Dark and antidark solitons in the modified nonlinear Schrödinger equation accounting for the self-steepening effect

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In this paper, the modified nonlinear Schrödinger equation is investigated, which describes the femtosecond optical pulse propagation in a monomodal optical fiber. Based on the Wadati-Konno-Ichikawa system, another type of Lax pair and infinitely many conservation laws are derived. Dark and antidark soliton solutions in the normal dispersion regime are obtained by means of the Hirota method. Parametric regions for the existence of the dark and antidark soliton solutions are given. Asymptotic analysis of the two-soliton solution shows that collisions between two solitons (two antidark solitons, two dark solitons, and dark and antidark solitons) are elastic. In addition, collision between dark and antidark solitons reveals that dark and antidark solitons can co-exist on the same background in the normal dispersion regime.

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I. INTRODUCTION

Since the theoretical prediction [1] and experimental demonstration [2] of the optical soliton in a single-mode fiber, interest has been aroused in the studies of optical solitons over the past decades [3–5]. Soliton can propagate over a long distance without the amplitude attenuation and shape change in the uniform nonlinear fiber under the condition that the group velocity dispersion (GVD) balances the self-phase modulation [1]. In the picosecond domain, the propagation of optical soliton pulses in the single-mode optical fibers is governed by the nonlinear Schrödinger (NLS) equation [6]. In the regime of anomalous dispersion, the bright soliton exists, while the dark soliton could arise in the regime of normal dispersion [2,7,8]. Compared with the bright soliton which is a pulse on a zero-intensity background, the dark soliton appears as an intensity dip in an infinitely extended constant background [7]. Dark solitons can be applied in the optical logic devices [9] and waveguide optics as the dynamic switches [10]. Additionally, recent studies have shown that dark solitons are more resistant to the perturbations than the bright ones [11].

However, it should be noted that in the normal dispersion regime there is another type of soliton similar to the bright soliton [12–16]. Such a type of soliton is called the *antidark* soliton which exists in the form of a bright pulse on a nonzero continuous wave background [12]. In this paper, we plan to study the formation mechanism and collision dynamics of the dark and antidark solitons in the normal dispersion regime of the following modified nonlinear Schrödinger equation [17],

$$iU_\zeta + \frac{1}{2}\sigma U_{\tau\tau} + N^2|U|^2U + isN^2(|U|^2U)_\tau = 0, \qquad (1)$$

which involves the time derivative of the pulse envelope into the conventional theory of nonlinear self-phase modulation [17], where U is a normalized complex amplitude of the pulse envelope, ζ is a normalized distance along the fiber, τ is the normalized time within the frame of the reference moving along the fiber at the group velocity, the sign of σ represents the normal ($\sigma<0$) and anomalous ($\sigma>0$) regimes, the integer value of N is related to the soliton order and s is an arbitrary real constant [17]. The last term in Eq. (1), which represents the self-steepening (SS) effect, plays a role for the short-pulse propagation in the long optical fibers or waveguides [18]. Equation (1) can be used to describe the subpicosecond or femtosecond optical pulse propagation in a monomodal optical fiber [19,20]. In addition, the modulated Alfvén wave propagation along a magnetic field in cold plasmas can be described by Eq. (1) [21].

In the past decades, studies have been made on the solutions and integrability of Eq. (1) [22–27]. In Refs. [22,23], the Lax pair of Eq. (1) has been obtained though the Wadati-Konno-Ichikawa (WKI) scheme. By means of the technique of determinant calculation, the analytic N-soliton solution of Eq. (1) has been given [24]. Reference [25] has derived the N-soliton solution of Eq. (1) by use of the Hirota method even though the standard bilinear form has not been presented. In Ref. [26], a family of analytic solutions of Eq. (1) including the Jacobi elliptic function solutions, stationary periodic solutions and algebraic soliton solutions have been obtained through a traveling-wave method and the possibility of the dark soliton existing in the normal dispersion regime has also been mentioned. By virtue of an ansatz, an analytic solution of Eq. (1) has been given, which can exhibit the phenomena that the inclusion of the term $(|U|^2U)_\tau$ produces a temporal pulse distortion leading to the development of an optical shock unless balanced by the dispersion [27].

In view of those prevenient studies, aspects motivating us to make investigation can be summarized as follows: (a) since Eq. (1) has not been solved by the inverse scattering transform, we can construct another type of Lax pair and consequently derive another class of infinitely many conservation laws as further evidence for the integrability of Eq.

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(1); (b) although some analytic soliton solutions have been obtained in Refs. [24,25,27], there has been an absence of certain analysis on the formation mechanism of the solitons of Eq. (1) in the normal dispersion regime. Therefore, we can construct one- and multisoliton solutions of Eq. (1) with $\sigma < 0$ by the Hirota method, in order to show that both dark and antidark solitons can be formed in the normal dispersion regime, and to make clear which parametric conditions allow the appearance of dark and antidark solitons, respectively.

With the aid of symbolic computation [3,4,28,29], the structure of this paper will be arranged as follows. In Sec. II, another type of Lax pair of Eq. (1) will be constructed by means of the WKI scheme and infinitely many conservation laws will be derived via the obtained Lax pair. In Sec. III, one-, two- and three-soliton solutions of Eq. (1) in the normal dispersion will be derived via the Hirota method. Parametric regions for the existence of dark and antidark solitons, and the graphical analysis of types of soliton collisions will be given in Sec. IV. Section V will be our conclusions.

