Maximum or minimum entropy production? How to select a necessary criterion of stability for a dissipative fluid or plasma

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Ten necessary criteria for stability of various dissipative fluids and plasmas are derived from the first and the second principle of thermodynamics applied to a generic small mass element of the system, under the assumption that local thermodynamic equilibrium holds everywhere at all times. We investigate the stability of steady states of a mixture of different chemical species at the same temperature against volume-preserving perturbations. We neglect both electric and magnetic polarization, and assume negligible net mass sources and particle diffusion. We assume that both conduction- and radiation-induced heat losses increase with increasing temperature. We invoke no Onsager symmetry, no detailed model of heat transport and production, no "Extended Thermodynamics," no "Maxent" method, and no "new" universal criterion of stability for steady states of systems with dissipation. Each criterion takes the form of—or is a consequence of—a variational principle. We retrieve maximization of entropy for isolated systems at thermodynamic equilibrium, as expected. If the boundary conditions keep the relaxed state far from thermodynamic equilibrium, the stability criterion we retrieve depends also on the detailed balance of momentum of a small mass element. This balance may include the ∇p-related force, the Lorenz force of electromagnetism and the forces which are gradients of potentials. In order to be stable, the solution of the steady-state equations of motion for a given problem should satisfy the relevant stability criterion. Retrieved criteria include (among others) Taylor's minimization of magnetic energy with the constraint of given magnetic helicity in relaxed, turbulent plasmas, Rayleigh's criterion of stability in thermoacoustics, Paltridge *et al.*'s maximum entropy production principle for Earth's atmosphere, Chandrasekhar' minimization of the adverse temperature gradient in Bénard's convective cells, and Malkus' maximization of viscous power with the constraint of given mean velocity for turbulent shear flow in channels. It turns out that characterization of systems far from equilibrium, e.g., by maximum entropy production is not a general property but—just like minimum entropy production—is reserved to special systems. A taxonomy of stability criteria is derived, which clarifies what is to be minimized, what is to be maximized and with which constraint for each problem.

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I. PROBLEM

Gibbs' statistical mechanics describes thermodynamic equilibrium. Depending on the problem, equilibrium is described either through maximization or minimization of some macroscopic quantity. For example, entropy is maximized in isolated systems, and Gibbs' free energy is minimized in problems with given temperature and pressure. The stability properties of thermodynamic equilibrium are well established. In contrast, no general consensus exists for the description of stable steady states of systems with dissipation far from thermodynamic equilibrium.

As a starting point, some researchers have invoked—either implicitly [1] or explicitly [2]—the so-called "local thermodynamic equilibrium" (LTE) [3]. LTE means that—although the total system is not at equilibrium—the internal energy per unit mass u is the same function of the entropy s per unit mass, the pressure p, the mass density ρ, etc. as in real equilibrium; more generally, the relationships among thermodynamic quantities will be the same as in real equilibrium. Furthermore, it LTE holds within a small mass element followed along its center-of-mass motion, then all relationships among total differentials of thermodynamic quantities remain valid, provided that the total differential da of the generic quantity a is $da=(da/dt)dt$ where $da/dt=\partial a/\partial t+\mathbf{v}\cdot\nabla a$ and $\mathbf{v}$ is the center-of-mass velocity. Usually, short-range, interparticle collisions ensure validity of LTE in fluids; as for plasmas, see [4–6].

Historically, a milestone toward characterization of systems with dissipation at LTE far from thermodynamic equilibrium has been the introduction of variational principles in the particular case when Onsager symmetry holds (for a review, see [7]). This symmetry leads to the "least dissipation" principle. In spite of its name, this principle is a maximization (with respect to variation in thermodynamical fluxes at fixed thermodynamical forces) of the difference between the amount of entropy produced per unit time inside the system ("entropy production") and the so-called Rayleigh dissipation function—see Sec. IV.5 of [7]. Similar principles are also helpful in solving the linearized Boltzmann equation (see the reviews of [8,9]). The least dissipation principle reduces to maximization of entropy production at fixed thermodynamical forces as the dissipation function is one half of the entropy production for linear phenomenological relationships—see Sec. IV.1 of [7] and the problems described by Eqs. (9), (11)–(13), and (17)–(18) of [10], by Sec. 1 of [11], by Eqs. (1.4), (1.9), and (1.25) of [8], and by the discussion of Eq. (6) of [58]. Under the same assumptions of LTE and Onsager symmetry, further variational

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principles—in both global and local form—are also available, which are fully equivalent to the least dissipation principle, either with respect to variation in forces at fixed fluxes and to variation in forces and fluxes "independently from one another"—see p. 96 of [7]. Extension to nonsteady states is discussed by Sienutycz *et al.* [13]—compare their equation (34) with equation (4.35) of [7].

As for steady states, Onsager symmetry leads also to another result [2], the "minimum entropy production" principle: entropy production is a minimum with respect to "simultaneous variations" of fluxes and forces–see Sec. V.3 of [7] and Sec. 3.3 of [9]. At least one flux vanishes in the discussion of the principle for steady discontinuous systems, while the differential equations ruling selected continuous systems show that the entropy production has a negative time derivative near the steady state and behaves like a Lyapunov function, so that minimization of entropy production is a criterion for stability of the steady state. However, the minimum production principle and the least dissipation principle are equivalent in steady states where the entropy current density at the boundary is fixed—see equation (5.39) of [7]. Thus, the choice between maximization and minimization in the domain of validity of Onsager symmetry for steady states at LTE depends ultimately on what is kept fixed, just like in thermodynamic equilibrium.

However, the validity of both least dissipation and "minimum production" is limited to systems where both LTE and Onsager symmetry hold. This limitation rules out both fluids and plasmas, which violate Onsager symmetry [5,12]. In order to circumvent this limitation, some authors [14] drop the LTE assumption, and postulate that entropy depends locally not only on u, ρ etc. but also on the heat flux and the viscous stress tensor. Their approach [the so-called "Extended Irreversible Thermodynamics," (EIT)], together with Einstein's formula for the probability of fluctuations, leads to predictions which agree with well-known results of kinetic theory near thermodynamic equilibrium. The price to be paid is that the absolute temperature is no more the multiplying factor of the differential of entropy in the first principle of thermodynamics—see Eqs. (3.1)–(3.3) of [14]. Moreover, in spite of the alleged independence of EIT from LTE—see Sec. 3.1 of [14]—the latter remains somehow involved. For example, the contribution of heat conduction to EIT entropy is basically the product of a relaxation time and the corresponding term in the entropy production rate at LTE—see Eqs. (9.1)-(9.3) of [14].

Another approach outside LTE is the so-called "Information Thermodynamics" or "Maxent" [15–19]. Starting from Jaynes' investigation of the connection between information theory, reproducibility and statistical mechanics [15,16], Dewar [17,18] derives a principle of maximum entropy production. Niven [19] links Maxent and a control volume approach. Maxent has been applied to atmospheric physics [20,21] and to problems with convection [21,22]. However, Dewar's results are criticized in Sec. 2.3.4 of [8] and by [23]. Moreover, equation "$q_i = s^{-1}$" at page 021113-2 of [19] appears to be the Maxent equivalent of the basic tenet of Gibbs' statistical mechanics that an isolated system in equilibrium is equally likely to be in any of its accessible states. The latter tenet is related to Liouville equation; but no Maxent equivalent of Liouville equation is explicitly stated in [19].

Independently, Sawada invokes neither LTE nor the constraint of fixed forces and postulates "a principle of maximum increasing rate of the total entropy," even if his arguments lead rather to maximization the amount of entropy exchanged per unit time with external, large heat reservoirs—see Eq. (7) of [24]. Furthermore, Ziegler postulates what he calls "orthogonality principle," which is basically equivalent to maximization of entropy production at fixed forces [8,25]. Remarkably, Ziegler invokes LTE explicitly—see Sec. 2 of [25]. According to Sec. 2.4 of [8], Ziegler's principle "has its statistical substantiation only if the deviation from equilibrium is small."

Indeed, a huge collection of experimental data suggests that many systems in stable steady state far from thermodynamic equilibrium maximize the entropy production; "however, attempts to derive the principle under discussion are so far unconvincing since they often require introduction of additional hypotheses, which by themselves are less evident than the proved statement" [8]. Unfortunately, there are also many well-known examples of systems with no Onsager symmetry in stable, far-from-equilibrium steady state and with no obvious maximization of entropy production. For example, the absolute value of the gradient of temperature in the Bènard convective cells investigated analytically by Chandrasekhar [26] attains a constrained minimum, the constraint being provided by the time-averaged power balance between buoyancy and dissipation. Helmholtz-Korteweg' [27] and Kirchhoff's [28,29] principles prescribe minimization of the dissipated power in viscous fluids and Ohmic resistors, respectively. Taylor's minimization of magnetic energy with the constraint of fixed magnetic helicity [30] has been applied to turbulent plasmas in both laboratory [31] and space [32,33]. Finally, thermoacoustic stability in fluids with combustion relies on Rayleigh's criterion, which is not a variational principle either [34–36]. After years of debate, most researchers are likely to agree with Jaynes' sharp criticism to any dissipation-related stability principle outside Gibbs' statistical mechanics [28]: "if we have enough information to apply the principle with any confidence, then we have more than enough information to solve the steady-state problem without it."

Remarkably, however, the very fact that any arbitrary small mass element of the system satisfies LTE at all times puts a constraint on the evolution of the whole system [4,37,38]. If the latter relaxes to some final, steady ($\partial/\partial t = 0$) stable state (referred to as "relaxed state" below) then it is conceivable that LTE alone may provide us with useful information on the relaxed state, with no need of Onsager symmetry, of EIT, of Maxent and of any *ad hoc* postulate. (Admittedly, as far as relaxed fluids far from thermodynamic equilibrium are concerned, the notion of "steady state" is rather ambiguous: all the same, we maintain—as a working hypothesis—that it still makes sense—possibly after time-averaging on time scales $\gg$ turbulent time scales).

