Critical behavior of the pure and random-bond two-dimensional triangular Ising ferromagnet

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We investigate the effects of quenched bond randomness on the critical properties of the two-dimensional ferromagnetic Ising model embedded in a triangular lattice. The system is studied in both the pure and disordered versions by the same efficient two-stage Wang-Landau method. In the first part of our study, we present the finite-size scaling behavior of the pure model, for which we calculate the critical amplitude of the specific heat's logarithmic expansion. For the disordered system, the numerical data and the relevant detailed finite-size scaling analysis along the lines of the two well-known scenarios—logarithmic corrections versus weak universality—strongly support the field-theoretically predicted scenario of logarithmic corrections. A particular interest is paid to the sample-to-sample fluctuations of the random model and their scaling behavior that are used as a successful alternative approach to criticality.

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I. INTRODUCTION

Understanding the role played by impurities on the nature of phase transitions is of great importance, both from experimental and theoretical perspectives. First-order phase transitions are known to be dramatically softened under the presence of quenched randomness [1–10], while continuous transitions may have their exponents altered under random fields or random bonds [11,12]. There are some very useful phenomenological arguments and some, perturbative in nature, theoretical results, pertaining to the occurrence and nature of phase transitions under the presence of quenched randomness [2,5,13,14]. Historically, the most celebrated criterion is that suggested by Harris [11]. This criterion relates directly the persistence, under random bonds, of the non random behavior to the specific-heat exponent α_p of the pure system. According to this criterion, if $\alpha_p > 0$, then disorder will be relevant, i.e., under the effect of the disorder, the system will reach a critical behavior. Otherwise, if $\alpha_p < 0$, disorder is irrelevant and the critical behavior will not change.

Pure systems with a zero specific-heat exponent ($\alpha_p = 0$) are marginal cases of the Harris criterion and their study, upon the introduction of disorder, has been of particular interest [15]. The paradigmatic model of the marginal case is, of course, the general random two-dimensional (2D) Ising model (random site, random bond, and bond diluted) and this model has been extensively investigated and debated (see Ref. [16] and references therein). Several recent studies, both analytical (renormalization group and conformal field theories) and numerical [mainly Monte Carlo (MC) simulations] devoted to this model, have provided very strong evidence in favor of the so-called logarithmic corrections' scenario [17–21]. According to this, the effect of infinitesimal disorder gives rise to a marginal irrelevance of randomness and besides logarithmic corrections, the critical exponents maintain their 2D Ising values. In particular, the specific heat is expected to slowly diverge with a double-logarithmic dependence of the form $C \propto \ln(\ln t)$, where $t = |T - T_c|/T_c$ is the reduced critical temperature [17–21]. Here, we should mention that there is not full agreement in the literature and a different scenario, the so-called weak universality scenario [22–25], predicts that critical quantities, such as the magnetization, zero-field susceptibility, and correlation length display power-law singularities, with the corresponding exponents β, γ, and ν changing continuously with the disorder strength; however this variation is such that the ratios β/ν and γ/ν remain constant at the pure system's value. The specific heat of the disordered system is, in this case, expected to saturate, with a corresponding correlation length's exponent $\nu \geq 2/d$ [12].

In general, a unitary and rigorous physical description of critical phenomena in disordered systems still lacks and certainly, lacking such a description, the study of further models for which there is a general agreement in the behavior of the corresponding pure cases is very important. In this sense, the triangular Ising ferromagnet is a further suitable candidate for testing the above predictions that has not been previously investigated in the literature. Thus, our investigation will be related to the extensive relevant literature concerning the critical properties of the disordered 2D square Ising model [15–23,26–59]. In particular, our discussion will focus on the main point of dispute over the last two decades, concerning the two well-known conflicting scenarios mentioned above and we will provide additional new evidence in favor of the well-established scenario of strong universality. We should note here that, the theoretically predicted scenario of strong universality has been confirmed by several MC studies on the square lattice starting from the early 90s to nowadays (see Ref. [16] for a detailed historical review).

As mentioned above, it is always important to consider further models for which the critical properties of the corresponding pure versions are exactly known. This is also the case for the present model under consideration, namely, the triangular Ising model, called hereafter as the triangular Ising model (TrIM), defined as usual by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} s_i s_j, \qquad (1)$$

where the spin variables s_i take on the values $-1, +1$, $\langle ij \rangle$ indicates summation over all nearest-neighbor pairs of sites, and $J > 0$ is the ferromagnetic exchange interaction. The

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TrIM belongs to the same universality class with the corresponding square Ising model (SqIM), sharing the same values of critical exponents and a logarithmic behavior of the specific heat [60,61]. Additionally, the critical temperature of the model and also the critical amplitude A_0 of Ferdinand and Fisher's [62] specific heat's logarithmic expansion [see also the discussion in Sec. III and Eq. (7)] are exactly known from the early work of Houtappel [63] to be $T_c=4/\ln 3=3.6409\cdots$ and $A_0=0.499069\cdots$, respectively. Nevertheless, it appears that for the TrIM a verification of the finite-size scaling (FSS) properties of the model using high quality data from MC simulation is still lacking. Thus, the first part of this work is devoted to the investigation of the FSS behavior of the model, especially the estimation of the amplitudes and other relevant coefficients in the specific heat's logarithmic expansion and also to the estimation of the critical exponents. In this sense, the aim of this first part is twofold: first, to provide the first detailed FSS analysis of the pure model and, second, to present a concrete reliability test of the proposed numerical scheme.