II. LAX PAIR AND INFINITELY MANY CONSERVATION LAWS

A given nonlinear evolution equation (NLEE) can be considered integrable when it is equivalent to the compatibility condition for the associated Lax pair [30]. Lax pair can be used not only to demonstrate the integrability but also to construct the soliton solutions via the Darboux transformation [30]. WKI inverse scattering problem provides us with a procedure to obtain the Lax pairs of a class of the NLEEs [31]. Here, we will make use of the WKI scheme to construct another form of Lax pair of Eq. (1), which is different from those in Refs. [22,23].

Following such a scheme, the Lax pair of Eq. (1) can be written as

$$\Psi_\tau = L\Psi, \quad \Psi_\zeta = M\Psi, \tag{2}$$

where $\Psi=(\Psi_1,\Psi_2)^T$ is the vector eigenfunction, T denotes the transpose of the matrix, and L and M are expressible in the form

$$L = \begin{pmatrix} \frac{i}{2s} + \frac{i\sigma\lambda^2}{2N^2 s} & \lambda U \\ \lambda U^* & -\frac{i}{2s} - \frac{i\sigma\lambda^2}{2N^2 s} \end{pmatrix},$$

$$M = \begin{pmatrix} A(\zeta,\tau,\lambda) & B(\zeta,\tau,\lambda) \\ C(\zeta,\tau,\lambda) & -A(\zeta,\tau,\lambda) \end{pmatrix},$$

with

$$A(\zeta,\tau,\lambda) = -\frac{i\sigma^3}{4N^4 s^2}\lambda^4 - \left(\frac{i\sigma^2}{2N^2 s^2} + \frac{1}{2} i\sigma |U|^2\right)\lambda^2 - \frac{i\sigma}{4s^2},$$

$$B(\zeta,\tau,\lambda) = -\frac{\sigma^2 U}{2N^2 s}\lambda^3 - \left(\frac{\sigma}{2s}U + N^2 s|U|^2 U - \frac{i\sigma}{2}U_\tau\right)\lambda,$$

$$C(\zeta,\tau,\lambda) = -\frac{\sigma^2 U^*}{2N^2 s}\lambda^3 - \left(\frac{\sigma}{2s}U^* + N^2 s|U|^2 U^* + \frac{i\sigma}{2}U^*_\tau\right)\lambda,$$

where λ is the spectral parameter and the asterisk means the complex conjugate. One can check that the compatibility condition $L_\zeta - M_\tau + LM - ML = 0$ is exactly equivalent to Eq. (1).

The derivation of conserved quantities of a given NLEE is considered as a key step of solving the initial-value problem by the inverse scattering transform [30,32], so we will derive the infinitely many conservation laws. Methods to obtain the conservation laws of a continuous system can be through the Lax pair [33], Bäcklund transformation [33] and formal solution of the eigenfunction [34,35].

According to the procedure in Ref. [22], by introducing $\Gamma = \Psi_2/\Psi_1$, we get the Riccati-type equation from Expression (2),

$$\left(-\frac{i}{2s} - \frac{i\sigma\lambda^2}{2N^2 s}\right)\Omega = -\lambda|U|^2 + \lambda\Omega^2 + U\Omega_\tau \quad \text{with} \quad \Omega = U\Gamma. \tag{3}$$

Expanding $\Omega = \sum_{n=1}^{\infty} \Omega_n \lambda^{-n}$ in Expression (3) and equating the coefficients of the same power of λ to zero, we have the recurrence relations,

$$\Omega_1 = -\frac{iN^2 s}{\sigma}|U|^2, \quad \Omega_2 = 0, \tag{4}$$

$$\Omega_{n+2} = \frac{iN^2 s}{\sigma}\left(\sum_{k=1}^{n} \Omega_k \Omega_{n-k+1} + \Omega_{n,\tau} - \frac{U_\tau}{U}\Omega_n + \frac{i}{s}\Omega_n\right) \quad (n \geq 1). \tag{5}$$

According to the compatibility condition $(\ln \Psi_1)_{\tau\zeta} = (\ln \Psi_1)_{\zeta\tau}$, we obtain the infinitely many conservation laws:

$$\frac{\partial}{\partial\zeta}\rho_j + \frac{\partial}{\partial\tau}J_j = 0 \quad (j = 1,2,\ldots), \tag{6}$$

with

$$\rho_1 = -\frac{iN^2 s}{\sigma}|U|^2, \tag{7}$$

$$J_1 = \frac{N^2 s}{2\sigma}(-3iN^2 s|U|^4 - \sigma U^* U_\tau + \sigma U U^*_\tau), \tag{8}$$

$$\rho_2 = \frac{N^4 s}{\sigma^3}(i\sigma|U|^2 - iN^2 s^2|U|^4 + s\sigma U U^*_\tau), \tag{9}$$

$$J_2 = \frac{N^4 s}{2\sigma^3}(4iN^2 s\sigma|U|^4 - 4iN^4 s^3|U|^6 + \sigma^2 U^* U_\tau - \sigma^2 U U^*_\tau + 6N^2 s^2 \sigma|U|^2 U U^*_\tau - is\sigma^2 U_\tau U^*_\tau + is\sigma^2 U U^*_{\tau\tau}), \tag{10}$$

