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Duality and Fisher zeros in the two-dimensional Potts model on a square lattice

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A phenomenological approach to the ferromagnetic two-dimensional (2D) Potts model on square lattice is proposed. Our goal is to present a simple functional form that obeys the known properties possessed by the free energy of the q-state Potts model. The duality symmetry of the 2D Potts model together with the known results on its critical exponent α allows us to fix consistently the details of the proposed expression for the free energy. The agreement of the analytic ansatz with numerical data in the $q=3$ case is very good at high and low temperatures as well as at the critical point. It is shown that the $q>4$ cases naturally fit into the same scheme and that one should also expect a good agreement with numerical data. The limiting $q=4$ case is shortly discussed.

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I. INTRODUCTION

The Potts model[1] is the most natural generalization of the two-dimensional (2D) Ising model, and it is deeply connected with many areas of both statistical physics and mathematics (for nice detailed reviews, see [1–5]; for the relation with the problem of color confinement, see [6] and review [7]).

It has been very difficult at the moment to compute directly the analytic free energy of the ferromagnetic Potts model in two dimensions on square lattice (which is the main object of the present paper) as it was done by Onsager in the Ising case [8]. Thus, it will be proposed here a less direct approach which is, nevertheless, able to give nontrivial analytic information on the free energy. The main purpose of this approach is not to give an exact solution for the free energy but rather to present a simple functional form that obeys the known properties possessed by the free energy of the q-state Potts model. In particular, this method may give a reasonable approximation of the free-energy function in all the range of temperatures even the not yet explored ones, i.e., outside the results of high and low temperatures or the neighborhood of critical point. Following the point of view of the phenomenological Regge theory of scattering (see [9,10]; two detailed reviews are [11,12]), a reasonable analytic form for the free energy in terms of one free parameter will be derived. In the present case it plays the role of Regge's trajectories in high-energy physics since it parametrizes some analytical features of the Potts model such as its duality properties and the Fisher zeros, in an analog way as the Regge trajectories encode the nonperturbative duality properties of scattering amplitudes, as was first observed in [13]. This can be done by requiring that the sought analytic expression for the free energy of the 2D Potts model should be compatible (in a suitable sense explained in the next section) with the proposal made in [14–16] for the free energy of the three-dimensional Ising (3DI) model. The proposal of [14–16] is in a very good agreement with numerical data so that one should expect that this approach should provide one with a formula for the free energy of 2D Potts which fit equally well the available numerical data. It will be shown that this is indeed the case. A nontrivial by-product of the present approach is that the proposed free energy of the 2D Potts model has the locus of the Fisher zeros which coincides with the well-known and well-tested conjecture in [17].

The paper is organized as follows: in the second section, it will be discussed a suitable consistency condition (called here "dimensional compatibility") which allows to derive an analytic ansatz for the free energy of the 2D Potts model in terms of few q-dependent curves. In the third section, it will be analyzed how the known duality symmetry of the Potts model and its known critical behavior fix all but one curve. In the fourth section, the proposed ansatz will be compared with the available numerical data both at high and low temperatures as well as at the critical point in the case $q=3$. In the fifth section, the $q\geq 4$ cases will be discussed. In the sixth section it will be shown how the proposed ansatz automatically predicts the locus of Fisher zeros consistent with the well-known conjecture.

II. DIMENSIONAL COMPATIBILITY

It is well known that the main thermodynamical quantities are combinatorial in nature. This fact makes the analysis of three-dimensional lattices even more difficult than the two-dimensional cases. In this respect, it is a rather surprising result that one can even imagine to write down a simple explicit functional form for the free energy of the Ising model in three dimensions which obeys known physical properties (in particular, good agreement with the available numerical data). Such simple expression is based on the assumption that the change in the combinatorial complexity when going from one to two dimensions is formally quite similar to the analogous change in the combinatorial complexity when going from two to three dimensions. One can write the exact Onsager solution for the free energy in two

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[1]In the present paper we will only consider spin models on regular square lattices.

dimensions as a convolution integral of the known exact free energy in one dimension with a suitable kernel. This integral kernel, which can be found explicitly, is the mathematical object responsible for the change in the combinatorial complexity when going from one to two dimensions [14]. Then, one argues that the change in many combinatorial quantities on hypercubic lattices (such as, for instance, the scaling of the number of states of a given energy with the size of the lattice) when going from one to two dimensions are similar to the analogous change when going from two to three dimensions (this is somehow confirmed by a comparison with the available numerical data [15,16]). Thus, it is not unnatural to assume that a suitable integral kernel exists such that when one consider its convolution with the Onsager solution one gets the (still unknown) exact free energy of the Ising model in three dimensions. Our assumption based on the above considerations is that such kernel is as similar as possible to the already known one allowing the jump from one to two dimensions.

The Kallen-Lehmann representation [14] gives rise to an ansatz for the free energy of the 3DI model of the following form[2]:

$$F_{3\mathrm{D}}^{(\zeta_i,\lambda)}(\beta) = F_{2\mathrm{D}}(\beta) + \frac{\lambda}{(2\pi)^2}\int_0^\pi dz \int_0^\pi dy \cdot \log\left\{\frac{1}{2}\left[1 + \left(1 - \left[2\frac{(\Delta(z)-1)^{\zeta_1}}{\Delta(z)}\right]^{\zeta_2}\sin^2 y\right)^{\zeta_3}\right]\right\},$$

$$\Delta(z) = \{1 + [1 - k_{eff}(\beta)^2 \sin^2 z]^{\zeta_0}\}^2, \quad \zeta_0,\zeta_1,\zeta_2,\zeta_3 > 0,$$

$$0 \le [k_{eff}(\beta)]^2 \le 1, \quad 1 \le \Delta(z) \le 4, \tag{1}$$

where the values of the parameters in the case of the two-dimensional Ising model would be $\lambda=1$, $\zeta_i=1/2$ for $i=0,\ldots,3$.

