ and therefore $x = 1.0$. Since the phase coupling function is slightly asymmetric, i.e., $\alpha \neq 0$, we use a numerical root finding method to find $y = 13.5 \times 10^{-2}$ such that Eqs. (12) are satisfied. At the transition point to synchronization the effective noise amplitude in units of the critical coupling strength is $\sigma_{\text{cr}} = 38.7 \times 10^{-2}$ so that the critical coupling strength is $K_{\text{cr}} = \sigma_{\text{eff}}/\sigma_{\text{cr}} = 2.6 \times 10^{-2}$. In Fig. 3(b) this value is compared to direct simulations. The critical coupling strength for quenched noise $\gamma \to 0$ of same standard deviation σ_0 is significantly higher at $K_{\text{cr}} = 3.5 \times 10^{-2}$. Both critical coupling strengths are accurately predicted by our theory.

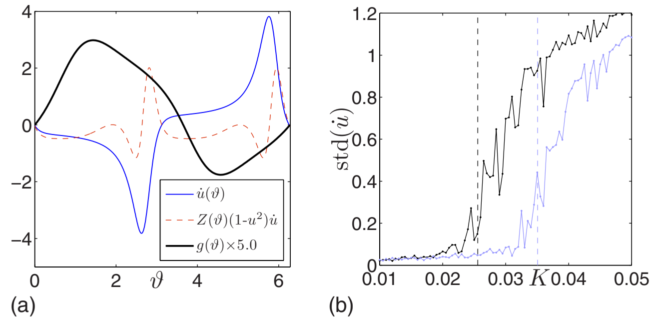

FIG. 3. (Color online) Phase reduction and synchronization transition in the system (15) of globally coupled van der Pol oscillators subject to parametric colored noise. In (a) we show the velocity $\dot{u}$ (solid thin line), the response of the phase velocity to a constant perturbation of the parameter ϵ (dashed) and the phase coupling function $g(\Delta\vartheta)$ (solid bold, amplified $\times 5$). In (b) we present time averaged values of the ensemble standard deviation std($\dot{u}$) as a function of the coupling strength K for $N = 5000$ oscillators, $\sigma_0 = 0.081$ and $\gamma = 0.01$ (black) and $\gamma = 0.0$ (light blue). The vertical dashed lines mark the critical coupling strength obtained by our method.

ACKNOWLEDGMENTS

The author thanks H. Kori for valuable feedback. This work was supported by JST Special Coordination Funds for Promoting Science and Technology.

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