$$\rho_3 = \frac{N^6 s}{\sigma^5}(-i\sigma^2|U|^2 + 3i\sigma N^2 s^2|U|^4 - 2iN^4 s^4|U|^6 + N^2 s^3\sigma|U|^2 U^* U_\tau - 2s\sigma^2 UU^*_\tau + 4\sigma N^2 s^3|U|^2 UU^*_\tau + is^2\sigma^2 UU^*_{\tau\tau}), \quad (11)$$

$$J_3 = \frac{N^6}{2\sigma^5}(-i\sigma^3|U|^2 + iN^2 s^2\sigma^2|U|^4 + 4iN^4 s^4\sigma|U|^6 - 4iN^6 s^6|U|^8 - s\sigma^3 U^* U_\tau + 4N^2 s^3\sigma^2|U|^2 U^* U_\tau - iN^2 s^4\sigma^2 U^{*2} U_\tau^2 - 2s\sigma^3 UU^*_\tau + 8N^4 s^5\sigma|U|^4 UU^*_\tau + 2is^2\sigma^3 U_\tau U^*_\tau - 4iN^2 s^4\sigma^2|U|^2 U_\tau U^*_\tau + is^2\sigma^3 UU^*_{\tau\tau} + 2iN^2 s^4\sigma^2|U|^2 UU^*_{\tau\tau} + s^3\sigma^3 U_\tau U^*_{\tau\tau}),$$

$$\vdots \quad (12)$$

where ρ_j and J_j $(j=1,2,\ldots)$ are called the conserved densities and conserved fluxes, respectively. The first three ones represent the energy, momentum and Hamiltonian conservation laws, respectively.

III. DARK AND ANTIDARK SOLITON SOLUTIONS VIA THE HIROTA METHOD

The Hirota method has been used to obtain the soliton solutions of the NLEEs [36]. If the NLEE is bilinearized, one may get the soliton solutions, especially the multi-soliton solutions directly through the truncated formal perturbation expansion at different levels [36,37]. In the following part, we will employ this method to construct the soliton solutions by means of symbolic computation.

Through the following gauge transformation,

$$U = u\,\exp\left(-\frac{iN^2 s}{\sigma}\int |u|^2 d\tau\right), \quad (13)$$

Equation (1) can be transformed into

$$iu_\zeta + \frac{1}{2}\sigma u_{\tau\tau} + N^2|u|^2 u + isN^2(|u|^2)u_\tau = 0. \quad (14)$$

By introducing the rational dependent variable transformation $u=g/f$ where $g(\zeta,\tau)$ and $f(\zeta,\tau)$ are both complex functions, the bilinear form of Eq. (14) is obtained as follows:

$$\left(iD_\zeta + \frac{\sigma}{2}D_\tau^2 - \mu\right)(g\cdot f) = 0, \quad (15)$$

$$D_\tau(f^*\cdot f) = -\frac{isN^2}{\sigma}|g|^2, \quad (16)$$

$$\left(D_\tau^2 - \frac{2\mu}{\sigma}\right)(f^*\cdot f) = \frac{2N^2}{\sigma}|g|^2 - \frac{isN^2}{\sigma}D_\tau(g^*\cdot g), \quad (17)$$

where μ is a real constant to be determined, D_ζ and D_τ are the bilinear differential operators [36] defined by

$$D_\zeta^m D_\tau^n(f\cdot g) = \left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta'}\right)^m\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\tau'}\right)^n f(\zeta,\tau)g(\zeta,\tau)|_{\zeta'=\zeta,\tau'=\tau}.$$

According to the gauge equivalence between Eqs. (1) and (14), Eqs. (15)–(17) can lead to Eq. (1) by taking the dependent variable transformation $U=uf^*/f$. Therefore, Eqs. (15)–(17) can also be regarded as the bilinear form of Eq. (1).

The expansions of g and f with respect to a formal expansion parameter ε are of the form

$$g = g_0(1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \cdots), \quad (18)$$

$$f = f_0(1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots), \quad (19)$$

where g_n and f_n $(n=0,1,2,\cdots)$ are the complex functions to be determined. Substituting Expansions (18) and (19) into Eqs. (15)–(17) and truncating the perturbation expansion at different levels, we can yield one- and multisoliton solutions of Eq. (1).

A. One-soliton solution

To obtain the one-soliton solution of Eq. (1), we terminate the power series expansions as $g=g_0(1+\varepsilon g_1)$ and $f=f_0(1+\varepsilon f_1)$, and assume that

$$g_0 = \rho e^{-i\eta\zeta}, \quad f_0 = e^{-i\xi\tau},$$

$$g_1 = -\alpha e^\theta, \quad f_1 = \beta e^\theta \quad \text{with} \quad \theta = \kappa\tau + \omega\zeta, \quad (20)$$

where ρ, η, ξ, κ, and ω are all real constants, α and β are both complex ones. Substituting Expression (20) into Eqs. (15)–(17) and after some symbolic calculations, we have

$$\mu = -N^2\rho^2 - \frac{N^4 s^2\rho^4}{2\sigma}.$$

Without loss of generality, by setting $\varepsilon=1$, the one-soliton solution can be explicitly expressed as