Namely, one may find the operator $\mathbf{D}_{1,2}$ [see Eq. (2)] which "dresses" the trivial one-dimensional solution of the Ising model giving rise to the Onsager solution. Then, the Kallen-Lehmann free-energy for the three-dimensional Ising model is deduced [14] by modifying $\mathbf{D}_{1,2}$ in such a way that the parameters which in $\mathbf{D}_{1,2}$ are fixed to be 1 and 1/2 (namely, in the two-dimensional case, $\lambda=1$, $\zeta_i=1/2$) become the free parameters. Afterwards, one can use such a modified operator to dress the Onsager solution obtaining a useful ansatz for the free energy of the three-dimensional Ising model. To be more precise, it is possible to define a class of operators $\mathbf{D}_{D,q}$ such that when they act on the free energy $F_{D,q}$ of the q-state ferromagnetic Potts model on hypercubic lattices in D dimensions give rise to the free energy $F_{(D+1),q}$ of the q-state Potts model in $(D+1)$ dimensions

$$\mathbf{D}_{D,q}[F_{D,q}(\beta)] = F_{(D+1),q}(\beta). \tag{2}$$

To fix the arbitrariness of the above definition it necessary to specify the domain of $\mathbf{D}_{D,q}$ and how it acts. Since we want that the free energies of reasonable systems belong to the domain of $\mathbf{D}_{D,q}$, the domain of $\mathbf{D}_{D,q}$ will be the class of functions which are smooth on $\mathbb{R}^+$ besides at most a finite number of points in which some of the derivatives of the function may be discontinuous. The variable is the inverse temperature and the singular point (or points) represents the phase transition. The range of the operator coincides with the domain since, by definition, when this operator acts on the free energy of the Ising model on a hypercubic lattice it generates the free energy of an analogous system in one more dimension. A reasonable way to represent these operators is as integral Kernel [14]: the simplest of these integral kernels can be constructed explicitly by comparing the Onsager solution with the free energy of the one-dimensional Ising model. Then, one can use an integral Kernel of the same functional form to act on the Onsager solution obtaining an approximate ansatz for the functional form of the free energy of the Ising model in three dimensions (the comparison of the ansatz with the numerical data is quite promising [15,16]).

[2]The parameters appearing in these formulas are related with those appearing in [14] by the following identities $\zeta_1=\nu$, $\zeta_3=1/2=\zeta_2$, and $\zeta_0=\alpha$.

Consistency condition

Let $\mathbf{Q}_{D,q}$ be the operators which when applied to the free energy $F_{D,q}$ of the q-state ferromagnetic Potts model in D dimensions give rise to the free energy $F_{D,(q+1)}$ of the $(q+1)$-state Potts model in D dimensions

$$\mathbf{Q}_{D,q}[F_{D,q}(\beta)] = F_{D,(q+1)}(\beta). \tag{3}$$

The domain of $\mathbf{Q}_{D,q}$ is the class of functions which are smooth on $\mathbb{R}^+$ besides at most a finite number of points in which some of the derivatives of the function may be discontinuous (again the singular points represent the phase transitions of the system). Also in this case the range of the operator coincides with the domain. Therefore, being the ranges and the domains of the operators $\mathbf{D}_{D,q}$ and $\mathbf{Q}_{D,q}$ compatible, it makes sense to compose them. In particular, one can observe that they have to satisfy a sort of commutativity constraint:

$$\mathbf{Q}_{(D+1),q} \cdot \mathbf{D}_{D,q} = \mathbf{D}_{D,(q+1)} \cdot \mathbf{Q}_{D,q}. \tag{4}$$

If one supposes that Eq. (1) is the exact free energy of the three-dimensional Ising model for some values of the parameters, the simplest way to satisfy Eq. (4) is the following approximate analytic formula for the free energy of the two-dimensional ferromagnetic Potts model:

$$F_{2\mathrm{D}}(q,u) = C_q + \frac{\lambda(q)}{2\pi}\int_0^\pi dt \, \log\left\{\frac{1}{2}[1 + (1 - (k_{2\mathrm{D}}(q,u))^2 \sin^2 t)^{s(q)}]\right\},$$

$$(k_{2\mathrm{D}}(q,u))^2 \le 1; \quad C_q = \log\left(2\frac{\exp(\beta) + (q-1)}{q}\right);$$

$$u = \exp(-\beta), \tag{5}$$

where $k_{2\mathrm{D}}(q,u)$ is the function encoding the duality properties of the model, the function C_q can be found by comparing the high-temperature expansion of Ising and Potts models in two dimensions (see, for instance, [1]); $\lambda(q)$ and $\varsigma(q)$ are two parameters.[3] It is worth to note that we are still free to add a constant (which depends on q) to the above free energy. Unfortunately this fact prevents us from using the Baxter's results on the free energy on the critical point to fix $\lambda(q)$. It is trivial to verify that the critical point u_c corresponding to the free energy in Eq. (5) is determined by the equation

$$k_{2D}(q,u_c) = 1 \tag{6}$$

as one expects on the basis of the Onsager solution.

When $q<4$, the curve $\varsigma(q)$ and $k_{2D}(q,u)$ can be fixed *a priori* using theoretical arguments related to the duality symmetry and to the known results on the critical behavior. While we will fix the normalization $\lambda(q)$ by a comparison with the numerical data.