Our main focus, on the other hand, is the case with bond disorder given by the bimodal distribution

$$P(J_{ij}) = \frac{1}{2}[\delta(J_{ij}-J_1) + \delta(J_{ij}-J_2)];$$

$$\frac{J_1+J_2}{2} = 1; \quad J_1 > J_2 > 0; \quad r = \frac{J_2}{J_1}, \tag{2}$$

so that r reflects the strength of the bond randomness and we fix $2k_B/(J_1+J_2)=1$ to set the temperature scale. The value of the disorder strength considered throughout this paper is $r=1/3$. The resulting quenched disordered (random-bond) version of the Hamiltonian defined in Eq. (1) reads now as

$$H = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \tag{3}$$

and will be referred in the sequel as the random-bond triangular Ising model (RBTrIM). The corresponding random-bond SqIM will be denoted hereafter respectively as RBSqIM. The model on the square lattice has the advantage that the critical temperature is exactly known as a function of the disorder strength r by duality relations [64]. For the RBTrIM there exist only several approximations for the critical frontier of the site- and bond-diluted cases, obtained via renormalization-group techniques [65] and a study of the critical behavior of the model is lacking.

The rest of the paper is laid out as follows: in Sec. II we present the necessary simulation details of our numerical scheme. In Sec. III we discuss the FSS behavior of the pure model, testing with our high accuracy numerical data the exact expansion of the critical specific heat. Then, in Sec. IV we present a detailed FSS analysis for the random version of the model, including—apart from the classical FSS techniques—concepts from the scaling theory of disordered systems. Our results and the relevant discussion clearly favor the scenario of strong universality in marginal disordered systems. Finally, Sec. V summarizes our conclusions.

II. SIMULATION DETAILS

Resorting to large scale MC simulations is often necessary [66], especially for the study of the critical behavior of disordered systems. It is also well known [67] that for such complex systems traditional methods become very inefficient and that in the last few years several sophisticated algorithms, some of them are based on entropic iterative schemes, have been proven to be very effective. The present numerical study of the RBTrIM will be carried out by applying our recent and efficient entropic scheme [59,68,69]. In this approach we follow a two-stage strategy of a restricted entropic sampling, which is described in our study of random-bond Ising models (RBIM) in 2D [59] and is very similar to the one applied also in our numerical approach to the 3d random-field Ising model (RFIM) [69]. In these papers, we have presented in detail the various sophisticated routes used for the restriction of the energy subspace and the implementation of the Wang-Landau (WL) algorithm [70]. Further details and an up to date implementation of this approach, especially for the study of disordered systems, is provided in our recent paper on the universality aspects of the pure and random-bond 2D Blume-Capel model [71].

We do not wish to reproduce here the details of our two-stage implementation and the practice followed in our scheme for improving accuracy by repeating the simulations. However, we should like to include a brief discussion on the approximate nature of the WL method. The usual WL recursion proceeds by modifying the density of states $G(E)$ according to the rule $G(E)\rightarrow fG(E)$ and initially one chooses $G(E)=1$ and $f=f_0=e$. Once the accumulative energy histogram is sufficiently flat, the modification factor f is redefined as: $f_{j+1}=\sqrt{f_j}$, with $j=0,1,\ldots$ and the energy histogram reset to zero until f is very close to unity (i.e., $f=e^{10^{-8}}\approx 1.000\ 000\ 01$). As has been reported by many authors in the study of several models, once f is close enough to unity, systematic deviations become negligible.

However, the WL recursion violates the detailed balance from the early stages of the process and care is necessary in setting up a proper protocol of the recursion. In spite of the fact that the WL method has produced very accurate results in several models, it is fair to say that there is not a safe way to access possible systematic deviations in the general case. This has been pointed out and critiqued in a recent review by Janke [72]. For the 2D Ising model (where exact results are available for judgement) the WL method has been shown to converge very rapidly. Furthermore, from our experience and especially from our recent study of the 2D RBSqIM [59], for which the exact phase diagram is known by duality relations, our high-level WL implementation has produced excellent results, enabling us to discriminate between competing theoretical predictions on that model. Since the RBTrIM is expected to have a similar "entropy structure" to the corresponding square model, we anticipate and it will be verified in the sequel, that, our WL scheme produces sufficiently accurate estimates enabling us, also in this case, to distinguish between competing theoretical predictions, as we have already done in the corresponding model on the square lattice.

Using this scheme we performed extensive simulations for several lattice sizes in the range $L=20-200$, over large

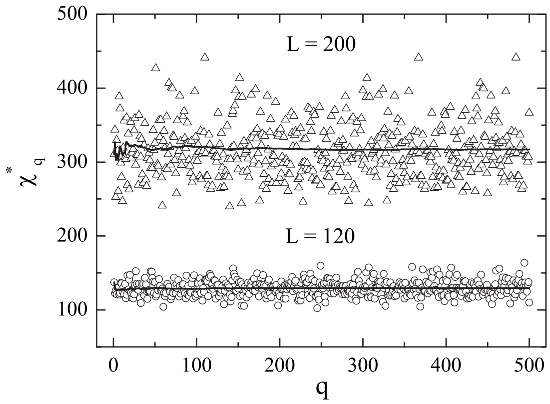

FIG. 1. Disorder distribution of the susceptibility maxima of two lattices with linear sizes $L=120$ and 200 of the RBTrIM. The running averages over the samples are shown by the thick solid lines.