In particular, the first and the second principle of thermodynamics in a nonpolarized mixture of N chemical species with the same temperature T for all chemical species at LTE lead to

$$du = Tds - pd(\rho^{-1}) + \Sigma_z \mu^{\circ}_z dc_z, \tag{1.1}$$

$$(\partial u/\partial T)_{\rho,N} > 0, \tag{1.2}$$

$$(\partial \rho^{-1}/\partial p)_{T,N} < 0, \tag{1.3}$$

$$\Sigma_{jz}(\partial \mu^{\circ}_z/\partial c_j)_{p,T} dc_z dc_j \geq 0 \tag{1.4}$$

(see [39] Sec. XV 5,12). Here j, $z=1,\ldots,N$, $\mu^{\circ}_z \equiv \mu_z m_z^{-1}$, $c_z \equiv N_z m_z (\Sigma_z N_z m_z)^{-1}$, where μ_z, m_z, and N_z are the chemical potential, the mass of one particle and the number of particles of the z-th chemical species, respectively. Finally, $(\,)_N$ means that all c_z's are kept fixed, and the sign $\geq$ is replaced by $=$ only for $dc_z=0$. Furthermore, if LTE holds within a small mass element followed along its center-of-mass motion, relationships (1.1)–(1.4) lead to

$$\begin{aligned}[d(T^{-1})/dt][d(\rho u)/dt] \leq{} & \rho\Sigma_z[d(\mu^{\circ}_z T^{-1})/dt](dc_z/dt) \\ & + [\rho^{-1}T^{-1}(dp/dt) + (u + p\rho^{-1}) \\ & \times (dT^{-1}/dt)](d\rho/dt)\end{aligned} \tag{1.5}$$

(see Appendix A). Basically, Eq. (1.5) is the so-called "general evolution criterion" (GEC) [37,38]. GEC provides a constraint on the evolution of a generic small mass element and is the starting point of our discussion. Physically, if LTE holds everywhere at all times during relaxation, then the relaxed state is the final outcome of the GEC-constrained evolution of many small mass elements. Our aim is to gain information concerning the relaxed state from GEC.

To this purpose, we generalize the arguments of [4] and [38]. Since we are interested in relaxed states of the system as a whole, we focus our attention on volume integrals like the volume $V=\int d\mathbf{x}$ of the system ($d\mathbf{x}$ volume element, $d\mathbf{a}$ surface element of the boundary), its internal energy $E=\int d\mathbf{x}\rho u$ and its entropy $S=\int d\mathbf{x}\rho s$. We define also the spatial average $\langle\langle a\rangle\rangle \equiv V^{-1}\int d\mathbf{x} a$ and the temporal average $\langle a\rangle \equiv \lim_{T\to\infty} T^{-1}\int_t^{t+T} dt' a(t')$ of a. Sec. II contains our assumptions. In Sec. III, suitable integration on the volume of the system will allow us to derive from GEC some inequalities involving the time derivatives of these volume integrals in the neighborhood of a relaxed state with the help of some simplifying assumptions and of the balances of mass and energy. In Sec. IV, these inequalities lead to necessary conditions for the stability of the relaxed state against volume-preserving perturbations. We will show that different criteria for stability of relaxed states in various problems of fluid mechanics and plasma physics follow from GEC as particular cases in Secs. V–IX, depending on the—insofar neglected—detailed momentum balance of the small mass element—i.e., on the detailed nature of the forces acting on the latter. Conclusions are given in Sec. X, which includes a synoptic table of our results.

II. ASSUMPTIONS

We assume no net mass source. Then, the mass balance reads $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v})=0$, i.e.,

$$\rho^{-1}d\rho/dt + \nabla\cdot\mathbf{v} = 0. \tag{2.1}$$

As for the balance of momentum, we are going to discuss problems where the only forces acting on the generic small mass element are the ∇p-related force, the forces which are gradients of potentials, the viscous force and the Lorenz force of electromagnetism. Other forces (e.g., due to magnetization, elasticity, etc.) would add new terms to ρdu in Eq. (1.1), and are neglected in this work. The ∇p-related force affects Eq. (1.1) through $pd(\rho^{-1})$. The potentials add a new contribution to the μ°_z's. The viscous force and the Lorenz force add a viscous heating power density and an Ohmic heating power density, respectively, to the heating power density P_h (≥ 0; we neglect endothermal reactions in the following), which is defined as follows:

$$P_h \equiv \nabla\cdot\mathbf{q} + P. \tag{2.2}$$

Here the quantity $\mathbf{q}$ is the heat flux. It is a local quantity and therefore does not include convective transport: convection—if any—is due to flow patterns on a spatial scale$\gg$the spatial scale a small mass element. The quantity P is defined as

$$P \equiv \rho(du/dt) + \rho p(d\rho^{-1}/dt). \tag{2.3}$$

Once an explicit expression for both P_h and $\mathbf{q}$ is provided, Eqs. (2.2) and (2.3) give the energy balance of the small mass element. The quantities $\int d\mathbf{x} P_h$ and $\int d\mathbf{x} T^{-1} P_h$ are the total heating power and the amount of entropy produced per unit time by heating processes, respectively. In the following, the only nonviscous and non-Ohmic heating mechanism (if any) is due to short-range, interparticle reactions that do not alter the momentum of the small mass element. For simplicity, we assume that the amount W of heat released per particle is $\gg \mu_z \approx T$. Nuclear fusion of deuterons ($T \approx 100$ KeV, $W/T \approx 30$) and oxygen-methane combustion ($T \approx 0.2$ eV, $W/T \approx 40$) are examples of such reactions. Then, we neglect terms $\propto dc_z/dt$ in P_h. This assumption does not imply that these reactions do not occur (their impact on the chemical composition of the mixture may be compensated by suitable diffusion flows even at $dc_z/dt \approx 0$, according to Eq. (2.13) of [2]), but puts in evidence that their occurrence affects our arguments below only weakly. Equations (1.1) and (2.3) give

$$P = \rho T ds/dt. \tag{2.4}$$

Of course, Eq. (2.4) holds also in chemically pure ($N=1$) substances, nonreacting mixtures, and quasineutral, hot ($T \geq$ KeV) plasmas where both recombination and ionization are negligible. Moreover, neglecting terms $\propto dc_z/dt$ in P_h allows us to correct an error in the proof discussed in Ref. [38].

We invoke no Onsager symmetry [2], no extended thermodynamics [14], no maximization postulate [17,24,25], and no detailed model of heat production and transport. We assume that a relaxed state exists and write $a(\mathbf{x},t)=a_0(\mathbf{x})+a_1(\mathbf{x},t)$, $P_0=0$. Stability of the relaxed state implies that the perturbation amplitude never diverges regardless of its physics (e.g., damped oscillations, etc.), i.e., it remains upper-bounded everywhere across the system at all times. Then, $a(\mathbf{x},t)$ has a finite upper bound everywhere at all times, i.e., a

$(\mathbf{x},t) \le A_M(t)$ everywhere, where $A_M(t) \le A_{\max}$ for arbitrary t. Analogous arguments hold for the lower bound $A_{\min}$ $[a(\mathbf{x},t) \ge A_m(t) \ge A_{\min}]$. We are free to take $A_{\min} = -A_{\max}$. Accordingly, stability of the relaxed state implies that for a generic quantity $a(\mathbf{x},t)$ a positive constant $A_{\max}$ and two functions of time $A_M(t)$, $A_m(t)$ exist such that the following inequality (invoked in the Appendices) holds:

$$A_{\max} \ge A_M(t) \ge a(\mathbf{x},t) \ge A_m(t) \ge -A_{\max}. \tag{2.5}$$

Furthermore, since we are interested in necessary condition of stability, we are free to select the perturbation a_1, which we invoke in order to test stability. We limit ourselves to smooth perturbations which relax gently back to the unperturbed state. By "smooth" and "gently" we mean that we neglect terms $\propto |\nabla(da_1/dt)|$ and $\propto |da_1/dt|^{-1}|d^2a_1/dt^2|$, respectively. Our choice includes no small-scale, fast (e.g., turbulent) fluctuations of a. For example, if $a_0(\mathbf{x})$ is the macroscopic electric current density flowing across a stable, steady, large tokamak plasma (lifetime $\approx$ tens of seconds) then our $a_1(\mathbf{x},t)$ includes slow perturbations related to resistive decay (lifetime $\approx$ plasma inductance/plasma resistance ≈ 1 second) but no magnetohydrodynamic (MHD) fluctuations (lifetime $\le 10^{-3}$ s). We do not say that fast fluctuations are negligible; but no stability exists if the steady state is unstable against smooth and gentle perturbations, and the relative criteria for stability are also necessary conditions of stability. Should the discussion include fast fluctuations too, stronger necessary criteria would result. We do not need such stronger criteria in the following. As shown in the Appendices, our choice of smooth and gentle perturbations is due to mathematical convenience. In particular, LTE implies that Gibbs' statistical mechanics holds inside a small fluid element. Then, Gibbs' statistical mechanics rules the fluctuations in each small fluid element near its own local thermodynamical equilibrium (e.g., with the help of Einstein's formula). As these fluctuations occur inside the small fluid element, we consider them as small-scale phenomena. As such, we may safely assume their time scale is a short one, and consider them as examples of those fast fluctuations we do not deal with in the present work.

Finally, physical intuition—in agreement with Le Châtelier's principle—suggests that a state where an increase of T induces a decrease in energy losses is a bad candidate for stability, as any decrease in energy losses is likely to induce further increase of T. Analogous arguments hold for cooling processes. Accordingly, we assume [38]

$$\int d\mathbf{x}(\nabla \cdot \mathbf{q}_1)[d(T^{-1})/dt] \le 0. \tag{2.6}$$

Inequality Eq. (2.6) involves just a volume integral; we say nothing about the sign of the integrand on the LHS of Eq. (2.6) at a given location in the system at a given time during the relaxation process. Since we are not interested in the fast fluctuations of each small fluid element near its own local thermodynamical equilibrium, we take into account in Eq. (2.6) just those large-scale perturbations which are not bound to satisfy Gibbs' statistical mechanics (precisely because the system as a whole is far from thermodynamic equilibrium).