$$U = \rho\,\exp[i(3\xi\tau - \eta\zeta)]\frac{(1-\alpha e^\theta)(1+\beta^* e^\theta)}{(1+\beta e^\theta)^2} = \rho\sqrt{1 - \frac{\alpha+\alpha^*+\beta+\beta^*}{2|\beta|\cosh(\theta+\ln|\beta|)+\beta+\beta^*}}e^{i\varphi}, \quad (21)$$

with

$$\xi = -\frac{N^2 s\rho^2}{2\sigma}, \quad \eta = -N^2\rho^2 - \frac{3N^4 s^2\rho^4}{8\sigma},$$

$$\beta = \varsigma(1 + i\varrho), \quad \varrho = -\frac{2\kappa^3 N^2 s\rho^2\sigma}{4\omega^2 - \kappa^2 N^4 s^2\rho^4 + \kappa^4\sigma^2},$$

$$\alpha = -\frac{N^2 s\rho^2\kappa - i\sigma\kappa^2 - 2\omega}{N^2 s\rho^2\kappa + i\sigma\kappa^2 - 2\omega}\beta,$$

$$\varphi = 3\xi\tau - \eta\zeta - \frac{i}{2}\ln\left[\left(\frac{1-\alpha e^\theta}{1-\alpha^* e^\theta}\right)\left(\frac{1+\beta^* e^\theta}{1+\beta e^\theta}\right)^3\right].$$

where ς is a real arbitrary constant.

The dispersion relation is given by

$$\omega^-:\ \ \omega = \frac{1}{2}\left(-\kappa N^2 s\rho^2 - \sqrt{-2\kappa^2 N^4 s^2 \rho^4 - 4\kappa^2 N^2 \rho^2 \sigma - \kappa^4 \sigma^2}\right), \tag{22}$$

$$\omega^+:\ \ \omega = \frac{1}{2}\left(-\kappa N^2 s\rho^2 + \sqrt{-2\kappa^2 N^4 s^2 \rho^4 - 4\kappa^2 N^2 \rho^2 \sigma - \kappa^4 \sigma^2}\right). \tag{23}$$

B. Two-soliton solution

By truncating g and f as $g=g_0(1+g_1+g_2)$ and $f=f_0(1+f_1+f_2)$, the two-soliton solution of Eq. (1) is presented as

$$U = \frac{g_0(1+g_1+g_2)f_0^*(1+f_1^*+f_2^*)}{[f_0(1+f_1+f_2)]^2}, \tag{24}$$

where

$$g_0 = \rho\exp(-i\eta\zeta),$$

$$f_0 = \exp(-i\xi\tau),$$

$$g_1 = -\alpha_1\exp(\theta_1) - \alpha_2\exp(\theta_2),\quad f_1 = \beta_1\exp(\theta_1) + \beta_2\exp(\theta_2),$$

$$g_2 = \chi_{12}\alpha_1\alpha_2\exp(\theta_1+\theta_2),\quad f_2 = \chi_{12}\beta_1\beta_2\exp(\theta_1+\theta_2),$$

with

$$\theta_j = \kappa_j\tau + \omega_j\zeta,\quad \omega_j = \frac{1}{2}\left(-N^2 s\rho^2\kappa_j \pm \sqrt{-2N^4 s^2\rho^4\kappa_j^2 - 4N^2\rho^2\sigma\kappa_j^2 - \sigma^2\kappa_j^4}\right),$$

$$\alpha_j = -\frac{N^2 s\rho^2\kappa_j - i\sigma\kappa_j^2 - 2\omega_j}{N^2 s\rho^2\kappa_j + i\sigma\kappa_j^2 - 2\omega_j}\beta_j,$$

$$\beta_j = \varsigma_j(1+i\varrho_j),\quad \varrho_j = -\frac{2N^2 s\rho^2\sigma\kappa_j^3}{4\omega_j^2 - N^4 s^2\rho^4\kappa_j^2 + \sigma^2\kappa_j^4}\quad (j=1,2),$$

$$\chi_{12} = \frac{\sigma^2\kappa_1^4\kappa_2^2 - 2\sigma^2\kappa_1^3\kappa_2^3 + \sigma^2\kappa_1^2\kappa_2^4 + 4\kappa_2^2\omega_1^2 - 8\kappa_1\kappa_2\omega_1\omega_2 + 4\kappa_1^2\omega_2^2}{\sigma^2\kappa_1^4\kappa_2^2 + 2\sigma^2\kappa_1^3\kappa_2^3 + \sigma^2\kappa_1^2\kappa_2^4 + 4\kappa_2^2\omega_1^2 - 8\kappa_1\kappa_2\omega_1\omega_2 + 4\kappa_1^2\omega_2^2},$$

where ς_j's $(j=1,2)$ and κ_j's $(j=1,2)$ are all arbitrary real constants.