As it will be shown in the next sections, $\varsigma(q)$ is related to the critical exponent $\alpha(q)$. Through the well-known critical behavior of the Potts model, one can get an implicit functional relation between ς and α

$$\varsigma(q) = \Phi[\alpha(q)]. \tag{7}$$

In the $q>4$ cases (in which the transition is first order) one can fix *a priori* $\varsigma(q)$ in terms of $\lambda(q)$ using the known exact results of Baxter on the latent heat. Indeed, the above formula may look an *ad hoc* approximation or, at least, not very natural at a first glance. On the other hand from the arguments at the beginning of section 1 it stems a constraint on the free energy of the Potts model in two dimensions. Namely, we are interested in finding a functional form that exactly obeys certain properties known to be possessed by the true free energy. So, we search for an expression for the free energy which is compatible with the properties of the integral kernels discussed above. This gives a further constraint on the form of the free energy of the Potts model in two dimensions. From the combinatorial point of view, this makes the proposed form in Eq. (5) natural. In other words, besides the known constraints on the free energy of the two-dimensional Potts model (such as the critical exponent and the duality symmetry), the proposed form in Eq. (5) is also the simplest compatible with the recursive structure proposed in [14] (which has been proved to be in good agreement with known results [15,16]). This explains why it is quite useful to look at the recursive structure connecting the Ising models in two and in three dimensionals to obtain an additional bit of information in order to fix the residual arbitrariness left in the choice of the free energy.

III. DUALITY AND $k_{2D}(q,u)$

The duality transformation in the case of the two-dimensional Ising model was discovered in [18] before the exact solution of Onsager [8]. In the case of the Potts model we are considering the following Hamiltonian:

$$H = -J\sum_{\langle ij\rangle} \delta_{\sigma_i\sigma_j}, \tag{8}$$

and the duality is

$$D(u) = \frac{1-u}{1+(q-1)u}. \tag{9}$$

Note that this transformation is not a symmetry of the full free energy *per site* in the thermodynamic limit: it leaves invariant the nonanalytic part of the free energy, while the trivial term $\log(\frac{e^{\beta J}+q-1}{\sqrt{q}})$ is not invariant. Anyway the critical point is determined by the properties of the nonanalytic part. The fixed point of the duality transformation $u_c=u_c(q)$ is

$$D(u_c) = u_c \Rightarrow u_c(q) = \frac{1}{1+\sqrt{q}}. \tag{10}$$

Thus, one has to find a function $k_{2D}(q,u)$ which encodes the duality properties of the two-dimensional Potts model and which reduces to the known result when $q=2$: that is

$$k_{2D}(q,D(u)) = \pm k_{2D}(q,u), \tag{11}$$

where the $\pm$ signs appear because the Kallen-Lehmann free energy in Eq. (5) depends on $[k_{2D}(q,\beta)]^2$.

The simplest solution (let us call it $\tilde{k}$) of Eq. (11) is

$$\tilde{k}(q,u) = A\frac{u[1-(q-1)u^2+(q-2)u]}{[1+(q-1)u^2]^2}. \tag{12}$$

It can be easily seen that $\tilde{k}(q,u)$ fulfills Eq. (11) for any value of the constant A which we will fix with the normalization condition at the critical point in Eq. (10)

$$\tilde{k}(q,u_c) = 1 \Rightarrow \frac{1}{A(q)} = \frac{u_c[1-(q-1)u_c^2+(q-2)u_c]}{[1+(q-1)u_c^2]^2}.$$

Furthermore the critical point is located at $u=u_c$ in Eq. (10) as it should be and it can be easily seen that when $q=2$ it reduces to the expression of k_{2D} in terms of the low-temperature variable u.

Indeed, once one has found the simplest $\tilde{k}$ invariant under duality transformation and which reduces to the Ising case when $q=2$, one can construct many more solutions by simply taking functions f of the $\tilde{k}$ in Eq. (12)

$$k_{2D}(q,u) = f\left\{A(q)\frac{u[1-(q-1)u^2+(q-2)u]}{[1+(q-1)u^2]^2}\right\},$$

such that $f(x)$ has the maximum when $x=1$ and $f(1)=1$. The simplest possibility is to consider $f(x)$ of the form

$$f(x) = x^{E(q)}.$$

One procedure to determine $E(q)$ for $q=3$ is to look at the coefficients of the low-temperature expansions (see [20,21]). Using Eq. (5) one can compute the ratios δ_n for very small u

$$\delta_n(q,u) = \frac{a_n(q,u)}{a_{n+1}(q,u)},$$

[3] $\lambda(q)$ is related to the overall normalization of the nonanalytic part of the partition function while $\varsigma(q)$ is related to the critical behavior.

$$a_n(q,u) = \left.\frac{\partial^n F_{2D}(q,u)}{\partial u^n}\right|_{u=0}$$

for some small n and verify that it is possible to fulfill the expected scaling in the cases $q=3$ with the choice

$$E(3) = 2, \qquad (13)$$

so that we will take

$$k_{2D}(3,u) = \left(A(3)\frac{u(1-2u^2+u)}{(1+2u^2)^2}\right)^2,$$

while, of course, the compatibility with the Onsager solution tells that $E(2)=1$. It is also interesting to observe that consistency with the $q\to 1^+$ limit (where the u dependence disappears) would suggest $E(1)=0$. Thus, from now on, we will fix $E(3)$ as in Eq. (13).

Critical behavior

The critical exponent $\alpha(q)$ for two-dimensional Potts model (when $q\leq 4$) is known to be (see [19])

$$\alpha(q) = \frac{2(1-2x)}{3(1-x)},$$

$$x = \frac{2}{\pi}\arccos\left(\frac{\sqrt{q}}{2}\right),$$

where the positive values of x correspond to the tricritical point while the negative values correspond to the critical point.