ensembles $\{1,\cdots,q,\cdots,Q\}$ of random realizations ($Q=500$). Let us note here that for the pure model we simulated for each lattice size, 200 independent runs (WL random walks). It is well known that, extensive disorder averaging is necessary when studying random systems, where usually broad distributions are expected leading to a strong violation of self-averaging [73,74]. A measure from the scaling theory of disordered systems, whose limiting behavior is directly related to the issue of self-averaging [73,74] may be defined with the help of the relative variance of the sample-to-sample fluctuations of any relevant singular extensive thermodynamic property Z as follows: $R_Z=([Z^2]_{av}-[Z]_{av}^2)/[Z]_{av}^2$. Figure 1 presents evidence that the above number of random realizations is sufficient in order to obtain the true average behavior and not a typical one. In particular, we plot in this figure (for lattice sizes $L=120$ and 200) the disorder distribution of the susceptibility maxima χ_q^* and the corresponding running average, i.e., a series of averages of different subsets of the full data set—each of which is the average of the corresponding subset of a larger set of data points, over the samples for the simulated ensemble of $Q=500$ disorder realizations. A first striking observation from this figure is the existence of very large variance of the values of χ_q^*, indicating the expected violation of self-averaging for this quantity. This figure illustrates that the simulated number of random realizations is sufficient in order to probe correctly the average behavior of the system, since already for $Q\approx 300$ the average value of χ_q^* appears quite stable.

Closely related to the above issue of self-averaging in disordered systems is the manner of averaging over the disorder. This nontrivial process may be performed in two distinct ways when identifying the finite-size anomalies, such as the peaks of the magnetic susceptibility. The first way corresponds to the average over disorder realizations ($[\ldots]_{av}$) and then taking the maxima ($[\ldots]_{av}^*$), or taking the maxima in each individual realization first, and then taking the average ($[\ldots^*]_{av}$). In the present paper, we have undertaken our FSS analysis using both ways of averaging and have found comparable results for the values of the critical exponents, as will be discussed in more detail below. Closing this brief outline, let us comment on the statistical errors of our numerical data.

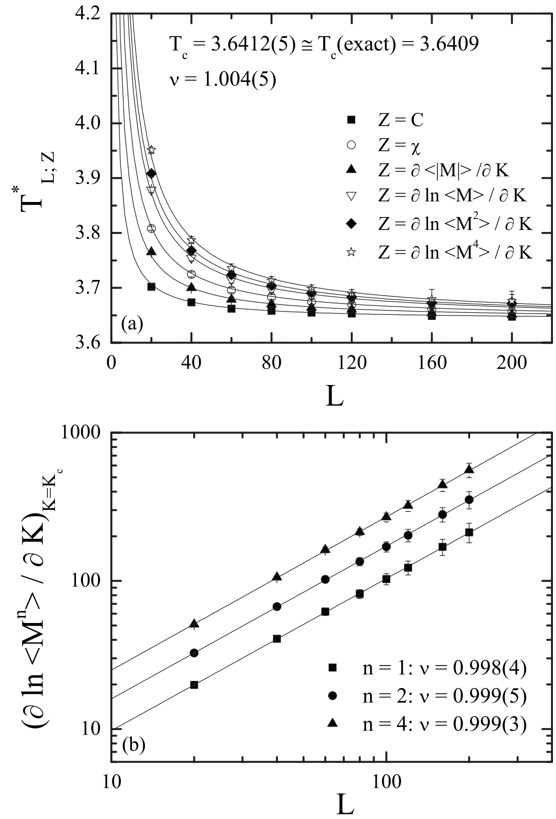

FIG. 2. (a) Simultaneous fitting of the form (6) of six pseudocritical temperatures defined in the text. (b) FSS of several powers of the logarithmic derivatives [Eq. (5)] of the order parameter at the critical temperature in a log-log scale. The solid lines are linear fittings giving the value $\nu=1$ for the correlation length's exponent.

The statistical errors of our WL scheme on the observed average behavior, were found to be of relatively small magnitude (of the order of the symbol sizes) when compared to the relevant disorder-sampling errors (due to the finite number of simulated realizations). Thus, the error bars in most of our figures below concerning the average $[\ldots]_{av}^*$ and used also in the corresponding fitting attempts, reflect the disorder-sampling errors and have been estimated using groups of 50 realizations via the jackknife method [67]. On the other hand for the case $[\ldots^*]_{av}$ the error bars shown reflect the sample-to-sample fluctuations.

III. PURE MODEL

In this section, we proceed to investigate the critical behavior of the pure TrIM defined in Eq. (1). Our aim is to observe the exact critical behavior of the model and also to estimate the whole set of critical exponents, paying particular attention to the FSS behavior of the critical specific heat. As mentioned above, the numerical data shown below in Figs. 2–4 have been estimated as averages over 200 independent runs together with the corresponding error bars.