C. Three-soliton solution

Following the procedure above, the three-soliton solution of Eq. (1) can also be obtained as

$$U = \frac{g_0(1+g_1+g_2+g_3)f_0^*(1+f_1^*+f_2^*+f_3^*)}{[f_0(1+f_1+f_2+f_3)]^2}, \tag{25}$$

where

$$g_0 = \rho\exp(-i\eta\zeta),\quad f_0 = \exp(-i\xi\tau),$$

$$g_1 = -\alpha_1\exp(\theta_1) - \alpha_2\exp(\theta_2) - \alpha_3\exp(\theta_3),$$

$$f_1 = \beta_1\exp(\theta_1) + \beta_2\exp(\theta_2) + \beta_3\exp(\theta_3),$$

$$g_2 = \chi_{12}\alpha_1\alpha_2\exp(\theta_1+\theta_2) + \chi_{13}\alpha_1\alpha_3\exp(\theta_1+\theta_3) + \chi_{23}\alpha_2\alpha_3\exp(\theta_2+\theta_3),$$

$$f_2 = \chi_{12}\beta_1\beta_2\exp(\theta_1+\theta_2) + \chi_{13}\beta_1\beta_3\exp(\theta_1+\theta_3) + \chi_{23}\beta_2\beta_3\exp(\theta_2+\theta_3),$$

$$g_3 = -\chi_{123}\alpha_1\alpha_2\alpha_3\exp(\theta_1+\theta_2+\theta_3),\quad f_3 = \chi_{123}\beta_1\beta_2\beta_3\exp(\theta_1+\theta_2+\theta_3),$$

with

$$\theta_j = \kappa_j\tau + \omega_j\zeta,$$

$$\omega_j = \frac{1}{2}\left(-N^2 s\rho^2\kappa_j \pm \sqrt{-2N^4 s^2\rho^4\kappa_j^2 - 4N^2\rho^2\sigma\kappa_j^2 - \sigma^2\kappa_j^4}\right),$$

$$\alpha_j = -\frac{N^2 s\rho^2\kappa_j - i\sigma\kappa_j^2 - 2\omega_j}{N^2 s\rho^2\kappa_j + i\sigma\kappa_j^2 - 2\omega_j}\beta_j,\quad \beta_j = \varsigma_j(1+i\varrho_j),$$

$$\varrho_j = -\frac{2N^2 s\rho^2\sigma\kappa_j^3}{4\omega_j^2 - N^4 s^2\rho^4\kappa_j^2 + \sigma^2\kappa_j^4} \quad \text{for} \quad (j=1,2,3), \quad \chi_{123} = \chi_{12}\chi_{13}\chi_{23},$$

$$\chi_{12} = \frac{\sigma^2\kappa_1^4\kappa_2^2 - 2\sigma^2\kappa_1^3\kappa_2^3 + \sigma^2\kappa_1^2\kappa_2^4 + 4\kappa_2^2\omega_1^2 - 8\kappa_1\kappa_2\omega_1\omega_2 + 4\kappa_1^2\omega_2^2}{\sigma^2\kappa_1^4\kappa_2^2 + 2\sigma^2\kappa_1^3\kappa_2^3 + \sigma^2\kappa_1^2\kappa_2^4 + 4\kappa_2^2\omega_1^2 - 8\kappa_1\kappa_2\omega_1\omega_2 + 4\kappa_1^2\omega_2^2},$$

$$\chi_{13} = \frac{\sigma^2\kappa_1^4\kappa_3^2 - 2\sigma^2\kappa_1^3\kappa_3^3 + \sigma^2\kappa_1^2\kappa_3^4 + 4\kappa_3^2\omega_1^2 - 8\kappa_1\kappa_3\omega_1\omega_3 + 4\kappa_1^2\omega_3^2}{\sigma^2\kappa_1^4\kappa_3^2 + 2\sigma^2\kappa_1^3\kappa_3^3 + \sigma^2\kappa_1^2\kappa_3^4 + 4\kappa_3^2\omega_1^2 - 8\kappa_1\kappa_3\omega_1\omega_3 + 4\kappa_1^2\omega_3^2},$$

$$\chi_{23} = \frac{\sigma^2\kappa_2^4\kappa_3^2 - 2\sigma^2\kappa_2^3\kappa_3^3 + \sigma^2\kappa_2^2\kappa_3^4 + 4\kappa_3^2\omega_2^2 - 8\kappa_2\kappa_3\omega_2\omega_3 + 4\kappa_2^2\omega_3^2}{\sigma^2\kappa_2^4\kappa_3^2 + 2\sigma^2\kappa_2^3\kappa_3^3 + \sigma^2\kappa_2^2\kappa_3^4 + 4\kappa_3^2\omega_2^2 - 8\kappa_2\kappa_3\omega_2\omega_3 + 4\kappa_2^2\omega_3^2},$$

where ς_j's $(j=1,2,3)$ and κ_j's $(j=1,2,3)$ are arbitrary real constants.

IV. ANALYSIS OF THE SOLITON SOLUTIONS

A. Parametric regions for the existence of dark and antidark soliton solutions

From Solution (21), in order to ensure a real dispersion relation and a continuation of the dark soliton regime, we must require that the GVD parameter σ be a negative constant and have a certain range, which further determines the choice of the wave number κ. After the algebraic manipulations, we have

$$\sigma + \frac{1}{2}N^2 s^2\rho^2 < 0, \tag{26}$$

$$\frac{\sqrt{2}}{\sigma}\sqrt{-N^4 s^2\rho^4 - 2N^2\rho^2\sigma} < \kappa < -\frac{\sqrt{2}}{\sigma}\sqrt{-N^4 s^2\rho^4 - 2N^2\rho^2\sigma}, \tag{27}$$

which are the basic conditions to ensure the existence of the soliton solutions of Eq. (1).