One can fix *a priori* the curve $\varsigma(q)$ by looking at the critical behavior of the model: the specific heat is known to have (see [22,23]) the following forms for $q=3$:

$$C_{div}(q=3,u\approx u_c) \propto |u-u_c(3)|^{-1/3}. \qquad (14)$$

On the other hand, the second derivative of the (nonanalytic part of the) free energy F_{2D} in Eq. (5) reads as

$$\partial_u^2 F_{2D} = -\frac{\varsigma(q)\lambda(q)}{2\pi}(\partial_u^2 H)\int_0^\pi \frac{(\sin^2 x)\Delta^{\varsigma(q)-1}dx}{1+\Delta^{\varsigma(q)}} + \frac{\varsigma(q)\lambda(q)}{2\pi}(\partial_u H)^2\int_0^\pi \frac{\Delta^{\varsigma(q)-2}(\sin^4 x)dx}{1+\Delta^{\varsigma(q)}} \times\left\{[\varsigma(q)-1]-\frac{\varsigma(q)\Delta^{\varsigma(q)}}{1+\Delta^{\varsigma(q)}}\right\}, \qquad (15)$$

where

$$\Delta = 1 - H\sin^2 x, \quad 0\leq \varsigma(q) < 1,$$

$$H = [k_{2D}(u)]^2.$$

Near the critical point, $H\approx 1$ and $\partial_u H\approx 0$ since, as it has been already discussed, the critical point is a smooth maximum of $k_{2D}(u)$. For this reason, the most singular term is the first one:

$$C_{div} \approx -\frac{\varsigma(q)\lambda(q)}{2\pi}(\partial_u^2 H)\int_0^\pi \frac{(\sin^2 x)\Delta^{\varsigma(q)-1}dx}{1+\Delta^{\varsigma(q)}}, \qquad (16)$$

since the divergent term in the integral of the second term in Eq. (15) are compensated by the vanishing first derivative of H at the critical point (while $\partial_u^2 H$ is of order 1). By imposing that the singular part of the specific heat in Eq. (16) reproduces the known critical behavior in Eq. (14) one gets an implicit relation between α and ς which fixes $\varsigma(3)$ to be

$$\varsigma(3) = 0.4. \qquad (17)$$

In the next sections we will draw a picture of the critical part of the free energy in Eq. (5) against the known results at the critical point which shows a excellent agreement.

IV. COMPARISON WITH NUMERICAL DATA FOR $q=3$

An explicit analytic expression for the free energy in Eq. (5) of the two-dimensional ferromagnetic Potts model has been constructed in which theoretical arguments (basically, duality and the known critical behavior) can fix everything but one curve $\lambda(q)$. Indeed, it is easy to see that $\lambda(3)$ can fixed in terms of the numerical expansion data at low temperatures giving rise to a good agreement. We would like to further emphasize that the main purpose of our work is to obtain an unique approximate functional form for free energy in the widest possible range of temperatures. This is conceptually different and complementary point of view with respect to the known numerical analysis and perturbative expansions. So we have not just assumed the soundness of the low temperature and the critical point numerical studies, but we used them to fix the only free parameter of the ansatz. As it is explained below the precision of our numerical algorithm is not sensitive as the numerical data themselves, neither have we pretended to compete with the precision of these methods. Rather we mean to give a global description for the behavior of the free energy in a unified scheme compatible with the known results on their respective different domains of applicability. Despite the functional simplicity of the free energy in Eq. (5) a low-temperature Taylor expansion, which involves many numerical derivations and integrations, using the standard commercial software (available to us) is very inaccurate and anyway beyond the goal of this work.

A. Low and high temperatures

Because of the built-in duality invariance of the model, we will only need to check the agreement at low temperatures since the agreement at high follows from duality. Our analysis is based on Refs. [20,21]: to be more precise, we checked that the normalization in [21] is consistent with (and reduces to) the normalization of [20] in the $q=2$ case of the two-dimensional Ising model. In particular, the expansion of the partition function in the above references corresponds to only consider the interesting nonanalytic term (neglecting the trivial term log cosh 2β in the Ising case). Therefore, in the Potts case one has to compare the low-temperature expansions for $q=3$ with the second term on the right-hand side of Eq. (5). We consider the low-temperature expansion, up the 14th order, valid no more than $u_c(3)/100$, since the Onsager