Figure 2(a) gives the shift behavior of the pseudocritical temperatures corresponding to the peaks of the following six quantities: specific heat C, magnetic susceptibility χ, deriva-

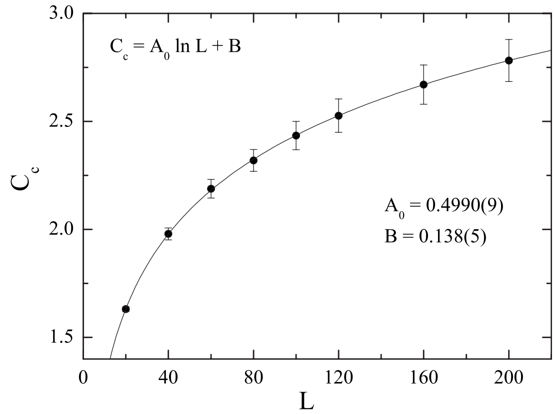

FIG. 3. FSS of the specific-heat data at the exact critical temperature.

tive of the absolute order parameter with respect to inverse temperature $K=1/T$ [75]

$$\frac{\partial \langle |M| \rangle}{\partial K} = \langle |M| H \rangle - \langle |M| \rangle \langle H \rangle, \tag{4}$$

and logarithmic derivatives of the first ($n=1$), second ($n=2$), and fourth ($n=4$) powers of the order parameter with respect to inverse temperature [75]

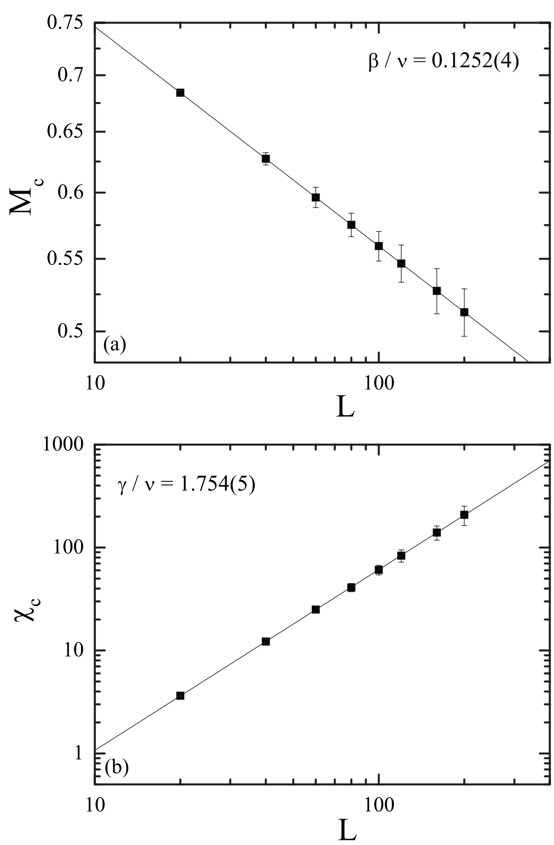

FIG. 4. Magnetic exponent ratios of the pure TrIM: FSS in a log-log scale of (a) the order parameter and (b) the magnetic susceptibility at the critical temperature. In both cases linear fittings are applied.

$$\frac{\partial \ln \langle M^n \rangle}{\partial K} = \frac{\langle M^n H \rangle}{\langle M^n \rangle} - \langle H \rangle. \tag{5}$$

Fitting our data for the whole lattice range to the expected power-law behavior

$$T^*_{L;Z} = T_c + bL^{-1/\nu}, \tag{6}$$

where Z stands for the different thermodynamic quantities mentioned above, we find the critical temperature to be $T_c = 3.6412(5)$ which is in excellent agreement with the exact value $3.6409\cdots$. Additionally, our estimate of the critical exponent ν of the correlation length is $\nu=1.004(5)$, also in excellent agreement with the value $\nu=1$.

Additional estimates for the critical exponent ν may be obtained from the scaling behavior of the logarithmic derivatives (5), which scale as $\sim L^{1/\nu}$ with the lattice size [75]. The FSS of these logarithmic derivatives at the critical temperature is shown in Fig. 2(b) and in all three cases a value $\nu \approx 1$ is obtained consistent with the estimate from panel (a) of Fig. 2 and with the exact value $\nu=1$.

We know turn to the most interesting issue in the study of the pure model, which is the specific heat's logarithmic expansion, as mentioned in the introduction. For the square lattice, it was shown in 1969 in the pioneering work of Ferdinand and Fisher [62] that close to the critical point the specific heat obeys the following FSS expansion

$$C_L(T) = A_0 \ln L + B(T) + B_1(T)\frac{\ln L}{L} + B_2(T)\frac{1}{L} + \cdots, \tag{7}$$

where the value of the critical amplitude A_0 is $0.494358\cdots$ As pointed out in Ref. [62], this is the same with the amplitude A_0 in the temperature expansion of the specific heat close to the critical point and this was already known from the original paper of Onsager [76]. The first B coefficients are given in Ref. [62] and further details have been presented in Refs. [68,77–79]. In particular, at the critical temperature the constant term B is $0.138149\cdots$ and as it is also well-known [62,78] the coefficient B_1 is zero.

The universality of the above expansion has been already pointed out and discussed by Fisher [60]. For the three most common 2D lattices, i.e., the square, plane triangular, and honeycomb, a unified approach has been presented by Wu *et al.* [79], from which one can find also for the plane triangular lattice the first B coefficients and the amplitude A_0 at the critical temperature.