III. INEQUALITIES

We derive in Appendix B the following inequalities from Eqs. (1.5)-(2.6) (c_1, c_2, c_3, c_4 are constant quantities)

$$d\left\{\int d\mathbf{x} T^{-1} P_h\right\}/dt \le c_1 dV/dt + c_2 d\left(\int d\mathbf{x} P_h\right)/dt, \tag{3.1}$$

$$-d\left\{\int d\mathbf{x}[\nabla \cdot (\rho s \mathbf{v}) + \mathbf{q} \cdot \nabla(T^{-1})]\right\}/dt \le c_1 dV/dt + c_2 d\left(\int d\mathbf{x} P_h\right)/dt, \tag{3.2}$$

$$d^2S/dt^2 \le c_3 d^2V/dt^2 + c_4 d^2E/dt^2. \tag{3.3}$$

IV. CRITERIA FOR STABILITY

According to Eqs. (3.1) and (3.2), a necessary condition for the stability of a relaxed state is that the latter satisfies the variational principles (we skip the subscript "0" below, unless otherwise stated)

$$\int d\mathbf{x} T^{-1} P_h = \text{min.} \quad \text{with fixed } V \text{ and } \int d\mathbf{x} P_h, \tag{4.1}$$

$$\int d\mathbf{x}[\nabla \cdot (\rho s \mathbf{v}) + \mathbf{q} \cdot \nabla(T^{-1})] = \text{max.} \quad \text{with fixed } V \text{ and } \int d\mathbf{x} P_h. \tag{4.2}$$

In fact, if the unperturbed state violates Eq. (4.1), then Eq. (3.1) forbids stability against perturbations which conserve both V and $\int d\mathbf{x} P_h$ but lower the value of $\int d\mathbf{x} T^{-1} P$. Similar arguments hold for Eqs. (3.2) and (4.2). In case of large competing energy transport, it is useful to discuss the limit $\nabla T \to 0$ (i.e., $T(\mathbf{x}) \to T_{\text{boundary}}$ everywhere) even if $P_h > 0$. In this case, it is possible to show (see Sec. 3 of [38]) that the solutions of the Euler-Lagrange equations of Eq. (4.1) solve also the Euler-Lagrange equations of the variational principle

$$\int d\mathbf{x} P_h = \text{min.} \quad \text{with fixed } V, \text{ and } T(\mathbf{x}) = T_{\text{boundary}} \text{ everywhere.} \tag{4.3}$$

For a relaxation process where both V and E are constant at all times (like, e.g., in isolated systems), Eq. (3.3) implies $d^2S/dt^2 \le 0$. Evolution toward thermodynamic equilibrium (S = max.) of isolated systems (E = const., V = const.) provides a well-known example of such relaxation. (As for nonisolated systems, suitable boundary conditions—e.g. an applied voltage—may keep the relaxed state far from thermodynamic equilibrium). Another consequence of Eq. (3.3) is discussed in Sec. IX.

V. APPLICATIONS: GENERALITIES

Generally speaking, relationships (3.1)–(4.3) lead to different results for different particular relaxed states, depending on the detailed heating mechanism (Ohmic, viscous, exothermal reactions, none), on the detailed nature of the forces acting on the system (the ∇p-related force, the viscous force, the Lorenz force of electromagnetism, and the forces that are gradients of potentials) and on the boundary conditions. As for the latter, in the following we analyze stability against perturbations of relaxed states which conserve volume at all times, so that we may skip the constraint of fixed volume. We are going to derive some results of hydrodynamics, plasma physics and thermoacoustics, which have been often suggested in the literature without rigorous proof in order to cope with experiments. To this purpose, we introduce the Joule heating power density $P_J \equiv (\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\cdot\mathbf{j}$ and the viscous heating power density $P_V \equiv \eta\Sigma_{ik}[\partial v_i/\partial x_k + \partial v_k/\partial x_i - 2\delta_{ik}(\nabla\cdot\mathbf{v})/3](\partial v_i/\partial x_k) + \zeta(\nabla\cdot\mathbf{v})^2$, where $i,k=1,2,3$, and $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$, η, and ζ are the electric field, the magnetic field, the electric current density, the dynamic viscosity and the second viscosity, respectively [1]. In the general case, we write $P_h = P_J + P_V +$ the contribution ($\propto W$) of short-range (chemical, nuclear) reactions, which conserve the momentum of the small mass element. Noteworthy, Eqs. (3.1)–(4.3) imply per se no minimization of the amount of entropy produced per unit time by all irreversible processes; this is in agreement with the fact that we did not invoke Onsager symmetry anywhere, unlike [2].

In the following, we invoke simultaneous validity of Eqs. (4.1) and (4.2) for the same relaxed state only occasionally, as we discuss some applications of each criterion separately for simplicity. As a result, the list of retrieved results is bound to be incomplete. Of course, the full set of conservation laws provides complete description of relaxed states and leaves no room for any further variational principle [28]. If relaxation occurs, then the description of the relaxed state which is provided with the help of the variational principle must agree with the description which is provided by the full set of equations of motion: "the energy equation, the equation of motion and the entropy equation are not independent of one another […] only two out of the three need to be used in any problem"—see [40] p. 189. A relaxed state should therefore both solve the steady-state equations of motion *and* satisfy the variational principle. In other words, the former provide the latter with further constraints. In the following, we mean that all maxima and minima we are going to derive are constrained not only by the constraints we shall write down explicitly in each case, but also by the steady-state equations of motion (this fact has been utilized explicitly, e.g., in Ref. [4]).

VI. APPLICATIONS OF EQ. (4.1)

(A) Strong stabilizing magnetic field allows stable tokamak plasmas at MHD equilibrium to exist with $\nabla T \neq 0$ (by "strong" we mean that the field is much larger than the magnetic field created by the currents flowing across the plasma). We neglect both recombination and ionization in the plasma bulk at least; Eq. (2.4) follows, and Eq. (4.1) allows determination of electric conductivity profiles in tokamaks starting from experimental data [41], under the assumption $P_h = P_J$. Conservation of total mass—assumed in [41]—follows from Eqs. (2.1) and (B.1) of Appendix B for $a=\rho$. Our assumption that T is the same for all chemical species is relaxed in [41], as T is replaced by the electron temperature.

(B) Our discussion requires two preliminary, auxiliary relationships. Generally speaking, $|\nabla T| \geq 0$ everywhere, and $\langle\langle|\nabla T|\rangle\rangle \geq 0$. Moreover, we show in Appendix B that

$$\int d\mathbf{x} T^{-1} P_h = \int d\mathbf{x}[T^{-1}\nabla\cdot\mathbf{q} + \nabla\cdot(\rho s\mathbf{v})] \qquad (6.1)$$

in steady state. Note that the LHS of Eq. (6.1) is non-negative. If the net amount of entropy advected across the boundary of the system is negligible, $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v}) = \int d\mathbf{a}\cdot(\rho s\mathbf{v}) = 0$. According to Eq. (6.1), Eq. (4.1) is compatible with

$$\int d\mathbf{x} T^{-1}\nabla\cdot\mathbf{q} = \text{min. with fixed} \int d\mathbf{x} P_h. \qquad (6.2)$$

Admittedly, this is no rigorous proof of Eq. (6.2). For example, should $\mathbf{q}$ vanish, then Eq. (6.1) would imply $\int d\mathbf{x} T^{-1} P_h = \int d\mathbf{a}\cdot\rho s\mathbf{v}$, but the values of the L.H.S. and the R.H.S. would be a minimum and a maximum under the same constraint of fixed $\int d\mathbf{x} P_h$ according to Eqs. (4.2) and (4.3), respectively. Broadly speaking, however, the larger $|\nabla T(\mathbf{x})|$ the larger the difference of temperature between a small mass element located at $\mathbf{x}$ with given temperature $T(\mathbf{x})$ and its surroundings, the larger the amount of heat exchanged per unit time by the small mass element through radiation and convection. In particular, $\int d\mathbf{x} T^{-1}\nabla\cdot\mathbf{q} = 0$ if $|\nabla T| = 0$ everywhere. In turn, $|\nabla T| = 0$ everywhere if and only if $\langle\langle|\nabla T|\rangle\rangle = 0$. Since $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v}) = 0$, Eq. (6.1) implies that $\int d\mathbf{x} T^{-1}\nabla\cdot\mathbf{q}$ is non-negative, just like $\langle\langle|\nabla T|\rangle\rangle$. Then, $\int d\mathbf{x} T^{-1}\nabla\cdot\mathbf{q}$ is an increasing function of $\langle\langle|\nabla T|\rangle\rangle$ in a neighborhood of $\langle\langle|\nabla T|\rangle\rangle = 0$, i.e., if $\langle\langle|\nabla T|\rangle\rangle$ is "not too large" (a precise meaning is given below) and Eq. (6.2) leads to

$$\langle\langle|\nabla T|\rangle\rangle = \text{min.} \quad \text{with fixed} \int d\mathbf{x} P_h. \qquad (6.3)$$

Indeed, this is in agreement with Chandrasekhar's results–see Secs. 33, 43 of [26]. Chandrasekhar notes that Eq. (6.3) applies to (possibly time-averaged) relaxed states near the onset of Bénard convection cells in hydrodynamics ($P_h = P_V$) and MHD ($P_h = P_J + P_V$). In the former case, viscous effects are not negligible, e.g., when the Rayleigh number $\mathrm{Ra} \approx \mathrm{Ra}_{\mathrm{thr}}$ where $\mathrm{Ra}_{\mathrm{thr}}$ is the threshold value of Ra, which corresponds to the onset of convection. Since $\langle\langle|\nabla T|\rangle\rangle \propto \mathrm{Ra}$ in the problems solved in [26], "not too large" means just $\mathrm{Ra} \approx \mathrm{Ra}_{\mathrm{thr}}$. A time-averaged balance between the mechanical power delivered by the buoyancy force and $\int d\mathbf{x} P_h$ gives the nonzero value of the latter, in agreement with the equations of motion.