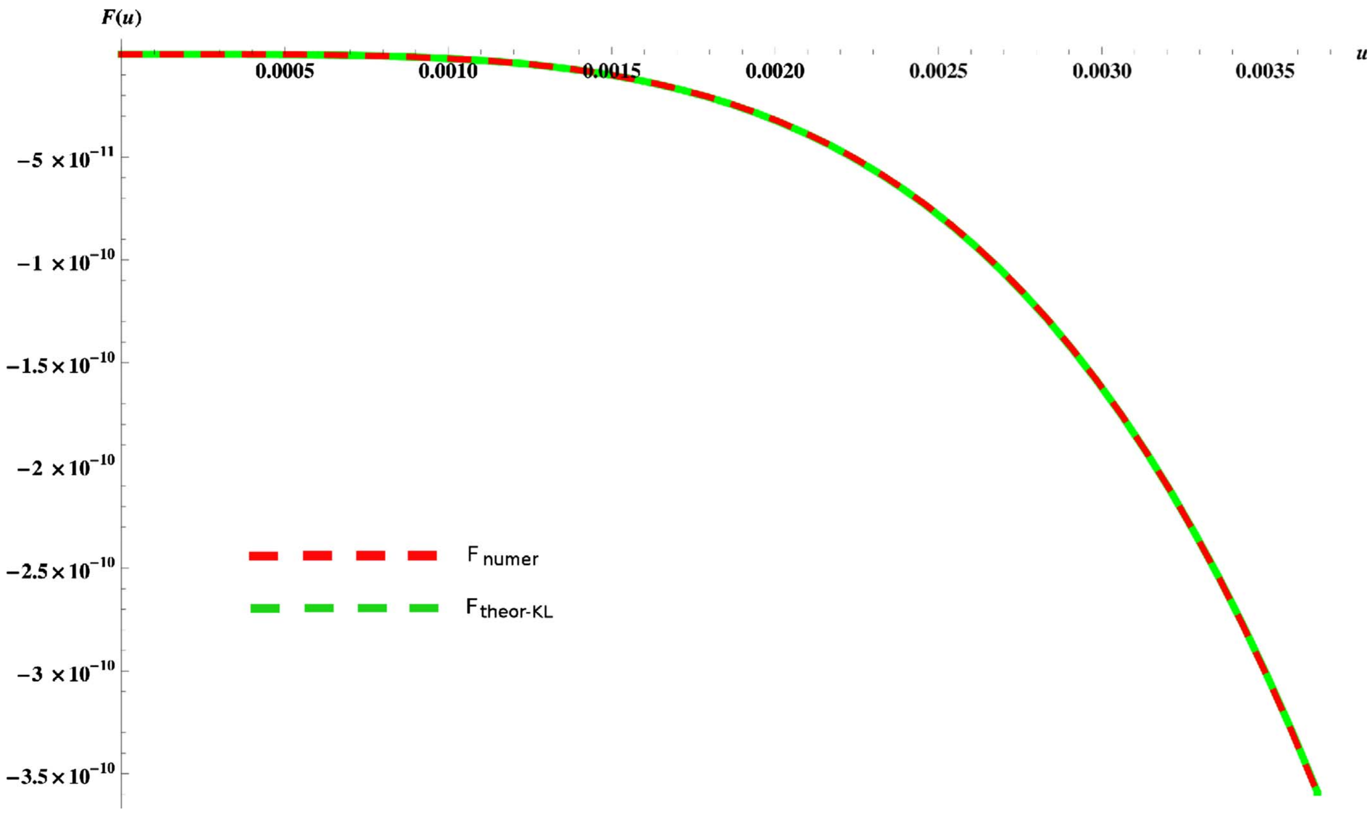

FIG. 1. (Color online) Numerical $F_{nm}(u)$ and Kallen-Lehmann $F_{2D}(u)$ free energies at low temperatures for $q=3$.

solution already in the 2D Ising model deviates significantly from the numerical expansion when $u>u_c(2)/100$. The low-temperature expansion of the free energy in [20,21] for the $q=3$ case is

$$F_{mc}(u) = -\log(1 + 2u^4 + 4u^6 + 4u^7 + 6u^8 + 24u^9 + 24u^{10} + 68u^{11} + 190u^{12} + 192u^{13} + 904u^{14})$$

and the best value we have found for $\lambda(3)$ is

$$\lambda(3) = 0.1543.$$

For this value of $\lambda(3)$ at low (and therefore high) temperatures the "precision" of the Kallen-Lehmann ansatz versus the numerical free energy may be measured in many different ways; for instance, one can use the following:

$$p(F_{mc},F_{2D}) \coloneqq \sqrt{\frac{\int (F_{mc}-F_{2D})^2 du}{\int (F_{mc})^2 du}} < 1\%.$$

As one can see from Fig. 1 the agreement is very good.

B. Critical behavior

Because of the simplicity of the free energy in Eq. (5), one can perform the numerical graph using MATHEMATICA®. Fig. 2, in which we plot the Kallen-Lehmann free energy $F_{2D}(3,u)$ against the critical free energy

$$F_{critic} = a + b|u - u_c(3)|^{2-(1/3)}$$

in a neighborhood [sized 1% of $u_c(3)$] of the critical point show a remarkable agreement. Also in this region the precision $p(F_{critic},F_{2D})$ of the result maintains well below 1%.

To get a clearer idea on the precision of the proposed ansatz, one could compare, using MATHEMATICA®, the exact Onsager solution at high temperature versus the corresponding high-temperature expansion in [20] in an interval of u from 0 to $u_c(2)/100$ [the Onsager solution deviates significantly from the numerical small u expansion when $u>u_c(2)/100$]. If one would do this, one would recognize how well the proposed ansatz describes the available numerical data in the 2D Potts case for $q=3$.

V. $q>4$ CASES

The $q>4$ cases are qualitative very different from the cases $q\leq 4$ since, when $q>4$, the phase transition is of first order. Thus, one could expect that even if the ansatz in Eq. (5) works very well for $q<4$, it is not at all obvious if it can also work when $q>4$. Remarkably enough, it can be shown that the free energy in Eq. (5) does indeed describe first-order phase transition at the critical point $u_c(q)$ in Eq. (10) provided $\varsigma(q)$ becomes negative,

$$\varsigma(q) < 0.$$

When $\varsigma(q)$ is negative $(\varsigma(q)=-|\varsigma(q)|)$, the leading behavior of the internal energy corresponding to the free energy in Eq. (5) is