As shown by Wu *et al.* [79] for all the three lattices, the coefficient B_1 is zero at the critical temperature and as in the square lattice, also for the triangular lattice, the critical amplitude of the expansion is $A_0=0.499069\cdots$, equal to the amplitude of the temperature expansion close to the critical point obtained by Houtappel [63]. From the work of Wu *et al.* [79] we find, using their closed form expressions, for the plane triangular lattice (with periodic boundary conditions and aspect ratio $R=1$) the following values $B=B(T_c) = 0.14185\cdots$ and $B_2=B_2(T_c)=-0.15003\cdots$

It is of interest at this point to examine the compatibility of our numerical data with the above expansion for the case of the critical specific-heat data, i.e., the data of the specific heat at the exact critical temperature of the TrIM. In Fig. 3 we consider only the first two terms in the above expansion (i.e., we set also $B_2=0$) and pay attention in estimating the critical amplitude A_0 and the constant B contribution. From the results of the fitting in the total lattice range $L=20-200$, as also shown in the figure, we observe that the estimated value for the critical amplitude $A_0=0.4990(9)$ is very close to the exact value $A_0=0.499069\cdots$ [63]. For the first B coefficient we find the value $B=0.138(5)$ which is in good agreement with the exact result $0.14185\cdots$ Let us note here that if we try to fit the data of Fig. 3 including also the coefficient B_2 of the expansion [7] we get the estimates 0.4952(59), 0.153(16), and −0.20(12) for the critical amplitude A_0 and the coefficients B and B_2, respectively. However, if we fix the values of A_0 and B to their exact ones, the estimate for B_2 we get from the fitting is −0.158(11), in excellent agreement with the exact value, whereas if we fix the value of B and B_2 to their exact ones we get the estimate 0.4990(3) for the critical amplitude A_0.

Finally, Figs. 4(a) and 4(b) present our estimations for the magnetic exponent ratios β/ν and γ/ν. For the estimation of β/ν we use the values of the order parameter at the exact critical temperature. As shown in panel (a), in a log-log scale, the linear fitting provides the estimate $\beta/\nu=0.1252(4)$. In panel (b) we show the FSS of the critical susceptibility, also in a log-log scale. The straight line is a linear fitting for $L\geq 20$ giving the estimate $\gamma/\nu=1.754(5)$. Thus, our results presented in this Section for the pure 2D TrIM model are in excellent agreement with the exact results and also with the expected logarithmic expansion of the specific heat [62,79]. This consists a very strong accuracy test of the proposed two-stage WL entropic sampling in restricted energy spaces.

IV. RANDOM MODEL

We now present our numerical results for the random-bond version of the triangular Ising model for disorder strength $r=1/3$. From simple universality-type theoretical arguments, this system is also expected to undergo a second-order transition between the ferromagnetic and paramagnetic phases and in particular it should be also expected that this transition will be in the same universality class as the RBSqIM.

We start our analysis of the RBTrIM by presenting the general shift behavior of various pseudocritical temperatures of the model. Figure 5(a) illustrates the shift behavior of seven pseudocritical temperatures defined as the temperature where the corresponding average thermodynamic property attains its maximum. The first six are as those defined in Sec. II for the corresponding pure model. The last pseudocritical temperature is introduced temperature, defined as the temperature where the ratio $R_{[\chi^*]_{av}}$ defined in Sec. II, becomes maximum. The solid lines show an excellent simultaneous power-law fitting attempt of the form (6) giving the value $T_c=3.4642(52)$ for the critical temperature of the random model and a value $\nu=0.997(6)$ for the critical exponent ν of the correlation length. The fitting shown in Fig. 5(a) has been performed for all lattice sizes and it was very stable when

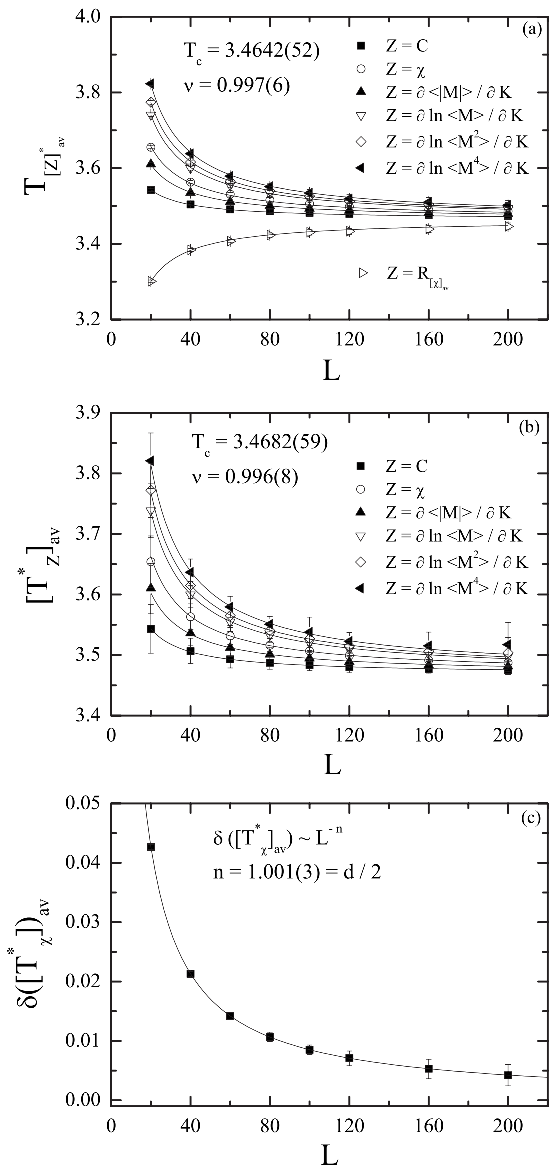

FIG. 5. (a)–(b) Shift behavior of several pseudocritical temperatures defined in the text. The error bars in panel (b) reflect the sample-to-sample fluctuations. (c) FSS of the sample-to-sample fluctuations of the pseudocritical temperature of the magnetic susceptibility shown in panel (b).

shifting the L range to larger values. Note also that a simple fitting attempt using only the defined pseudocritical temperature defined with the help of the sample-to-sample fluctuations gives a value $T_c=3.4677(47)$ for the critical temperature and a value $\nu=0.994(8)$ for the correlation length's exponent. These overall estimates for the exponent ν consist a strong indication that the RBTrIM shares the same value of ν as the pure version, thus reinforcing the scenario of logarithmic corrections (strong universality).