(C) Our result (6.3) is also in agreement with Malkus' analysis [42] of steady-state turbulent shear flows in viscous ($P_h = P_V$), incompressible ($\nabla\cdot\mathbf{v} = 0$) fluids moving with mean velocity $U = U(z)$ in the $\mathbf{x}$ direction between fixed parallel surfaces of infinite extent $z = +z_0$ and $z = -z_0$ in the Cartesian frame of coordinates $\{x,y,z\}$ and with $U(z=+z_0)=U(z=-z_0)=0$. Here "mean" is defined as an average in the y direction. Periodic boundary conditions allow us to write $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v})=0$. Malkus [Eqs. (1.12)–(1.14)] maximizes the total rate of dissipation per unit mass (written in the form $\nu\Sigma_{ik}\langle\langle|\partial v_i/\partial x_k|^2\rangle\rangle$, $\nu\equiv\rho^{-1}\eta$ kinematic viscosity), with fixed $U_m\equiv(2z_0)^{-1}\int_{-z_0}^{+z_0}Udz$. This is equivalent to

$$\int d\mathbf{x}P_h=\text{max.} \quad \text{with fixed} \quad \nabla\cdot\mathbf{v}(=0) \quad \text{and fixed} \quad U_m \tag{6.4}$$

as the fluid is incompressible and our treatment ensures conservation of total mass (see Sec. VIA). We are going to proof that the solutions of Eq. (6.3) solve also Eq. (6.4). To this purpose, it is enough to proof that $\langle\langle|\nabla T|\rangle\rangle$ is an increasing function of U_m. If this is true, in fact, U_m too is an increasing function of $\langle\langle|\nabla T|\rangle\rangle$, so that Eq. (6.3) leads to

$$U_m=\text{min.} \quad \text{with fixed} \quad \nabla\cdot\mathbf{v}(=0) \quad \text{and fixed} \quad \int d\mathbf{x}P_h \tag{6.5}$$

and a lemma of variational calculus, the reciprocity principle for isoperimetric problems (see Sec. IX.3 of [43]), ensures that the solutions of Eq. (6.5) solve also Eq. (6.4). It is customary to assume that $U(z)$ changes sign nowhere in the range $+z_0\geq z\geq -z_0$; with no loss of generality we take $U(z)\geq 0$ everywhere, so that $U_m\geq 0$; $U_m=0$ if and only if $U=0$ everywhere, which in turn is equivalent to $P_h=0$ everywhere. For simplicity, let us assume $T(z=+z_0)=T(z=-z_0)=T_{\text{boundary}}$ at all times. In this case, if $P_h=0$ everywhere then $|\nabla T|=0$ everywhere and $\langle\langle|\nabla T|\rangle\rangle=0$. Correspondingly, $U_m=0$ if and only if $\langle\langle|\nabla T|\rangle\rangle=0$, and $U_m>0$ if and only if $\langle\langle|\nabla T|\rangle\rangle>0$. Then, $\langle\langle|\nabla T|\rangle\rangle$ is an increasing function of U_m in a neighborhood of $U_m=0$, as required. Broadly speaking, the larger U_m the larger $|dU/dz|$ (as U vanishes at $\pm z_0$) the larger the viscous heating, the larger $|\nabla T|$ (as T_{boundary} is fixed). A similar behavior is expected in the more interesting case when the Reynolds number $\text{Re}\equiv\nu^{-1}z_0U_m$ exceeds the threshold value Re_{thr} corresponding to the onset of turbulence, as this onset corresponds to a sharp increase in viscous dissipation: the larger $\text{Re}-\text{Re}_{\text{thr}}$, the larger $U_m-\text{Re}_{\text{thr}}\, z_0^{-1}\nu$, the larger $\int d\mathbf{x}P_h$ and $\langle\langle|\nabla T|\rangle\rangle$, so that once again $\langle\langle|\nabla T|\rangle\rangle$ is an increasing function of U_m in a right neighborhood of Re_{thr} at least. Admittedly, the actual values of both $\int d\mathbf{x}P_h$ and $\langle\langle|\nabla T|\rangle\rangle$ are usually neglected in most practical applications. However, what is crucial here is their mutual dependence. In particular, nonzero values of U_m (hence $\int d\mathbf{x}P_h$) prevent $\langle\langle|\nabla T|\rangle\rangle$ from vanishing. Finally, Eq. (6.5) is equivalent to the variational principle

$$\text{Re}=\text{min.} \quad \text{with fixed} \quad \nabla\cdot\mathbf{v}(=0) \quad \text{and fixed} \quad \int d\mathbf{x}P_h \tag{6.6}$$

for constant values of ν and z_0. Indeed, it is shown in Ref. [42] that Eq. (6.6) is equivalent to Eq. (6.4). Formally, Eq. (6.6) is similar to Chandrasekhar's constrained minimization of $\langle\langle|\nabla T|\rangle\rangle\propto\text{Ra}$ discussed above in Sec. VIB for Bénard cells.

VII. APPLICATIONS OF EQ. (4.2)

(A) If we neglect heat conduction and radiation (i.e., $\mathbf{q}$) Eq. (4.2) reduces to $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v})=\text{max.}$ with fixed $\int d\mathbf{x}P_h$. If our relaxed system contains a strong shock wave [44] in a viscous fluid ($P_h=P_V$) then the shock thickness shrinks to zero for large Mach number—see Sec. 87 of [1]—and $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v})=\int_A d\mathbf{f}\cdot\rho s\mathbf{v}$, $(\mathbf{v}\cdot d\mathbf{f})_{\text{I}}=-v_{\perp\text{I}}|d\mathbf{f}|$, and $(\mathbf{v}\cdot d\mathbf{f})_{\text{II}}=v_{\perp\text{II}}|d\mathbf{f}|$ where A is a closed surface embedding the shock wave with unit surface $d\mathbf{f}$, $\int_A d\mathbf{f}\cdot\rho s\mathbf{v}$ is the amount of entropy exchanged by the strong shock wave with the external world through convection per unit time, subscripts "I" and "II" refer to the region in front and behind the shock wave and the subscript $\perp$ refers to the direction perpendicular to the shock wave, respectively. While $\int d\mathbf{x}P_h\to 0$ as the shock thickness vanishes, discontinuity $T_{\text{II}}\neq T_{\text{I}}$ prevents $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v})=\int d\mathbf{x}T^{-1}P_h$ from vanishing in Eq. (6.1). If $\rho s\mathbf{v}$ changes mainly in the direction which is orthogonal to the shock wave, then $\int_A d\mathbf{f}\cdot\rho s\mathbf{v}\propto(\rho s v_\perp)_{\text{II}}-(\rho s v_\perp)_{\text{I}}$. Mass balance and irreversibility require $(\rho v_\perp)_{\text{II}}=(\rho v_\perp)_{\text{I}}$ and $s_{\text{II}}>s_{\text{I}}$, respectively; Eq. (4.2) leads to

$$(\rho s v_\perp)_{\text{II}}-(\rho s v_\perp)_{\text{I}}=\text{max.} \tag{7.1}$$

Rebhan has shown that Eq. (7.1) holds [see Eq. (17) of [44]] and that it may *replace* one conservation equation.

(B) If convection across the boundary is negligible, then $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v})=0$ and Eq. (4.2) reduces to $\int d\mathbf{x}\mathbf{q}\cdot\nabla(T^{-1})=\text{max.}$ with the constraint of fixed $\int d\mathbf{x}P_h$. This result is in agreement with Eq. (15) of [17] and has a simple physical meaning provided that $\mathbf{q}=-\chi(\mathbf{x})\nabla T(\mathbf{x})$, with $\chi(\mathbf{x})$ thermal conductivity. In fact, $\mathbf{q}\cdot\nabla(T^{-1})=+\chi(\mathbf{x})[\nabla(\ln T)(\mathbf{x})]^2$; in turn, if $T(\mathbf{x})$ is given—e.g., from the solution of Eq. (6.3)—then maximization of $\chi(\mathbf{x})$ follows everywhere once $\int d\mathbf{x}P_h$ is provided, e.g., from the energy balance of Sec. VI. Physically, this result agrees with Eq. (6.3), as maximization of thermal conductivity (due, e.g., to turbulence, as in plasmas) leads naturally to minimize $|\nabla T|$ everywhere.

(C) If both $\int d\mathbf{x}P_h$ and $\int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v})$ vanish then heating mechanisms provide no more constraint and Eqs. (4.2) and (6.1), the relationship $\mathbf{q}\cdot\nabla(T^{-1})=\nabla\cdot(T^{-1}\mathbf{q})-T^{-1}\nabla\cdot\mathbf{q}$ and Gauss' divergence theorem give

$$\int d\mathbf{a}\cdot T^{-1}\mathbf{q}=\text{max.} \tag{7.2}$$

The LHS of Eq. (7.2) is the rate of entropy increase in the surrounding world due to conductive and/or radiative energy transport across the boundary of our system. Generalizing Paltridge's results [20], Ozawa and co-workers [21,22] apply Eq. (7.2) to the general circulation of Earth's global fluid (the atmosphere and the ocean), as no net convection occurs across the boundary and P_h is neglected everywhere inside the system. Our $\mathbf{q}$ and Eq. (4.2) correspond to the quantity $\mathbf{F}$ of Sec. 2.4 and to Eq. (8.b) of [22], respectively. $\mathbf{F}$ includes no large-scale convection (quoting [22]: it "does not in principle include the advective heat flux") but includes heat conduction ruled by small-scale turbulence as well as radiation. In particular, Eq. (7.2) describes well the overall outcome which results from the radiation power balances of many different regions of the global fluid (see the discussion of Eq. (9) in [22]). Ozawa *et al.* note also [21] that Eq. (7.2) applies reasonably well to the viscous ($P_h=P_V$), turbulent, time-averaged relaxed fluids with Bénard convection cells in the limit of large Rayleigh number: $\mathrm{Ra}\gg\mathrm{Ra}_{\mathrm{thr}}$ (see Fig. 4 of [21]: $\mathrm{Ra}\gtrsim 10^5\gg\mathrm{Ra}_{\mathrm{thr}}=1700$); in this limit both viscosity and P_V become negligible (in the fluid bulk at least).

(D) The experiments of Biwa *et al.* [45] provide us with a further, independent confirmation of Eq. (7.2). They supply to a fluid a given amount Q_H of heat per unit time through a heat exchanger H at temperature T_H and remove an amount Q_C of heat per unit time through another heat exchanger C at temperature $T_C\neq T_H$. Steady state is achieved when $Q_H=Q_C$. No heating occurs inside the system made of H and C and no convection occurs across the boundaries of this system. Accordingly, Eq. (7.2) holds and $\int d\mathbf{a}\cdot T^{-1}\mathbf{q}=(Q_H T_H^{-1}-Q_C T_C^{-1})=\mathrm{max.}$, or, equivalently

$$(Q_C T_C^{-1}-Q_H T_H^{-1})=\mathrm{min.} \qquad (7.3)$$

The contributions of H and C to $\int d\mathbf{a}\cdot T^{-1}\mathbf{q}$ are >0 and <0, respectively, as Q_H comes out from the system toward the fluid and Q_C comes into the system from the fluid. Observations [45] show that stable steady states satisfy Eq. (7.3). Our derivation of the 'maximum entropy principles' Eq. (7.1)–(7.3) invokes no Maxent method and no maximization postulate [17,24].