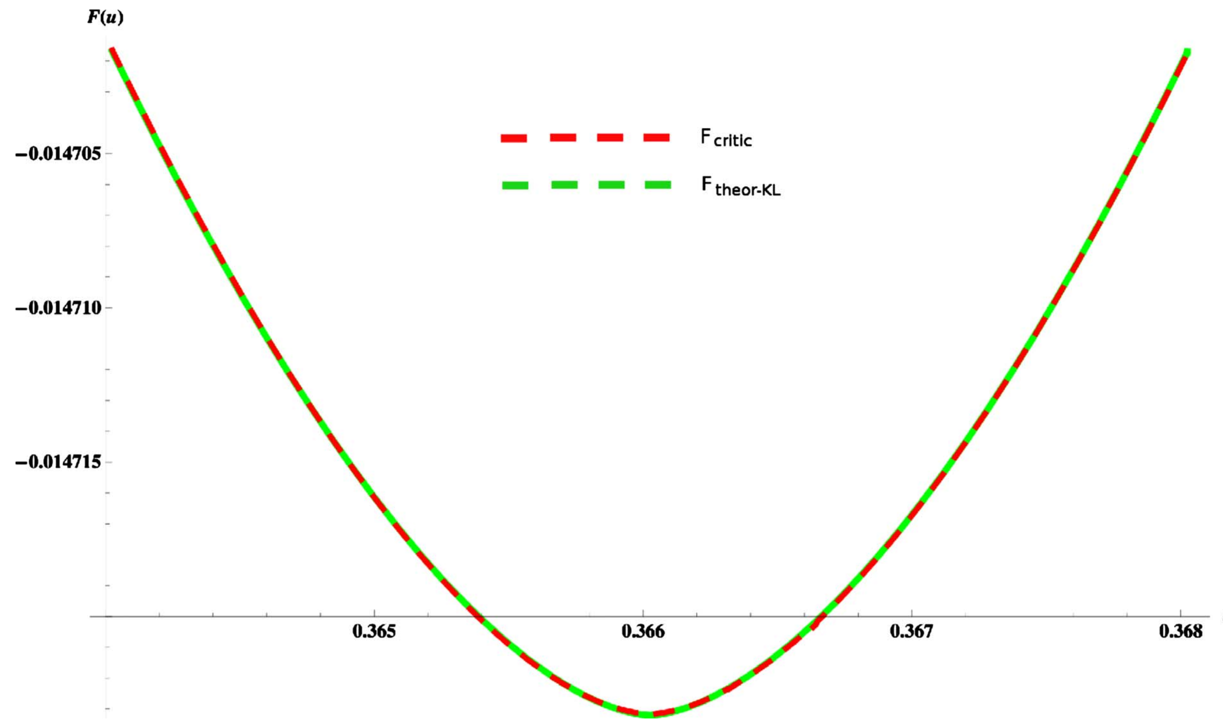

FIG. 2. (Color online) Critical behavior of free energy F_{critic} and Kallen-Lehmann $F_{2\mathrm{D}}(u)$ near the critical point ($q=3$, $a=-0.014\,723\,2$, and $b=0.677\,394$).

$$
\begin{aligned}
E_{2\mathrm{D}}(q,u\approx u_c) &\approx -\frac{\lambda(q)}{2\pi}(2k_{2\mathrm{D}}\partial_u k_{2\mathrm{D}}) \\
&\quad\times\int_0^{\pi}\frac{[\varsigma(q)\{1-[k_{2\mathrm{D}}(q,u)]^2\sin^2 t\}^{-|\varsigma(q)|-1}\sin^2 t]dt}{1+\{1-[k_{2\mathrm{D}}(q,u)]^2\sin^2 t\}^{-|\varsigma(q)|}} \\
&\approx -\frac{\lambda(q)}{2\pi}(2k_{2\mathrm{D}}\partial_u k_{2\mathrm{D}}) \\
&\quad\times\int_0^{\pi}[\varsigma(q)\{1-[k_{2\mathrm{D}}(q,u)]^2\sin^2 t\}^{-1}\sin^2 t]dt.
\end{aligned} \tag{18}
$$

Indeed, the derivative of $k_{2\mathrm{D}}(q,u)$ at the critical point is still zero but the divergence of the integral is stronger so that it gives rise to a finite contribution; for $u\approx u_c$ one has

$$|\partial_u k_{2\mathrm{D}}|\approx 2\varepsilon \ll 1, \quad k_{2\mathrm{D}}\approx 1-\varepsilon^2,$$

while the integral in Eq. (18) is dominated by the region $t \approx \pi/2$. Thus, one gets a finite discontinuity of the first derivative

$$
\begin{aligned}
&E_{2\mathrm{D}}(q,u\approx u_c) \\
&\quad\approx \mp\frac{\lambda(q)\varsigma(q)}{2\pi}(4\varepsilon)\int_{-a}^{a}\left[\frac{1}{1-(1-\varepsilon^2)(1-\delta^2)}\right]d\delta \\
&\quad\approx \mp\frac{2\lambda(q)\varsigma(q)}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+y^2}dy = \mp 2\lambda(q)\varsigma(q),
\end{aligned}
$$

where a is a small positive number, the − sign refers to taking the limit to the critical point from the left and the + from the right. Therefore the internal energy acquires a finite jump (namely, a nontrivial latent heat) at the phase transition as it should be. Furthermore, the jump is symmetric in agreement with the result of Baxter. With the same arguments one can see that when $\varsigma(q)$ is positive and less than one (as it is the case for $q<4$), both left and right derivatives are zero at the critical point so that the transition is second order. This is a very interesting fact in itself since it allows us to describe with the same analytical ansatz also the region in which a first-order transition takes place.

The above reasoning tells that to describe the first-order region $\varsigma(q)<0$ and to describe the second-order region $\varsigma(q)>0$. In the second-order region it has been shown that $\varsigma(q)$ is related to the critical exponent α. Below it will be discussed that in the first-order region one can fix the discontinuity of the first derivative of the free energy at the critical point [which, roughly, is proportional to $\lambda(q)\varsigma(q)$] using some exact results obtained by Baxter.