Figure 5(b) illustrates again the shift behavior of the six pseudocritical temperatures of panel (a), estimated now via the second way of averaging discussed in Sec. I, i.e., by taking the average over the individual pseudocritical temperatures. The error bars shown in this panel reflect the sample-to-sample fluctuations of the pseudocritical temperatures. Again, the results obtained from the simultaneous fitting attempt T_c=3.4682(59) and ν=0.996(8), as also shown in the figure, agree excellently with the estimates of panel (a), providing further evidence in favor of the accuracy of our numerical scheme and the strong universality hypothesis.

Noteworthy that, if we fix the exponent ν to the exact value ν=1 we get the most accurate estimates for the critical temperature to be 3.4663(16) and 3.4669(19) from the corresponding fittings of panels (a) and (b) of Fig. 5.

Using now the above sample-to-sample fluctuations of the pseudocritical temperatures and the theory of FSS in disordered systems as introduced by Aharony and Harris [73] and Wiseman and Domany [74], one may further examine the nature of the fixed point that controls the critical behavior of the disordered system. According to the theoretical predictions [73,74], the pseudocritical temperatures T_Z^* of the disordered system are distributed with a width $\delta[T_Z^*]_{av}$, that scales with the system size as

$$\delta([T_Z^*]_{av}) \sim L^{-n}, \quad (8)$$

where $n=d/2$ or $n=1/\nu_r$, depending on whether the disordered system is controlled by the pure or the random fixed point, respectively. In panel (c) of Fig. 5 we plot these sample-to-sample fluctuations of the pseudocritical temperature of the magnetic susceptibility.

The solid line shows a very good power-law fitting giving the value 1.001(3) for the exponent n of the theory, which is in agreement with the case $n=d/2$, indicating that the random model is controlled by the pure fixed point. We should note here that analogous results to those discussed here in panel (c) for the case of the site-diluted Ising model on the square lattice have been presented by Tomita and Okabe, using the probability-changing cluster algorithm [48].

As in the pure case, the second alternative estimation of ν is carried out by analyzing the divergency of the logarithmic derivatives of the order parameter. In Fig. 6(a) we illustrate in a double-logarithmic scale the size dependence of the first-(filled squares), second-(filled circles), and fourth-order (filled triangles) logarithmic derivatives (averaged over the individual maxima). The solid lines show linear fittings for the sizes $L \geq 60$. In all cases a value ν=1 is obtained for the critical exponent ν, providing further evidence to the strong universality scenario emerged from Fig. 5. Figure 6(b) illustrates our method to evaluate and discuss the stability of the estimation for the exponent ν from the scaling behavior of the logarithmic derivatives of panel (a). It shows values of effective exponents (ν_{eff}) determined by imposing a lower cutoff (L_{min}) and applying simultaneous fittings in windows ($L_{\text{min}}-L_{\text{max}}$), where as for the pure case, L_{max}=200 and L_{min}=20, 40, 60, 80, and 100 as a function of $1/L_{\text{min}}$. The effective estimates show a finite-size effect for small values of the lower cutoff, whereas and for $L \geq 60$ a clear trend toward the value ν=1 of the Ising universality class is obtained. Let us note here that the same picture emerged from the FSS of the disorder-averaged logarithmic derivatives of the form $[\partial \ln\langle M^n\rangle/\partial K]_{av}^*$ that corresponds to the first way of averaging, but is omitted here for brevity. We should note here that a similar cross-over behavior in the estimates of the critical exponent ν has been observed in the case of the 2D

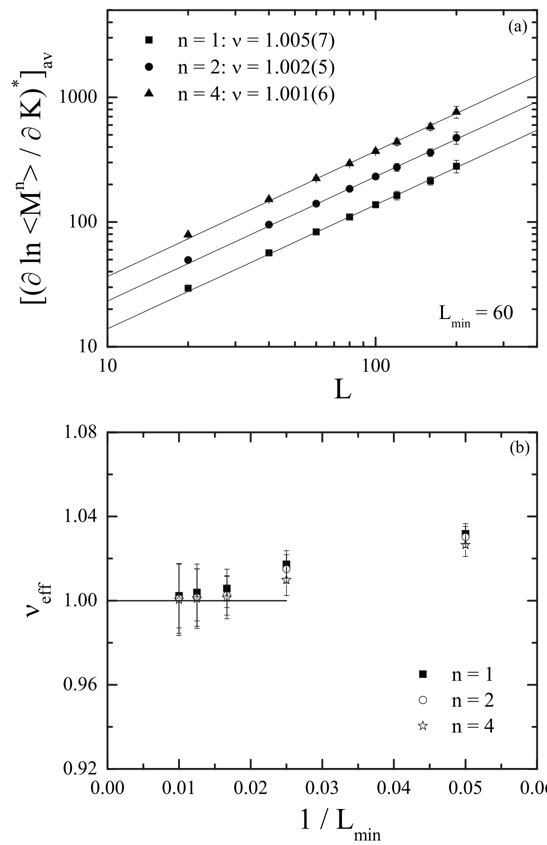

FIG. 6. (a) FSS of the logarithmic derivatives of the order parameter in a log-log scale. The solid lines are simple linear fittings for the larger lattice sizes (L_{min}=60). (b) Values of effective exponents ν_{eff} obtained from the data of panel (a) from several fitting attempts in the range ($L_{\text{min}}-L_{\text{max}}$). The solid line marks the proposed estimate ν=1.

site-diluted SqIM by Ballesteros *et al.* [39] and has been explained as logarithmic corrections.