VIII. APPLICATIONS OF EQ. (4.3)

(A) Kirchhoff's variational principle

$$\int d\mathbf{x}|\mathbf{j}|^2\eta_\Omega=\mathrm{min.}\quad\text{with fixed}\quad\nabla\cdot\mathbf{j}(=0) \qquad (8.1)$$

—see [28,29] and Probl. 3 Sec. 21 of [46]—follows from Eq. (4.3) provided that $P_h=P_J$ and that Ohm's law takes the usual form $\eta_\Omega\mathbf{j}=(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})$ [46] where $\eta_\Omega=\eta_\Omega(T)>0$ is the scalar electric resistivity. In fact, the constraint $T(\mathbf{x})=T_{\mathrm{boundary}}$ everywhere reduces to $\eta_\Omega=\eta_\Omega(T_{\mathrm{boundary}})$ everywhere; this fact allows us to take $\eta_\Omega(T)$ out from the volume integral and to factor it out, so that we skip the constraint $T=T_{\mathrm{boundary}}$ in Eq. (8.1) as it affects no more the result of minimization. Moreover, the constraint $\nabla\cdot\mathbf{j}=0$ (conservation of electric charge) follows from the steady-state equation of motion for $\mathbf{j}$, i.e., Ampère's law $\mu_0\mathbf{j}=\nabla\wedge\mathbf{B}$, with $\mu_0=4\pi\times10^{-7}$ SI units. A solution of Eq. (8.1) is $\mathbf{j}\eta_\Omega=\nabla h$, $\Delta h=0, h$, scalar$=h(\mathbf{x})$. Complex geometrical patterns associated to the solutions of Eq. (8.1) are discussed in Ref. [49].

(B) Radiative transport may flatten ∇T in relaxed plasmas—as in radiation-cooled, free-flowing electric arcs—in the plasma bulk at least. By "free-flowing" we mean that the arc is in contact with no solid walls but the electrodes, so that the exchange of matter between the arc bulk and the external world is reduced and $dc_z/dt=\mathbf{v}\cdot\nabla c_z=0$ in steady-state ($\partial/\partial t=0$). When applied to such an arc with negligible Lorenz force and voltage drop V, where the external world maintains an electric current I at a value I_{boundary}, Eq. (8.1) takes the form $IV=\mathrm{min.}$ with fixed I ($=I_{\mathrm{boundary}}$), i.e., reduces to the principle $V=\mathrm{min.}$ postulated by Steenbeck [47]. Our discussion shows that V is quite insensitive to arc temperature. An independent investigation of SF6 arcs confirms this result—see Fig. 8 and Sec. V of [48].

(C) The natural counterpart of Eq. (8.1) for $P_h=P_V$ is Helmholtz-Kortweg variational principle

$$\int d\mathbf{x}|\nabla\wedge\mathbf{v}|^2\eta=\mathrm{min.}\quad\text{with fixed}\quad\nabla\cdot\mathbf{v}(=0) \qquad (8.2)$$

—see Sec. 327 of [27], where Eq. (8.2) is derived for incompressible fluids, which are subject to the ∇p-related force, the viscous force and (possibly) to constant potential forces. Formally, Eq. (8.2) follows from Eq. (4.3) for incompressible, viscous fluids with dynamical viscosity $\eta=\eta(T)$, $T(\mathbf{x})=T_{\mathrm{boundary}}$ everywhere across the fluid and $\mathbf{v}=0$ everywhere on the boundary [in contrast to Malkus' problem of Sec. VIC, where $U_m\neq0$ implies $\langle\langle|\nabla T|\rangle\rangle\neq0$, hence $|\nabla T|\neq0$ somewhere, and Eq. (4.3) does not apply]. In fact, the constraint $T(\mathbf{x})=T_{\mathrm{boundary}}$ leads to $\eta=\eta(T_{\mathrm{boundary}})$ everywhere; this fact allows us to take $\eta(T)$ out from the volume integral and to factor it out, so that we skip the constraint $T=T_{\mathrm{boundary}}$ in Eq. (8.2)—see Sec. 4 of [37]. Moreover, Gauss' divergence theorem and the identity [50] $\Sigma_{ik}[\partial v_i/\partial x_k+\partial v_k/\partial x_i-2\delta_{ik}(\nabla\cdot\mathbf{v})/3](\partial v_i/\partial x_k)=(4/3)(\nabla\cdot\mathbf{v})^2+|\nabla\wedge\mathbf{v}|^2+2\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{v}(\nabla\cdot\mathbf{v})]$ leads to $\int d\mathbf{x}P_V=\int d\mathbf{x}\{[(4\eta/3)+\zeta](\nabla\cdot\mathbf{v})^2+\eta|\nabla\wedge\mathbf{v}|^2\}d\mathbf{x}$, which in turn reduces to $\int d\mathbf{x}|\nabla\wedge\mathbf{v}|^2\eta$ as $\nabla\cdot\mathbf{v}=0$. The analogy between Eqs. (8.1) and (8.2) is evident if we rewrite Eq. (8.1) with $\mathbf{B}$ as Lagrangian coordinate: $\int d\mathbf{x}|\nabla\wedge\mathbf{B}|^2\eta_\Omega=\mathrm{min.}$ with $\nabla\cdot\mathbf{B}=0$. Not surprisingly, a solution of Eq. (8.2) is $\mathbf{v}=\nabla\Phi$, $\Delta\Phi=0$, $\Phi(\mathbf{x})$ scalar (the "velocity potential"). In contrast with Helmholtz-Kortweg's result Eq. (8.2), Maxent method leads to *maximization* of $\int d\mathbf{x}P_V$ for a $\nabla T=0$, $N=1$ viscous fluid—see both Sec. 4.2 and Eq. (15) of Ref. [17], where the quantity $\Sigma_{mn}\langle\phi_{mn(\Gamma)}(\partial v_{mn(\Gamma)}/\partial x_n)\rangle=P_V$ and the term $\propto\nabla\mu_z$ vanishes after integration by parts.

(D) If relaxation occurs in plasmas where magnetic interactions occur, $P_h=P_V+P_J$ but no strong, stabilizing $\mathbf{B}$ is present (in contrast with tokamaks), then turbulent electromagnetic ("e.m.") fluctuations are likely to enhance energy transport and flatten ∇T. Accordingly, Eq. (4.3) applies—to the plasma bulk at least—provided that recombination and ionization are negligible so that Eq. (2.4) holds; this is often true as $T\approx$ KeV. Ohm's law of turbulent plasmas contains many terms [51], including the contribution $\propto\lambda$ of e.m. fluctuations, where $\lambda(\mathbf{x})$ is some non-negative quantity [53]. If we compute η with the simplifying approximation of unmagnetized ions, then P_V is the same as in fluids [51] ($\zeta=0$ in plasmas—see Sec. 58 of [52]). Then, the field $\mathbf{B}$ of the solution of the Euler-Lagrange equations of Eq. (4.3) solves also [4] the Euler-Lagrange equations of Taylor's principle [30] of minimization of magnetic energy with the constraint of fixed magnetic helicity $\int d\mathbf{x}\mathbf{A}\cdot\mathbf{B}$ ($\mathbf{B}=\nabla\wedge\mathbf{A},\mathbf{A}$ vector potential)

$$\int d\mathbf{x}|\mathbf{B}|^2 = \text{min.} \quad \text{with fixed} \int d\mathbf{x}\mathbf{A}\cdot\mathbf{B} \qquad (8.3)$$

provided that $\mathrm{Ha} > \mathrm{Ha}_{\mathrm{cr}}$, where $\mathrm{Ha} \equiv (\eta_{\Omega}\eta)^{-1/2}Bd$, , d,B and $\mathrm{Ha}_{\mathrm{cr}}$ are the Hartmann number, a typical linear size of the system, a typical value for $|\mathbf{B}|$ and a threshold value, respectively; if an electrical current I flows across a pure hydrogen plasma then $\mathrm{Ha}_{\mathrm{cr}}=4.4\cdot 10^3 I(\mathrm{MA})^{1/2}T(\mathrm{KeV})^{-1/2}$. In contrast, if $\mathrm{Ha} < \mathrm{Ha}_{\mathrm{cr}}$ then spontaneous filamentation of the plasma current occurs [54]. Independent computations confirm our conclusion qualitatively at least—see Sec. 3 and Fig. 3 of Ref. [55]. Our discussion shows that Eq. (8.3) fails to describe regions of plasma where $\nabla T \neq 0$—in agreement with experiments [56].

IX. APPLICATIONS OF EQ. (3.3)

We retrieve in Appendix C Rayleigh's stability criterion in thermoacoustics

$$\int d\mathbf{x}\langle P_{h1}p_1\rangle \leq (\gamma-1)^{-1}\rho c_s^{\,2}\int d\mathbf{a}\cdot\langle \mathbf{v}_1 p_1\rangle \qquad (9.1)$$

for a mixture of perfect gases which are contained within slowly moving boundaries and where $P_J=0$, $P_V=0$, P_h depends on short-range reactions and both heat conduction and radiation are negligible—see, e.g., [34], Eq. (1) of [35] and Eq. (2) of [36]. Here c_s is the speed of sound; γ is the specific heat ratio (which we assume to be the same for all gases of the mixture for simplicity); $\langle P_{h1}p_1\rangle$ and $\langle \mathbf{v}_1 p_1\rangle$ are the correlation between perturbations of p and P_h and the correlation between perturbations of $\mathbf{v}$ and p, respectively.

X. CONCLUSIONS

No universal criterion of stability for steady states of systems with dissipation exists but the second principle of thermodynamics [5]. When applied to a small mass element of a mixture of different chemical species with the same temperature at local thermodynamic equilibrium (LTE), the latter principle leads to an often overlooked inequality, the so-called general evolution criterion (GEC) [37,38], which puts a constraint on the time derivatives of thermodynamic quantities like pressure, mass density etc. We assume that LTE holds within an arbitrary small mass element followed along its center-of-mass motion at all times, and that the system as a whole evolves toward a final, steady, stable ("relaxed") state, where we maintain (as a working hypothesis) that the word "steady" makes sense—possibly after time-averaging on time scales$\gg$turbulent time scales. Since LTE holds everywhere at all times during relaxation, the relaxed state is the final outcome of the GEC-constrained evolution of many small mass elements.