A. Comparison with the Baxter results

Baxter (see [2]; for a review, see [1]) was able to compute the latent heat $L(q,u_c)$ at the critical point in the $q>4$ case:

$$L(q,u_c)=2(1+q^{-1/2})\Delta(q)\tanh\left(\frac{\Theta}{2}\right), \quad q>4,$$

$$\Delta(q)=\prod_{n=1}^{\infty}[\tanh(n\Theta)]^2, \quad \cosh\Theta=\frac{\sqrt{q}}{2}.$$

Thus, one can fix the discontinuity ΔE_c of the first derivative of the free energy in Eq. (5) at the critical point

$$\Delta E_c = E_{2D}(q,u_c^+) - E_{2D}(q,u_c^-),$$

in terms of the Baxter result:

$$\Delta E_c = L(q,u_c). \tag{19}$$

Such equation allows, in principle, to fix one of the two curves in terms of the other [for instance, one can choose to express $\lambda(q)$ in terms of $\varsigma(q)$] living us with only one curve which could be fixed by looking at the numerical expansion at low temperatures. Unfortunately, at least in the cases $q=5$ and $q=6$ (which we analyzed more closely), we have not been able to develop a suitable software to test at the same time the low temperatures and the critical behavior. Even if, at first glance, the numerical problems to be solved for $q>4$ are similar to the ones which appear for $q<4$, there are two important differences. The first is that for $q<4$ the known critical behavior is only related to $\varsigma(q)$ which therefore can be fixed, while for $q>4$ the Baxter result on the latent heat determines ΔE_c which is a rather complicated function of both $\lambda(q)$ and $\varsigma(q)$ and this does make the numerical analysis more involved. The second is that, at least in the cases $q=5$ and $q=6$, the numbers which arise in the low-temperature expansions are very small and this makes our software extremely slow.

Nevertheless, we have verified using the software MATHEMATICA® that looking at the low temperatures only, it is possible to achieves an almost perfect agreement with the series expansion of [20,21], both for $q=5$ and $q=6$. This is a strong indication that this framework also works in the case $q>4$ since, because of the built in invariance under the duality transformation in Eq. (9), an excellent matching at low temperatures by construction implies an equivalent agreement at high temperatures. Thus, taking into account that in the $\varsigma(q)<0$ region the free energy in Eq. (5) has a first-order phase transition, one should expect that the present method provides an explicit analytic description of the free energy both for $q<4$ and for $q>4$ in terms of only two q-dependent curves,[4] in excellent agreement with numerical expansion data.

B. $q=4$ case

The $q=4$ case is the more delicate. The first obvious reason is that $q=4$ is the boundary between the range in which the model exhibit a second-order phase transition ($q\leq 4$) and the range in which the transition is first order (that is, $q>4$). As a matter of fact, for $q=4$: the specific heat singularity has logarithmic correction (see [22] or [23]):

$$C_{div}(q=4,u-u_c) \approx \frac{|u-u_c(4)|^{-2/3}}{\log|u-u_c(4)|}. \tag{20}$$

The present formalism provides one with a very natural mechanism for the arising of logarithmic corrections.

Assuming that $\varsigma(q)$ is a continuous function of q, then one could argue that $\varsigma(4)=0$ since in the first-order region $\varsigma(q)<0$ while in the second-order region $0<\varsigma(q)<1$. The interpretation of the last sentence is that, when $\varsigma(q)=0$, the present framework provides one with the appearance of the expected "nested" logarithm inside the free energy:

$$F_{2D}(4,u) = C_4 + \frac{\lambda(4)}{2\pi}\int_0^{\pi} dt \log\left\{\frac{1}{2}[1+\Xi(q,u) \times\log\{1-[k_{2D}(4,u)]^2\sin^2 t\}]\right\}.$$

Unfortunately, we have still not found a theoretical argument to fix *a priori* Ξ in an analytic way. Nevertheless, it is worth to stress that the above formula does give rise *automatically* to a second-order phase transition with a logarithmic correction of the type in Eq. (20) and, therefore, to find theoretical arguments able to fix $\Xi(q,u)$ is an interesting open problem.

VI. FISHER ZEROS

A powerful theoretical tool is the analysis of the Fisher zeros [24] in which the inverse temperature β is extended to the whole complex β plane (in the same way as Yang and Lee complexified the magnetic field [25]). By looking at the distribution of zeros of the partition function in the complex β plane, one can determine the universal amplitude ratio A_+/A_- of the specific heat and write down simple expressions (which only involve the density of zeros and the angle which the line of zeros form with the real β axis at the critical point) for the free energy and the specific heat close to the critical point; see [26,27]. Such tools are also useful when analyzing the strength of the phase transitions (see, for instance, [28–30] and references therein). Therefore, the Fisher zeros contain very deep nonperturbative information on the corresponding systems.

It is expected that also in the case of the two-dimensional Potts model the Fisher zeros should lie on a circle: strong theoretical as well as numerical evidences have been provided in [17] (for some more recent evidences, see [31,32] and references therein). The circle is given by $|x|=1$, where $x=v/\sqrt{q}$ and $v=e^{\beta}-1=u^{-1}-1$. To study the locus of the Fisher zeros, it is convenient to express $\tilde{k}(q,u)$ as follows:

$$\tilde{k}(q,u) = 4\frac{uD(u)}{[u+D(u)]^2}. \tag{21}$$

It is easy to observe from Eq. (21) that[5] $\tilde{k}(q,u)\leq 1$ and $\tilde{k}(q,u)=1$ if and only if $u=D(u)$; this condition identifies the real fixed point of the duality map. On the other hand according to the conjecture, the locus of Fisher zeros of the 2D

[4]Furthermore, at least one of these two curves can be fixed *a priori* analytically using known analytical results at the critical point [see Eqs. (17) and (19)].