Thus, summarizing our estimates for the critical exponent ν, we feel that it is clear that it maintains the value ν=1 of the pure case, indicating again the validity of the strong universality scenario.

We continue the presentation of our results by showing in Fig. 7 the FSS of the specific-heat maxima averaged over disorder: $[C]_{av}^*$ (up filled triangles) and $[C^*]_{av}$ (down open triangles). Using these data for the larger sizes $L \geq 60$, we tried to observe the quality of the fittings, assuming a double-logarithmic divergence of the form

$$[C]_{av}^*;[C^*]_{av} \sim C_1 + C_2 \ln(\ln L), \quad (9)$$

or a simple power law

$$[C]_{av}^*;[C^*]_{av} \sim C_\infty + C_3 L^{\alpha/\nu}. \quad (10)$$

Although it is rather difficult to numerically distinguish between the above scenarios, our detailed fitting attempts indicated that the double-logarithmic scenario [Eq. (9)] applies better to the numerical data and this is generally true for both $[C]_{av}^*$ and $[C^*]_{av}$ data.

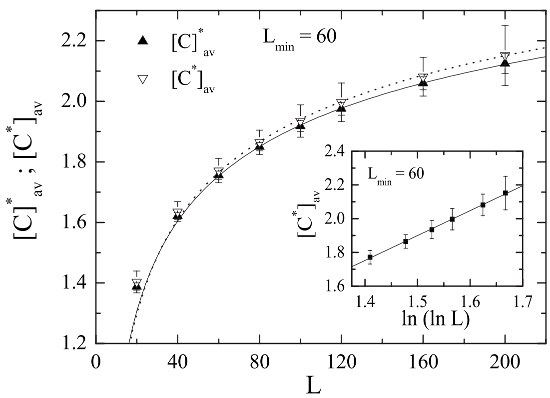

FIG. 7. FSS of the specific-heat maxima obtained via the two distinct ways of disorder averaging. The solid and dotted lines are double-logarithmic fittings of the form (9) for lattice sizes in the range L=60–200. The inset shows the data for the case $[C^*]_{av}$ as a function of the double logarithm of L. The solid line is an excellent linear fitting.

In fact, the double-logarithmic fitting is shown in the main panel, whereas in the corresponding inset of Fig. 7 the data of $[C^*]_{av}$ are plotted as a function of ln(ln L). The solid line shown is an excellent linear fit for $L \geq 60$. Let us now give some details on the quality of the applied fittings. We used the following sets of data points ($L_{\min}-L_{\max}$), with $L_{\max}$ =200 and $L_{\min}$=20, 40, 60, 80, and 100. The quality of the fittings indicated a very good trend for the values of χ^2/DoF for the double-logarithmic fittings (9) in the range: 0.2–0.7 and for both sets of data points. However, a strong reliability test in favor of the logarithmic corrections scenario is provided by the stability of the coefficient C_2, for both $[C]^*_{av}$ $[C_2 \approx 1.43(5)]$ and $[C^*]_{av}$ $[C_2 \approx 1.48(4)]$ data. On the other hand, the estimated values of the exponent α/ν of the power law (10), for both $[C]^*_{av}$ and $[C^*]_{av}$, fluctuate in the range $[-0.12(9), -0.05(6)]$ (as $L_{\min}$ increases) with the fitting procedure becoming rather unstable as we move to larger values of $L_{\min}$. The conclusion is that our numerical data are more properly described by the double-logarithmic form (9), in agreement with the MC findings of Selke *et al.* [43] and Ballesteros *et al.* [39] for the site-diluted SqIM and also with those of Wang *et al.* [28] for the strong disorder regime (r=1/4 and r=1/10) of the RBSqIM.

In Fig. 8 we provide estimates for the magnetic exponent ratios β/ν and γ/ν of the RBTrIM. In panel (a) we plot the average magnetization at the estimated critical temperature, as a function of the lattice size L in a log-log scale. The solid line is a linear fitting for $L \geq 20$ giving within error bars the value of the pure model, i.e., β/ν=0.1253(5)≈0.125. Additional estimate for the ratio β/ν can be obtained from the FSS of the derivative of the absolute order parameter with respect to inverse temperature defined in Eq. (4) which is expected to scale as $L^{(1-\beta)/\nu}$ with the system size [75]. Thus, in panel (b) of Fig. 8 we plot the data for $\partial\langle|M|\rangle/\partial K$ averaged over disorder as a function of L, also in a double-logarithmic scale. The solid line is a linear fitting for the larger lattice sizes $L \geq 60$, which combined with the value ν=1, gives an estimate of 0.1247(4) for the ratio β/ν. Finally, in panel (c) we present the FSS of the maxima of the average magnetic