Note that Solution (21) can be rewritten as

$$U = \rho\sqrt{1 - \frac{4\kappa^4\sigma^2\varsigma}{(4\omega^2 - \kappa^2 N^4 s^2\rho^4 + \kappa^4\sigma^2)[2\sqrt{\varsigma^2 + \varsigma^2\varrho^2}\cosh(\theta + \ln\sqrt{\varsigma^2 + \varrho^2\varsigma^2}) + 2\varsigma]}}\, e^{i\varphi}, \tag{28}$$

which can describe two types of solitons, namely, dark and antidark solitons, depending on the sign of $\Delta \equiv 4\omega^2 - \kappa^2 N^4 s^2\rho^4 + \kappa^4\sigma^2$. If the parameter $\varsigma > 0$, then for the case of $\Delta < 0$, Solution (28) represents the antidark soliton solution, while $\Delta > 0$ corresponds to the dark one. However, If the parameter $\varsigma < 0$, for the cases of $\Delta < 0$ and $\Delta > 0$, dark and antidark soliton solutions will be obtained. In the following part, we will take the instance of $\varsigma > 0$ to discuss the ranges of parameters with respect to each type.

It can be found that the case of ω^+ leads to $\Delta > 0$, which means one can only get the dark soliton solution. While for

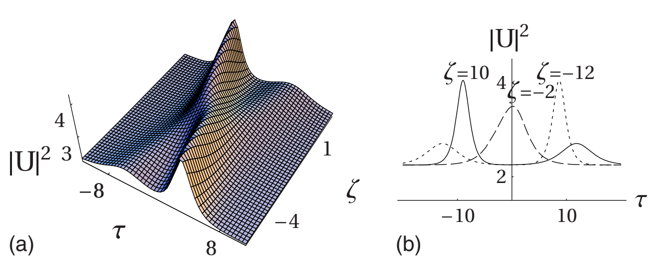

FIG. 1. (Color online) (a) Elastic collision between two antidark solitons via solution (24) with the parameters $\rho=1.5$, $s=-1$, $N=2$, $\sigma=-8$, $\kappa_1=0.5$, $\kappa_2=1$, and $\varsigma_1=\varsigma_2=1$. (b) Corresponding trajectories of (a) at: $\zeta=-12$ (short dashed curve), $\zeta=-2$ (long dashed curve) and $\zeta=10$ (solid curve).

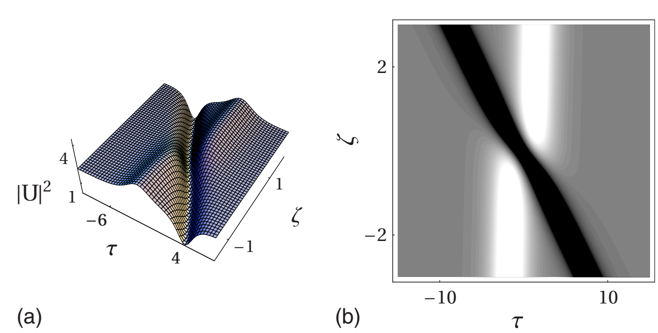

FIG. 2. (Color online) (a) Elastic collision between a dark soliton and an antidark soliton via solution (24) with the parameters $\rho=1.5$, $s=-1$, $N=2$, $\sigma=-8$, $\kappa_1=1.1$, $\kappa_2=-1.4$ and $\varsigma_1=\varsigma_2=1$. (b) Contour plot of (a).

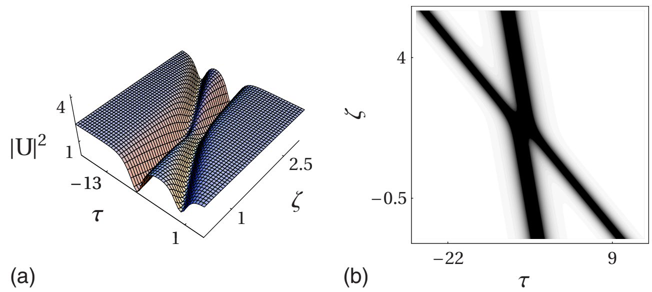

FIG. 3. (Color online) (a) Elastic collision between two dark solitons via solution (24) with the parameters $\rho=1.5$, $s=-1$, $N=2$, $\sigma=-8$, $\kappa_1=0.8$, $\kappa_2=1.3$ and $\varsigma_1=\varsigma_2=1$. (b) Contour plot of (a).

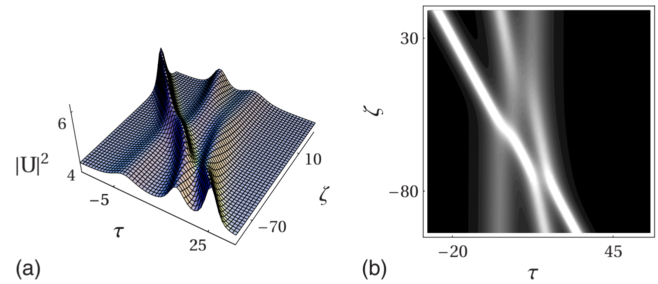

FIG. 4. (Color online) (a) Elastic collision between three antidark solitons via solution (25) with the parameters $\rho=2$, $s=-1$, $N=1$, $\sigma=-3.1$, $\kappa_1=0.6$, $\kappa_2=0.8$, $\kappa_3=0.4$, $\varsigma_1=\varsigma_3=1$ and $\varsigma_2=2000$. (b) Contour plot of (a).