Our aim is to gain information concerning the relaxed state from GEC. We neglect both net mass sources, particle diffusion, electric and magnetic polarization. We assume that both conduction- and radiation-induced heat losses increase with increasing temperature. We invoke also no Onsager symmetry (in agreement with [41] and in contrast with [2]), no detailed model of heat production and transport, no "extended thermodynamics" [14] and no "Maxent" method [17]. Suitable integrations on the volume of the system of both GEC and the balance of mass and energy of the small mass element lead to constraints on the evolution of smooth perturbations which relax gently back to the relaxed state. These constraints take the form of inequalities involving the time derivatives of quantities like, e.g., the volume of the system and the amount of entropy produced by heating processes. Each inequality takes the simple form $dA/dt \leq dB/dt + dC/dt$, where A, B, and C are volume integrals. Each inequality has its own triplet A,B,C. It forbids stability against perturbations which leave both B and C unaffected, unless the unperturbed state satisfies the variational principle $A = \text{min.}$ with the constraints of fixed B and C. The latter principle is therefore a necessary condition for stability, which turns out to be useful whenever relaxation satisfies $B = \text{const.}$ and $C = \text{const.}$ In all the inequalities we have found, C is either the volume of the system or its time derivative. Correspondingly, we limit ourselves to perturbations which conserve volume at all times, for simplicity, so that $dC/dt = 0$.

There is nothing special in our inequalities. More of them are likely to be found, e.g., after relaxing one or more of our assumptions. The only fundamental property we refer to is LTE, as it allows us to derive GEC (or a possible generalization of it) from the second principle. Straightforward inspection shows that our results lead—as particular cases—to many criteria of stability for relaxed states in hydrodynamics, plasma physics and thermoacoustics, which have been often suggested in the literature without rigorous proof in order to cope with experiments.

The criterion of stability to be adopted depends on the particular problem. Not surprisingly, for isolated systems we retrieve maximization of total entropy at thermodynamic equilibrium. If the boundary conditions keep the relaxed state far from thermodynamic equilibrium, the actual criterion of stability depends on the detailed nature of the momentum balance of the small mass element, i.e., of the forces acting on it (the ∇p-related force, the Lorenz force of electromagnetism and the forces which are gradients of potentials). Correspondingly, we deal with systems with various heating mechanism—Ohmic heating, viscous heating, heating due to short-range reactions (which conserve the momentum of the small mass element)—as well as with the no-heating case.

Each criterion of stability takes the form of—or is a consequence of—a variational principle: maximization or minimization, depending on the problem. In agreement with [44] and in contrast with [24], we obtain that characterization of systems far from equilibrium, e.g., by maximum entropy production is not a general property but—just like minimum entropy production—is reserved to special systems. We retrieve minimization of the amount of entropy produced per unit time by all irreversible processes in no case, as we invoked no Onsager symmetry. In each case, however, a relaxed state solves the steady-state equations of motion (in order to be a physically allowable steady state) *and* satisfies the relevant stability condition (in order to be stable). If the latter reduces, e.g., to a constrained minimization, the former adds new constraints (see e.g., [4]).

TABLE I. Classification of some necessary criteria of stability against volume-preserving perturbations. From the text, we recall that $\mathbf{A}$ is the vector potential, $\mathbf{B}$ is the magnetic field, c_s is the sound speed, $d\mathbf{a}$ is the surface vector element at the boundary, $d\mathbf{x}$ is the volume element, E is the total energy, $\mathbf{j}$ is the electric current density, P_h is the heating power density, $\mathbf{q}$ is the heat flux due to conduction and radiation, S is the total entropy, s is the entropy per unit mass, T is the temperature, U_m is the mean macroscopic velocity in sheared flow between parallel plates, $\mathbf{v}$ is the velocity, γ is the specific heat ratio, η is the dynamical viscosity, η_Ω is the electrical resistivity, $k=(\gamma-1)^{-1}\rho c_s^{\,2}$, λ is a non-negative quantity proportional to the turbulent correction to Ohm's law in MHD turbulence [53], ν is the kinematic viscosity, ρ is the mass density, and W is the energy released per particle by short-range (chemical, nuclear) reactions. For the generic quantity $a(\mathbf{x})$, $\langle\langle a\rangle\rangle$, and a_1 are its spatial average and its perturbation with respect to the relaxed state, respectively. For non-negative a, $\langle\langle a\rangle\rangle \neq 0$ if and only $a=0$ everywhere: thus, $\langle\langle \eta_\Omega\rangle\rangle=0$ implies no Joule heating anywhere, $\langle\langle \eta\rangle\rangle=0$ implies no viscous heating anywhere, and $\langle\langle \lambda\rangle\rangle=0$ implies no turbulent e.m. fluctuations anywhere.

	Thermod. eq. S=max. E=fixed	Ref. [41] $\int d\mathbf{x}P_h/T$ =min. $\int d\mathbf{x}P_h$ =fixed	Ref. [44] $\int d\mathbf{a}\cdot\rho s\mathbf{v}$ =max.	Refs. [45,20,22] $\int d\mathbf{a}\cdot\mathbf{q}/T$ =max.	Ref. [26] $\langle\langle\lvert\nabla T\rvert\rangle\rangle$ =min. $\int d\mathbf{x}P_h$ =fixed	Refs. [28,29] $\int d\mathbf{x}\lvert\mathbf{j}\rvert^2\eta_\Omega$ =min. $\nabla\cdot\mathbf{j}=0$	Ref. [42] $\nu\Sigma_{ik}\ll\lvert\partial v_i/\partial x_k\rvert^2$ =max. $\nabla\cdot\mathbf{v}=0$ U_m=fixed	Ref. [27] $\int d\mathbf{x}\lvert\nabla\wedge\mathbf{v}\rvert^2\eta$ =min. $\nabla\cdot\mathbf{v}=0$	Ref. [30][b] $\int d\mathbf{x}\lvert\mathbf{B}\rvert^2$ =min. $\int d\mathbf{x}\mathbf{A}\cdot\mathbf{B}$ =fixed	Refs. [35,36][c] $\int d\mathbf{x}\langle P_{h1}p_1\rangle$ $\leq$ $k\int d\mathbf{a}\langle\mathbf{v}_1p_1\rangle$
$\int d\mathbf{x}T^{-1}P_h$	=0	≠0	=0[a]	=0	≠0	≠0	≠0		≠0	≠0
$\int d\mathbf{a}\cdot\rho s\mathbf{v}$	=0		≠0	=0	=0		=0	=0		
$\int d\mathbf{x}\mathbf{q}\cdot\nabla(T^{-1})$	=0		=0[a]	≠0						=0
$\langle\langle\lvert\nabla T\rvert\rangle\rangle$	=0	≠0			≠0[d]	=0	≠0[e]	=0	=0	
$\langle\langle\eta_\Omega\rangle\rangle$		≠0				≠0	=0	=0	≠0	=0
$\langle\langle\eta\rangle\rangle$		=0				=0	≠0	≠0	≠0	=0
$\langle\langle\lambda\rangle\rangle$		=0				=0	=0	=0	≠0	=0
W		=0			=0	=0	=0	=0	=0	≠0

[a]In strong shocks, volume integrals vanish as the volume shrinks to zero for large Mach numbers.
[b]For a turbulent plasma with current I and Hartmann number $\mathrm{Ha}>\mathrm{Ha}_{\mathrm{cr}}$; spontaneous filamentation occurs [54] if $\mathrm{Ha}<\mathrm{Ha}_{\mathrm{cr}}$.; $\mathrm{Ha}_{\mathrm{cr}}=4.4\cdot 10^3 I(\mathrm{MA})^{1/2}T(\mathrm{KeV})^{-1/2}$ in pure hydrogen.
[c]Mixture of perfect gases with the same γ.
[d]Boundaries at different temperatures.
[e]Boundaries at the same T.

A taxonomy of stability criteria is derived (see Table I) which clarifies what is to be minimized, what is to be maximized and with which constraint for each problem. The list is not—and is not meant to be—complete. In fact, we did not investigate the consequences of possible simultaneous validity of more criteria for the same relaxed state, for simplicity. To write down a complete list is a task, which overcomes the limits of the present paper, and will be the matter of future work.

Each column of Table I refers to a different criterion of stability. Each criterion applies at least to the relaxed state of the physical system quoted in the corresponding bibliography and described in the corresponding column, where each of the quantities on the left may either vanish ('=0') of differ from zero ('≠0'). Blank cells mean that the quantity on the corresponding row has no effect on the validity of the criterion.

(i) The first column refers to the benchmark case of thermodynamic equilibrium, where there is no heating and no transport of heat and entropy.

(ii) The second column refers to a variational principle utilized in the reconstruction of current density profiles in tokamak plasmas at MHD equilibrium at JET [41].

(iii) The third column refers to the maximization of the entropy exchanged with the external world through convection per unit time. Rebhan [44] shows that this variational principle is equivalent to one of the conservation equations invoked in the description of strong shock waves.

(iv) The fourth column refers to the maximization of the rate of entropy supplied to the surrounding environment through conductive and/or radiative energy transport; it is postulated by Ozawa and co-workers [21,22]–who generalize Paltridge's results [20]–in order to describe both the large-scale structure of the general circulation of Earth's atmosphere and ocean and Bénard's convection cells at Rayleigh number Ra≫onset threshold between parallel walls at different temperatures. Experimenters in thermoacoustics observe that the same variational principle rules the onset of oscillations in their experimental set-up [45]. Our proof of the 'maximum' principles of the third and the fourth column invoked no Maxent method.

(v) The fifth column refers to the constrained minimization of adverse temperature gradient with the constraint of given heating power, a variational property which Chandrasekhar [26] shows to be enjoyed e.g., by Bénard convection cells at Ra≈onset threshold between parallel walls at different temperatures.

(vi) The sixth column refers to Kirchhoff's minimization [28] of Ohmic heating power—with the constraint of charge conservation—for steady currents flowing across electrical conductors with uniform resistivity and no turbulence. A particular case of this variational principle is Steenbeck's variational principle [47] V=min. in a free-flowing, radiation-cooled arc with voltage fall V. A weak, independently confirmed [48] dependence of V on T follows from Steenbeck's discussion.

(vii) The seventh column refers to Malkus' maximization of total rate of viscous dissipation per unit mass—with the constraint of fixed mean macroscopic velocity—in the turbulent sheared flow between parallel walls at the same temperature of incompressible, viscous fluids which are subject to the ∇p force.

(viii) The eighth column refers to Helmholz-Korteweg's [27] minimization of viscous power for incompressible, viscous fluids which are subject to the ∇p force and (possibly) to gradients of constant potentials, with $\mathbf{v}=0$ everywhere at the boundary and uniform T across the fluid. This result is in contrast with the maximization prescribed by the Maxent approach [17].