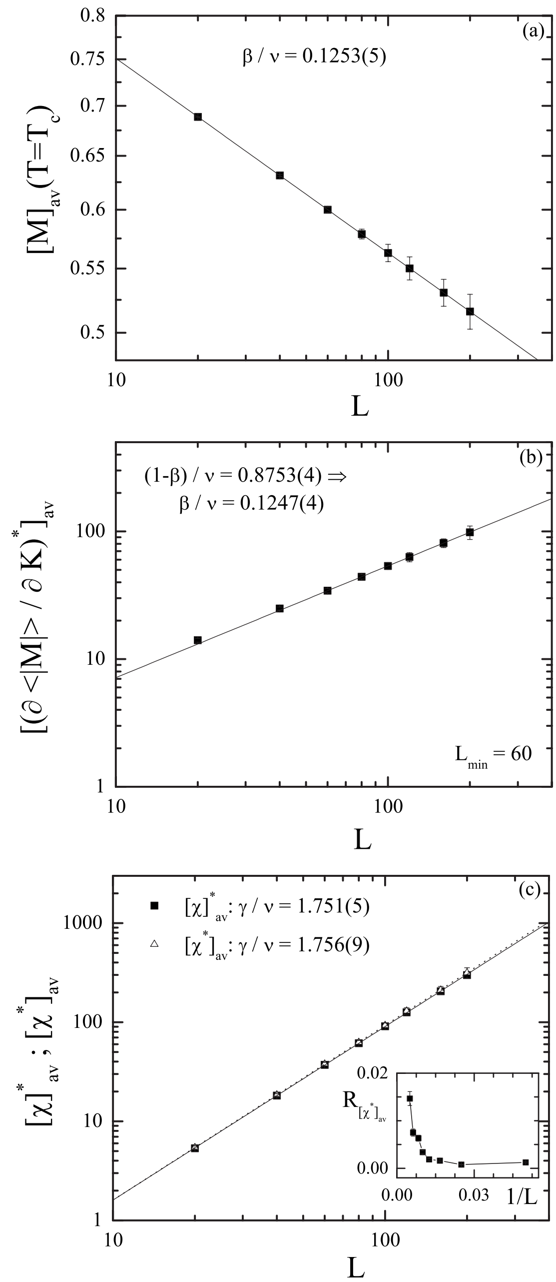

FIG. 8. (a) FSS of the average order parameter at the critical temperature. (b) FSS of the disorder-averaged inverse temperature derivative of the absolute order parameter. (c) FSS of the disorder-averaged magnetic susceptibility. The inset shows the limiting behavior of the ratio $R_{[\chi^*]_{av}}$. In all main panels (a)–(c) a double-logarithmic scale is considered. Additionally, all fitting attempts are performed in the complete lattice range L=20–200 since no deviation in the estimated values of the critical exponents was observed when shifting the value of the lower cutoff $L_{\min}$.

susceptibility $[\chi]^*_{av}$ (filled squares) and also the average of the individual maxima $[\chi^*]_{av}$ (open triangles). The solid and dotted lines present linear fittings using the total lattice range spectrum, giving the estimates 1.751(5) and 1.756(9) for the ratio γ/ν in very good agreement with the expected value 1.75 of the pure system. For the average $[\chi]^*_{av}$ the error bars indicate the statistical errors due to the finite number of the realizations, as discussed in Sec. I. For the average $[\chi^{*}]_{av}$ the errors bars shown reflect now the relatively large sample-to-sample fluctuations.

Finally, using the latter sample-to-sample fluctuations, we construct the ratio $R_{[\chi^{*}]_{av}}$ and plot it as a function of the inverse linear size, as shown in the inset of Fig. 8(c). Clearly, for the present model the limiting value of $R_{[\chi^{*}]_{av}}$ is nonzero, indicating, as expected also for marginal disordered systems [74], a strong violation of self-averaging.

V. CONCLUSIONS

The effects induced by the presence of quenched bond randomness on the critical behavior of the 2D Ising spin model embedded in the triangular lattice have been investigated by an efficient entropic scheme based on the Wang-Landau algorithm. In the first part of our study we presented the finite-size scaling behavior of the pure model, for which we calculated with high accuracy the critical exponents and the coefficients of the specific heat's logarithmic expansion at the critical point. Our results are in full agreement with the exact expansion presented by Wu *et al.* [79].

In the second part of our study, we investigated the critical properties of the disordered system. The presented detailed finite-size scaling analysis along the lines of the two existing scenarios—strong versus weak universality—strongly supports the scenario of strong universality. Thus, our results are in agreement with the behavior predicted originally on theoretical basis many years ago by Dotsenko and Dotsenko [17], Jug [18], Shalaev [19], Shankar [20], and Ludwig [21] and verified by simulations in recent years for the square random Ising model by several authors [39–41,43,44,48,58,59]. Particular interest was paid to the sample-to-sample fluctuations of the random model and their scaling behavior that were used as a successful alternative approach to estimate the critical temperature and the correlation length's exponent. Closing, we would like to note that another interesting candidate, that has not been studied before in the triangular lattice, is the three-state Potts model, which, in its pure version, has a positive specific-heat exponent. Disorder would be relevant in this case and could provide a further complementary study of the present work, analogous to the early transfer-matrix calculations of Derrida *et al.* [80] of the random-bond model on the square lattice.

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