the case of ω^-, one could obtain both dark and antidark solitons. According to the analysis of the one-soliton solution, we get the parametric conditions associated with dark and antidark solitons,

$$\text{Dark soliton}\begin{cases} s<0 \ \text{and} \ \dfrac{\sqrt{2}}{\sigma}\sqrt{\chi_2}<\kappa<0, \\ s<0 \ \text{and} \ \dfrac{1}{s\sigma}\sqrt{\chi_1}<\kappa<-\dfrac{\sqrt{2}}{\sigma}\sqrt{\chi_2}, \\ s>0 \ \text{and} \ 0<\kappa<-\dfrac{\sqrt{2}}{\sigma}\sqrt{\chi_2}, \\ s>0 \ \text{and} \ \dfrac{\sqrt{2}}{\sigma}\sqrt{\chi_2}<\kappa<\dfrac{1}{s\sigma}\sqrt{\chi_1}, \end{cases} \tag{29}$$

$$\text{Antidark soliton}\begin{cases} s<0 \ \text{and} \ 0<\kappa<\dfrac{1}{s\sigma}\sqrt{\chi_1}, \\ s>0 \ \text{and} \ \dfrac{1}{s\sigma}\sqrt{\chi_1}<\kappa<0, \end{cases} \tag{30}$$

with $\chi_1=-3N^4s^4\rho^4-8N^2s^2\rho^2\sigma-4\sigma^2$, $\chi_2=-N^4s^2\rho^4-2N^2\rho^2\sigma$, $\frac{-2\sigma}{3N^2s^2}<\rho^2<\frac{-2\sigma}{N^2s^2}$.

B. Soliton collisions

For the purpose of better understanding the collision dynamics between two solitons, we make an appropriate asymptotic analysis of Solution (24) as follows:

(a) S_1^- $(\theta_1\sim 0,\ \theta_2\to-\infty)$

$$U\to S_1^-=\rho\sqrt{1-\frac{\alpha_1+\alpha_1^*+\beta_1+\beta_1^*}{2|\beta_1|\cosh(\theta_1+\ln|\beta_1|)+\beta_1+\beta_1^*}}e^{i\varphi_1^-}, \tag{31}$$

with

$$\varphi_1^-=3\xi\tau-\eta\zeta-\frac{i}{2}\ln\left[\frac{1-\alpha_1e^{\theta_1}}{1-\alpha_1^*e^{\theta_1}}\left(\frac{1+\beta_1^*e^{\theta_1}}{1+\beta_1e^{\theta_1}}\right)^3\right].$$

(b) S_2^- $(\theta_2\sim 0,\ \theta_1\to+\infty)$

$$U\to S_2^- =\rho\left|\frac{\alpha_1\beta_1^*}{\beta_1^2}\right|\sqrt{1-\frac{\alpha_2+\alpha_2^*+\beta_2+\beta_2^*}{2|\beta_2|\cosh(\theta_2+\ln|\chi_{12}\beta_2|)+\beta_2+\beta_2^*}}e^{i\varphi_2^-}, \tag{32}$$

with

$$\varphi_2^-=3\xi\tau-\eta\zeta-\frac{i}{2}\ln\left[\frac{\alpha_2\beta_2^{*3}}{\alpha_2^*\beta_2^3}\frac{1-\alpha_2e^{\theta_2}}{1-\alpha_2^*e^{\theta_2}}\left(\frac{1+\beta_2^*e^{\theta_2}}{1+\beta_2e^{\theta_2}}\right)^3\right].$$

(c) S_1^+ $(\theta_1\sim 0,\ \theta_2\to+\infty)$

$$U\to S_1^+ =\rho\left|\frac{\alpha_2\beta_2^*}{\beta_2^2}\right|\sqrt{1-\frac{\alpha_1+\alpha_1^*+\beta_1+\beta_1^*}{2|\beta_1|\cosh(\theta_1+\ln|\chi_{12}\beta_1|)+\beta_1+\beta_1^*}}e^{i\varphi_1^+}, \tag{33}$$

with

$$\varphi_1^+=3\xi\tau-\eta\zeta-\frac{i}{2}\ln\left[\frac{\alpha_1\beta_1^{*3}}{\alpha_1^*\beta_1^3}\frac{1-\alpha_1e^{\theta_1}}{1-\alpha_1^*e^{\theta_1}}\left(\frac{1+\beta_1^*e^{\theta_1}}{1+\beta_1e^{\theta_1}}\right)^3\right].$$

(d) S_2^+ $(\theta_2\sim 0,\ \theta_1\to-\infty)$

$$U\to S_2^+=\rho\sqrt{1-\frac{\alpha_2+\alpha_2^*+\beta_2+\beta_2^*}{2|\beta_2|\cosh(\theta_2+\ln|\beta_2|)+\beta_2+\beta_2^*}}e^{i\varphi_2^+}, \tag{34}$$

with

$$\varphi_2^+=3\xi\tau-\eta\zeta-\frac{i}{2}\ln\left[\frac{1-\alpha_2e^{\theta_2}}{1-\alpha_2^*e^{\theta_2}}\left(\frac{1+\beta_2^*e^{\theta_2}}{1+\beta_2e^{\theta_2}}\right)^3\right].$$

Comparing expression (31) with Eq. (33), and expression (32) with Eq. (34), we can see that the physical quantities of the solitons do not change before and after the collision on account of $|\alpha_j|=|\beta_j|$ $(j=1,2)$, although there exist the small phase shifts, $(\ln|\chi_{12}\beta_1|-\ln|\beta_1|)/\kappa_1$ and $(\ln|\beta_2|-\ln|\chi_{12}\beta_2)/\kappa_2$, respectively. Therefore, the collision between two solitons is elastic.