(ix) The ninth column refers to Taylor's minimization of magnetic energy with the constraint of fixed magnetic helicity [30] in turbulent, relaxed, $T \geq$ KeV plasmas where fluctuations are not stabilized by external magnetic fields and fluctuation-enhanced transport flattens ∇T, provided that the Hartmann number > a threshold value $\mathrm{Ha}_{\mathrm{cr}}$. For pure hydrogen plasmas where a current I flows, $\mathrm{Ha}_{\mathrm{cr}} = 4.4 \cdot 10^3 I(\mathrm{MA})^{1/2} T(\mathrm{KeV})^{-1/2}$. Below this threshold, spontaneous filamentation occurs [54]. These results agree with independent calculations [55].

(x) The 10th column refers to the version of Rayleigh's criterion [34] which is discussed in [35] [36] for thermoacoustic oscillations in a mixture of perfect gases where heating is due to short-range (chemical, nuclear) reactions; if we replace the sign $\leq$ with $>$ we obtain a sufficient condition for instability.

We have shown that LTE (and its consequence, the general evolution criterion) is the common basis of many different criteria for stability for dissipative fluids and plasmas. Further extension of the method outlined here to other problems is conceivable, and will be the matter of future work.

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APPENDIX A

1. Proof of Eq. (1.5)

Here we rewrite the proof of [37] and of Appendix of [38], for the sake of completeness. We write $\rho^{-1} = \Sigma_z c_z v_z$, $h \equiv u + p\rho^{-1} = \Sigma_z c_z h_z$, $s = \Sigma_z c_z s_z$, $g \equiv h - Ts = \Sigma_z c_z \mu^\circ_z$. The following identities hold:

$$h_z = \mu^\circ_z + T s_z; \tag{A.1}$$

$$(\partial h/\partial p)_{T,N} = \rho^{-1} - T(\partial \rho^{-1}/\partial T)_{p,N}; \tag{A.2}$$

$$(\partial h/\partial T)_{p,N} = (\partial u/\partial T)_{\rho,N} - T(\partial \rho^{-1}/\partial T)^2{}_{p,N}(\partial \rho^{-1}/\partial p)^{-1}{}_{T,N}; \tag{A.3}$$

Equation (1.1) leads to Gibbs-Duhem relationship $\Sigma_z c_z d\mu^\circ_z = \rho^{-1} dp - s dT$, which in turn leads to two relationships

$$s_z = -(\partial \mu^\circ_z/\partial T)_{p,N}; \tag{A.4}$$

$$v_z = (\partial \mu^\circ_z/\partial p)_{T,N}; \tag{A.5}$$

Finally, we choose p, T and the c_j's as independent variables, invoke $da = (da/dt)dt$, and write

$$d(\rho u)/dt = -(dp/dt) + d(\rho h)/dt; \tag{A.6}$$

$$dh/dt = (\partial h/\partial p)_{T,N}(dp/dt) + (\partial h/\partial T)_{p,N}(dT/dt) + \Sigma_j h_j (dc_j/dt); \tag{A.7}$$

$$\begin{aligned} d\rho^{-1}/dt = {} & (\partial \rho^{-1}/\partial p)_{T,N}(dp/dt) + (\partial \rho^{-1}/\partial T)_{p,N}(dT/dt) \\ & + \Sigma_j v_j (dc_j/dt); \end{aligned} \tag{A.8}$$

$$\begin{aligned} d(\mu^\circ_k T^{-1})/dt = {} & T^{-1}(\partial \mu^\circ_k/\partial p)_{T,N}(dp/dt) \\ & + [\partial(\mu^\circ_k T^{-1})/\partial T]_{p,N}(dT/dt) \\ & + T^{-1}\Sigma_j(\partial \mu^\circ_k/\partial c_j)_{p,T}(dc_j/dt); \end{aligned} \tag{A.9}$$

Equations (A.1)–(A.9) lead to

$$\begin{aligned} & (dT^{-1}/dt)[d(\rho u)/dt] - \rho \Sigma_k [d(\mu^\circ_k T^{-1})/dt](dc_k/dt) \\ & \quad + \rho T^{-1}(dp/dt)(d\rho^{-1}/dt) - (dT^{-1}/dt) h (d\rho/dt) \\ & = -\rho T^{-2}(\partial u/\partial T)_{\rho,N}(dT/dt)^2 \\ & \quad + \rho T^{-1}(\partial \rho^{-1}/\partial p)^{-1}{}_{T,N}[(\partial \rho^{-1}/\partial T)_{p,N}(dT/dt) \\ & \quad + (\partial \rho^{-1}/\partial p)_{T,N}(dp/dt)]^2 - \Sigma_k \Sigma_j (\partial \mu^\circ_k/\partial c_j)_{p,T}(dc_k/dt) \\ & \quad \times (dc_j/dt). \end{aligned} \tag{A.10}$$

Inequalities (1.2)–(1.4) and (A.10) lead to Eq. (1.5).

APPENDIX B

1. Proof of Eqs. (3.1)–(3.3) and (6.1)

First, some intermediate steps. We invoke the identity

$$d\left[\int_\Omega d\mathbf{x} a\right]/dt = \int_\Omega d\mathbf{x}[\partial a/\partial t + \nabla \cdot (a\mathbf{u})], \tag{B.1}$$

(see Sec. II 4.116 of [57]) which holds for a domain Ω with moving boundary, where a point on the boundary moves locally at speed $\mathbf{u}$. If Ω is the region of space occupied by our system, then $\mathbf{u}=\mathbf{v}$ on its boundary, and we skip the subscript 'Ω' for simplicity. Equations (2.1) and (B.1) give

$$d\left[\int_\Omega d\mathbf{x} a\right]/dt = \int_\Omega d\mathbf{x}\{\rho d(\rho^{-1}a)/dt + \nabla \cdot [a(\mathbf{u}-\mathbf{v})]. \tag{B.2}$$

If Ω is the region of space occupied by our system, then $\mathbf{u} = \mathbf{v}$ on its boundary, and Eq. (B.2) reduces to

$$d\left[\int d\mathbf{x} a\right]/dt = \int d\mathbf{x} \rho d(\rho^{-1}a)/dt. \tag{B.3}$$

In turn, Eq. (B.3) leads to

$$d^2\left[\int d\mathbf{x}\rho a\right]/dt^2 = \int d\mathbf{x}\rho d^2 a/dt^2. \tag{B.4}$$

We limit ourselves to smooth perturbations, i.e., we neglect terms $\propto$ gradients of time derivatives and da/dt has the same sign everywhere. Then, we may safely assume that a function $g(\mathbf{x},t)$ of $\mathbf{x}$ and t, which is either the product of da/dt and a positive definite quantity or the product of two time derivatives (say da/dt and da'/dt) has the same sign of $\langle\langle g\rangle\rangle$ everywhere in Ω. For such $g(\mathbf{x},t)$ relationship (2.5) leads to

$$a_E\int d\mathbf{x}g(\mathbf{x},t) \le \int d\mathbf{x}a(\mathbf{x},t)g(\mathbf{x},t) \le a_e\int d\mathbf{x}g(\mathbf{x},t). \tag{B.5}$$

Here $a_E \equiv -a_e$; $a_e = A_{\max}$ if $\langle\langle g\rangle\rangle > 0$ and $a_e = A_{\min}$ if $\langle\langle g\rangle\rangle < 0$. In fact, if $\langle\langle g\rangle\rangle > 0$ then $g(\mathbf{x},t) \ge 0$ and $\int d\mathbf{x}a(\mathbf{x},t)g(\mathbf{x},t) \le |\int d\mathbf{x}a(\mathbf{x},t)g(\mathbf{x},t)| \le \int d\mathbf{x}|a(\mathbf{x},t)g(\mathbf{x},t)| = \int d\mathbf{x}|a(\mathbf{x},t)||g(\mathbf{x},t) \le A_M(t)\int d\mathbf{x}g(\mathbf{x},t) \le A_{\max}\int d\mathbf{x}g(\mathbf{x},t)$. Analogous arguments hold for $\langle\langle g\rangle\rangle \le 0$ and for the inequality concerning a_E.

Now, the proof of Eqs. (3.3) and (3.1) begins. Relationships (1.5), (2.3), and (2.4) give

$$\rho d(\rho^{-1}PT^{-1})/dt \le T^{-1}\rho d(\rho^{-1}P)/dt + \rho^{-1}T^{-1}(dp/dt)(d\rho/dt). \tag{B.6}$$

We define $\varphi \equiv (1/T)[1+(P_{ho}/P_h)d(\ln T^{-1})/d(\ln P_h)]$, $\psi \equiv P_h(T^{-1}-\varphi) - T^{-1}dp/dt$, $c_1 \equiv \psi_e$, $c_2 \equiv \varphi_e$, $c_4 \equiv T^{-1}{}_e$, and $c_3 \equiv p_e c_4$. Relationships (B.3), (B.5), and (B.6) give

$$d\left(\int d\mathbf{x}T^{-1}P\right)/dt \le c_4\int d\mathbf{x}[\rho d(\rho^{-1}P)/dt + \rho^{-1}(dp/dt)(d\rho/dt)]. \tag{B.7}$$

Relationships (2.3), (B.4), (B.5), and (B.7) lead to

$$\begin{aligned} d\left(\int d\mathbf{x}P/T\right)/dt &\le c_4\int d\mathbf{x}\rho p d^2(\rho^{-1})/dt^2 + c_4 d^2E/dt^2 \\ &\le c_3\int d\mathbf{x}\rho d^2(\rho^{-1})/dt^2 + c_4 d^2E/dt^2 \\ &= c_3 d^2V/dt^2 + c_4 d^2E/dt^2. \end{aligned} \tag{B.8}$$

Relationships (2.4), (B.3), and (B.4) lead to

$$d\left(\int d\mathbf{x}P/T\right)/dt = d^2S/dt^2. \tag{B.9}$$

Relationships (B.8) and (B.9) lead to Eq. (3.3).

Relationships (2.1), (2.2), and (B.6), $P_h = P_{h0} + P_{h1}$, and $P_0 = 0$ lead to

$$\begin{aligned} 0 &\ge Pd(T^{-1})/dt - \rho^{-1}T^{-1}(dp/dt)(d\rho/dt) \\ &= (P_h - \nabla\cdot\mathbf{q})_1 d(T^{-1})/dt - \rho^{-1}T^{-1}(dp/dt)(d\rho/dt) \\ &= \rho d(\rho^{-1}P_hT^{-1})/dt - \psi(\nabla\cdot\mathbf{v}) - (\nabla\cdot\mathbf{q})_1 d(T^{-1})/dt \\ &\quad - \varphi\rho d(\rho^{-1}P_h)/dt. \end{aligned} \tag{B.10}$$

After volume integration of both sides of Eq. (B.10), relationships (2.1), (2.6), (B.3), and (B.5) lead to Eq. (3.1).