[5]One may notice that in the above expression in Eq. (21) for $\tilde{k}(q,u)$ the dependence on q is implicit in the duality transformation $D(u)$. Furthermore, $\tilde{k}(q,u)$ reduces to the known expression for $q=2$. Therefore, when expressed in terms of u and $D(u)$, the equation determining the locus of Fisher zeros is formally exactly the same as in the Ising case provided one replaces the duality transformation of the Ising case with the corresponding Potts duality transformation.

Potts model on square lattice is the analytic extension of the equation $u_c = D(u_c)$. Therefore our proposal is consistent with the conjecture providing further support to this framework.

VII. CONCLUSIONS AND PERSPECTIVES

A phenomenological approach to the ferromagnetic two-dimensional Potts model on square lattice has been developed. After introducing the D-dressing and the q-dressing operators $\mathbf{D}_{D,q}$ and $\mathbf{Q}_{D,q}$, it has been described how the compatibility between $\mathbf{D}_{D,q}$ and $\mathbf{Q}_{D,q}$ allows one to write down an explicit analytic ansatz for the free energy in terms of one free parameter (for each q). The duality symmetries of the 2D Potts model together with the known theoretical results on its critical exponent allow to fix *a priori* all but one curve. The agreement of the proposed analytic free energy with low- and high-temperature expansion as well as the critical point is excellent for $q=3$. For $q=5$ and $q=6$ one can also see that the agreement with numerical data at low and high temperatures is also very good but we have not been able to test the corresponding critical points because of some subtle numerical problems. Nevertheless it has been proved that the corresponding phase transition when $\varsigma < 0$ is first order. The $q=4$ case remains basically opens but we have some indications that the present framework is also able to capture important features of such subtle case since it predicts automatically logarithmic correction to the power-law divergence of the specific heat.

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[1] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[2] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Dover Publications, New York, 2007).
[3] F. Y. Wu, Rev. Mod. Phys. **64**, 1099 (1992).
[4] *Braid Group, Knot Theory, and Statistical Mechanics II*, Advanced Series in Mathematical Physics Vol. 17, edited by C. N. Yang and M. L. Ge (World Scientific, Singapore, 1994).
[5] J. Cardy, e-print arXiv:math-ph/0103018.
[6] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B **210**, 423 (1982).
[7] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002).
[8] L. Onsager, Phys. Rev. **65**, 117 (1944).
[9] T. Regge, Nuovo Cimento **14**, 951 (1959).
[10] T. Regge, Nuovo Cimento **18**, 947 (1960).
[11] P. D. B. Collins Phys. Rep. **1**, 103 (1971).
[12] G. Veneziano Phys. Rep. **9**, 199 (1974).
[13] F. Canfora, L. Parisi, and G. Vilasi, Phys. Lett. B **638**, 85 (2006).
[14] F. Canfora, Phys. Lett. B **646**, 54 (2007).
[15] M. Astorino, F. Canfora, C. Martinez, and L. Parisi, Phys. Lett. B **664**, 139 (2008).
[16] M. Astorino, F. Canfora, and G. Giribet, Phys. Lett. B **671**, 291 (2009).
[17] P. P. Martin Nucl. Phys. B **225**, 497 (1983); J. M. Maillard and R. Rammal, J. Phys. A **16**, 353 (1983); P. P. Martin and J. M. Maillard, *ibid.* **19**, L547 (1986); V. Matveev and R. Shrock, Phys. Rev. E **54**, 6174 (1996); C. N. Chen, C. K. Hu, and F. Y. Wu, Phys. Rev. Lett.. **76**, 169 (1996); F. Y. Wu, G. Rollet, H. Y. Huang, J. M. Maillard, C. K. Hu, and C. N. Chen, Phys. Rev. Lett. **76**, 173 (1996); K. C. Lee, *ibid.* **73**, 2801 (1994); R. Kenna, Nucl. Phys., Proc. Suppl. **63**, 646 (1998).
[18] H. Kramers and G. Wannier, Phys. Rev. **60**, 252 (1941).
[19] R. B. Pearson, Phys. Rev. B **22**, 2579 (1980).
[20] M. E. Fisher and D. S. Gaunt, Phys. Rev. **133**, A224 (1964).
[21] K. M. Briggs, I. G. Enting, and A. J. Guttman, J. Phys. A **27**, 1503 (1994).
[22] M. Nauenberg and D. J. Scalapino, Phys. Rev. Lett. **44**, 837 (1980).
[23] J. L. Cardy, M. Nauenberg, and D. J. Scalapino, Phys. Rev. B **22**, 2560 (1980).
[24] M. E. Fisher, in *Lecture in Theoretical Physics*, edited by W. E. Brittin (Gordon and Breach, New York, 1968), Vol. VIIC, p. 1.
[25] C. N. Yang and T. D. Lee Phys. Rev. **87**, 404 (1952); T. D. Lee and C. N. Yang*ibid.* **87**, 410 (1952).
[26] R. Abe, Prog. Theor. Phys. **37**, 1070 (1967); **38**, 72 (1967); **38**, 322 (1967); **38**, 568 (1967).
[27] M. Suzuki Prog. Theor. Phys. **38**, 289 (1967); **38**, 1225 (1967); **38**, 1243 (1967); **39**, 349 (1968).
[28] W. Janke and R. Kenna, J. Stat. Phys. **102**, 1211 (2001).
[29] W. Janke, D. A. Johnston, and R. Kenna, Nucl. Phys. B **682**, 618 (2004).
[30] R. Kenna, J. Phys. A **31**, 9419 (1998).
[31] J.-L. Jacobsen, J.-F. Richard, and J. Salas, Nucl. Phys. B **743**, 153 (2006).
[32] S.-C. Chang, J. Salas, and R. Shrock, J. Stat. Phys. **107**, 1207 (2002).