In order to demonstrate the collision behavior between two solitons, we will graphically analyze the two-soliton solution through the choices of the parameters. Figure 1 display the collision between two antidark solitons with the

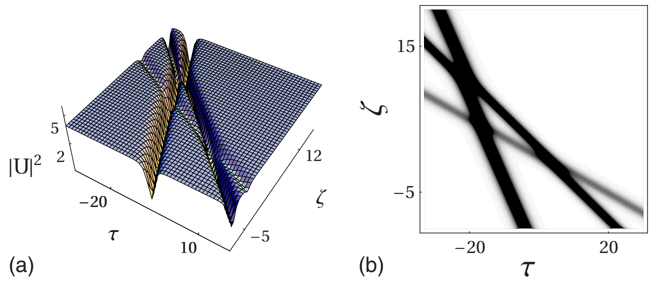

FIG. 5. (Color online) (a) Elastic collision between three dark solitons via solution (25) with the parameters $\rho=2$, $s=-1$, $N=1$, $\sigma=-3.1$, $\kappa_1=0.8$, $\kappa_2=-1.35$, $\kappa_3=1.16$, $\varsigma_1=\varsigma_3=1$ and $\varsigma_2=2000$. (b) Contour plot of (a).

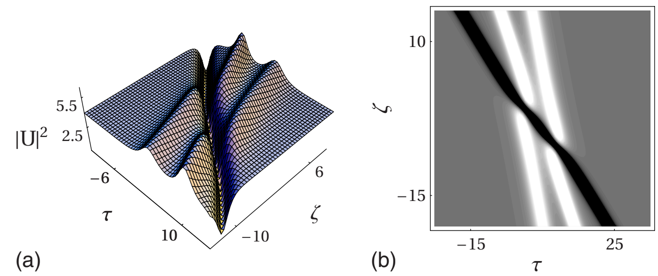

FIG. 6. (Color online) (a) Elastic collision between two antidark solitons and one dark soliton via solution (25) with the parameters $\rho=2$, $s=-1$, $N=1$, $\sigma=-3.1$, $\kappa_1=0.8$, $\kappa_2=1$, $\kappa_3=0.5$, and $\varsigma_1=\varsigma_2=\varsigma_3=1$. (b) Contour plot of (a).

wave numbers κ_1 and κ_2 both satisfying condition (30). Apart from the elastic collision features, an interesting phenomenon arises, namely, at the moment of the collision the two solitons emerge into one and the amplitude is lower than that of any one before the collision as shown in Fig. 1(b), which is different from the regular collision between two antidark solitons in Refs. [13,16]. Moreover, as we know, few studies have been done to discuss the interaction between two antidark solitons. If κ_1 and κ_2 satisfy conditions (29) and (30), respectively, the dark and antidark solitons of Eq. (1) coexist on the same background and exhibit the elastic collision which can be seen in Fig. 2. Similarly, the elastic collision between two dark solitons is presented in Fig. 3 when the wave numbers κ_1 and κ_2 both meet condition (29). Therefore, by adopting the values of the wave numbers κ_j's $(j=1,2)$, we can control the type of collision behavior between two solitons.

With respect to the three-soliton solution, we can also have the similar behavior of the soliton collision by modifying the parameters in the corresponding regions. Collisions among three solitons are all pairwise elastic as shown in Figs. 4–6, which are similar to the cases in Ref. [38]. Figure 4 displays the pairwise elastic collisions among three antidark solitons with different amplitudes and velocities at three different positions with the choices of parameters ς_j's and κ_j's $(j=1,2,3)$ which affect the velocities and initial phases of the solitons. The three solitons collide with one another without any change in the physical quantities except for the small phase shifts during the process of the collision. By choosing the parameters under condition (29), we get the pairwise elastic collisions among three dark solitons at three different spots in Fig. 5. Meanwhile, in Fig. 6, we present the phenomenon that the two antidark solitons collide elastically with one dark soliton.

V. CONCLUSIONS

In conclusion, we have investigated Eq. (1) which describes the femtosecond optical pulse propagation in a monomodal optical fiber. Another type of lax pair (2) has been constructed via the WKI scheme and infinitely many conservation laws in Expression (6) have been presented, which further prove the integrability of Eq. (1). In addition, bilinear form (15)–(17) has been derived via the Hirota method. Moreover, dark and antidark soliton solutions in expressions (21), (24), and (25) have been obtained through the bilinear form. Besides, conditions for the existence of the dark and antidark solitons have also been given in expressions (30) and (29). Through the asymptotic analysis of two-soliton solution in expression (24), collisions between solitons including two antidark solitons, two dark solitons, and dark and antidark solitons are found to be all elastic (as seen in Figs. 1–3). Meanwhile, the phenomenon that dark and antidark solitons can coexist on the same background has been verified to be existent for Eq. (1). Pairwise elastic collisions among three solitons have also been demonstrated through Figs. 4–6.

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