Let us proof Eq. (3.2). Relationships (2.2), (2.4), and (B.2) give the total time derivative of $S_\Omega \equiv \int_\Omega d\mathbf{x}\rho s$ in a generic region Ω

$$dS_\Omega/dt = \int_\Omega d\mathbf{x}\{T^{-1}P_h + \mathbf{q}\cdot\nabla T^{-1} + \nabla\cdot[\rho s(\mathbf{u}-\mathbf{v}) - T^{-1}\mathbf{q}]\}. \tag{B.11}$$

We apply Eq. (B.11) to two regions: our system (with $\mathbf{u}=\mathbf{v}$ on its boundary), and a region Ξ with fixed boundary ($\mathbf{u}=0$) where $P_h=0$ and $\nabla T=0$. Here and in the following, Ξ is just a dummy region, which is useful for explicit discussion of the role of boundary; Ξ will disappear in the final result. Accordingly, Eq. (B.11) gives

$$dS/dt = \int d\mathbf{x}[T^{-1}P_h + \mathbf{q}\cdot\nabla T^{-1} - \nabla\cdot(T^{-1}\mathbf{q})], \tag{B.12}$$

$$dS_\Xi/dt = -\int_\Xi d\mathbf{x}\nabla\cdot(\rho s\mathbf{v} + T^{-1}\mathbf{q}). \tag{B.13}$$

During the relaxation process, we assume that our system and Ξ interact with each other only, and across a common contact surface. (As a thought-experiment, think, e.g., of a small perturbation of a droplet of lukewarm coffee which is in contact with a cup Ξ, with negligible coffee evaporation during relaxation of the perturbation). Since Ξ is a dummy region, this assumption implies no loss of generality. Accordingly, $d\mathbf{a} + d\mathbf{a}_\Xi = 0$ along the common contact surface, where $d\mathbf{a}$ and $d\mathbf{a}_\Xi$ are the surface elements of the boundaries of our system and Ξ, respectively. Gauss' theorem and Eq. (B.13) lead to

$$\begin{aligned} dS_\Xi/dt &= -\int_\Xi d\mathbf{x}\nabla\cdot(\rho s\mathbf{v} + T^{-1}\mathbf{q}) = -\int_\Xi d\mathbf{a}_\Xi\cdot(\rho s\mathbf{v} + T^{-1}\mathbf{q}) \\ &= \int d\mathbf{a}\cdot(\rho s\mathbf{v} + T^{-1}\mathbf{q}) = \int d\mathbf{x}\nabla\cdot(\rho s\mathbf{v} + T^{-1}\mathbf{q}). \end{aligned} \tag{B.14}$$

Term-by-term sum of Eqs. (B.12) and (B.14) gives the total time derivative of the total entropy $S_{\text{total}} \equiv S + S_\Xi$

$$dS_{\text{total}}/dt = \int d\mathbf{x}[T^{-1}P_h - T^{-1}(\nabla\cdot\mathbf{q}) + \nabla\cdot(\rho s\mathbf{v} + T^{-1}\mathbf{q})]. \tag{B.15}$$

As anticipated, Ξ has disappeared on the RHS of Eq. (B.15). Let us take the total time derivative of both sides of Eq. (B.15))

$$d^2S_{\text{total}}/dt^2 = d\left(\int d\mathbf{x}T^{-1}P_h\right)/dt + d\left(\int d\mathbf{x}[\nabla\cdot(\rho s\mathbf{v}) + \mathbf{q}\cdot\nabla(T^{-1})]\right)/dt. \tag{B.16}$$

Relationships (3.1) and (B.16) give

$$d^2S_{\text{total}}/dt^2 \le c_1 dV/dt + c_2 d\left(\int d\mathbf{x}P_h\right)/dt + d\left(\int d\mathbf{x}[\nabla\cdot(\rho s\mathbf{v}) + \mathbf{q}\cdot\nabla(T^{-1})]\right)/dt. \tag{B.17}$$

We neglect the LHS of Eq. (B.17) in comparison with the RHS as our perturbation relaxes gently back to the unperturbed state; Eq. (3.2) follows.

Finally, (B.1) and (B.12) give Eq. (6.1) in steady state ($\partial/\partial t=0$).

APPENDIX C

1. Proof of Eq. (9.1)

The arguments of Sec. 4 and Eq. (B.8) show that perturbations which conserve E and V make the system to relax to states which minimize $\int d\mathbf{x}P/T$. Then, in a neighborhood of a relaxed state we may write

$$0 \le \left(\int d\mathbf{x}T^{-1}P\right)_1. \tag{C.1}$$

We invoke both $\mathbf{u}=\mathbf{v}$ on the boundary and the smallness of $a_1 \approx da=(da/dt)dt$. Together with Eqs. (2.3) and (B.1), time-averaging of both sides of Eq. (C.1) gives

$$\begin{aligned}0 &\le \left\langle\left(\int d\mathbf{x}T^{-1}P\right)_1\right\rangle = \left\langle dtd\left(\int d\mathbf{x}T^{-1}P\right)/dt\right\rangle \\ &= \left\langle\left(\int d\mathbf{x}dt\,\partial(T^{-1}P)/\partial t\right\rangle + \left\langle\int d\mathbf{x}\nabla\cdot(T^{-1}P\mathbf{v})dt\right\rangle \\ &= \left\langle\left(\int d\mathbf{x}dt\,\partial(T^{-1}P)/\partial t\right\rangle + \left\langle\int d\mathbf{a}\cdot T^{-1}P\mathbf{v}dt\right\rangle \\ &= \left\langle\left(\int d\mathbf{x}dt\,\partial(T^{-1}P)/\partial t\right\rangle + \left\langle\int d\mathbf{a}\cdot\mathbf{v}T^{-1}\rho(du + pd\rho^{-1})\right\rangle.\end{aligned} \tag{C.2}$$

If the time scale of the boundary motion is $\gg$ the time scale of perturbations, then $\langle(\int d\mathbf{x}c\rangle \approx \int d\mathbf{x}\langle a\rangle$ and $\langle\int d\mathbf{a}\cdot\mathbf{v}a\rangle \approx \int d\mathbf{a}\cdot\langle\mathbf{v}a\rangle$. Since we are dealing with perturbations near a minimum, we are interested in second-order quantities only. We neglect long-range correlations between P and T (this choice will be justified below) and write $\int d\mathbf{x}\langle dt\,\partial(T^{-1}P)/\partial t\rangle = \int d\mathbf{x}\langle(T^{-1})_1P_1\rangle$. Thus, Eq. (C.2) gives

$$-\int d\mathbf{x}\langle(T^{-1})_1P_1\rangle \le \int d\mathbf{a}\cdot\langle\mathbf{v}_1[T^{-1}\rho u_1 + T^{-1}\rho p(\rho^{-1})_1]\rangle. \tag{C.3}$$

In order to select the perturbation of interest, we take advantage of the smallness of P_1. According to $P_0=0$ and to Eq. (2.4) $P_1 \approx 0$ is satisfied whenever $ds\approx 0$. Since the perturbation is almost adiabatic and we are dealing with a mixture of perfect gases, we make a small error if we assume that relationships $p=d_1\rho T$, $p=d_2\rho^{\gamma}$, $\rho u=(1+\varepsilon)p/(\gamma-1)$ hold—approximately at least—during the relaxation; here d_1, d_2, γ, and ε are constant quantities, γ plays the role of a constant specific heat ratio and the quantity $\varepsilon \ll 1$ expresses the deviation from the adiabatic behavior (to the author's knowledge, the impact of this deviation on the evolution of our soundlike perturbation has been stressed for the first time in [34]). It follows that $(T^{-1})_1=-Gp_1$ and $T^{-1}\rho u_1 + T^{-1}\rho p(\rho^{-1})_1 = Fp_1$, where $G\equiv d_1 d_2^{-1/\gamma}(1-1/\gamma)p_0^{-2+1/\gamma}$, $F\equiv\varepsilon G(\gamma-1)^{-1}p_0$. Both—$G$ and F are upper bounded in a neighborhood of a stable state. It follows from Eq. (B.5) (with $a=F$ and $a=G$) that

$$G_E\int d\mathbf{x}\langle P_1p_1\rangle < -\int d\mathbf{x}\langle(T^{-1})_1P_1\rangle, \tag{C.4}$$

$$\int d\mathbf{a}\cdot\langle\mathbf{v}_1[T^{-1}\rho u_1 + T^{-1}\rho p(\rho^{-1})_1]\rangle < F_e\int d\mathbf{a}\cdot\langle\mathbf{v}_1p_1\rangle, \tag{C.5}$$

where we invoked $a_1\approx da=(da/dt)dt$, $\langle(\int d\mathbf{x}c\rangle \approx \int d\mathbf{x}\langle a\rangle$, $\langle\int d\mathbf{a}\cdot\mathbf{v}a\rangle \approx \int d\mathbf{a}\cdot\langle\mathbf{v}a\rangle$, and Eq. (B.5) with $g=(dP/dt)(dp/dt)$ and $g=[d(\mathbf{v}\cdot d\mathbf{a})/dt](dp/dt)$ in Eqs. (C.4) and (C.5), respectively. If $\nabla\cdot\mathbf{q}$ is negligible, then Eqs. (2.2) and (C.3)–(C.5) lead to the following necessary condition for stability:

$$\int d\mathbf{x}\langle P_{h1}p_1\rangle \le k\int d\mathbf{a}\cdot\langle\mathbf{v}_1p_1\rangle, \tag{C.6}$$

where $k\equiv F_e/G_E$. Since P_h depends on short-range reactions and $\nabla\cdot\mathbf{q}=0$ implies $P=P_h$ in Eq. (2.2), long-range correlations between P and T are indeed negligible. Physically, compression induces heating, hence $G>0$ everywhere, and $G_E>0$. We can always take $F_e>0$; then $k>0$. If the signs = and $\ge$ replace the sign $\le$ in Eq. (9.1), marginal stability and a sufficient condition for instability follow. Marginal stability corresponds to a balance between the power supplied by short-range reactions to the fluid and the power lost through convection across the boundary. Then [35] $k=(\gamma-1)^{-1}\rho c_s^2$, and Eq. (C.6) reduces to Eq. (9.1).

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