Polarization of an electromagnetic wave in a randomly birefringent medium: A stochastic theory of the Stokes parameters

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We present a framework for the stochastic features of the polarization state of an electromagnetic wave propagating through the optical medium with both deterministic (controlled) and disordered birefringence. In this case, the Stokes parameters obey a Langevin-type equation on the Poincaré sphere. The functional integral method provides for a natural tool to derive the Fokker-Planck equation for the probability distribution of the Stokes parameters. We solve the Fokker-Planck equation in the case of a random anisotropic active medium submitted to a homogeneous electromagnetic field. The possible dissipation and relaxation phenomena are studied in general and in various cases, and we give hints about how to validate experimentally the corresponding phenomenological equations.

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I. INTRODUCTION

The study of the polarization state of an electromagnetic wave propagating through an optical medium has a long history [1–3]. The wave-polarization state is commonly characterized by various forms of the Stokes parameters [4–7] and their geometrical realization as the Poincaré sphere [1,8].

Linear or nonlinear birefringence of the medium plays the central role in the proper evolution of the wave polarization. Beyond the perfect media, the stochastic behavior of the polarization state when randomness of linear birefringence is an issue. Then, starting from the work of Perrin [9], a number of previous works have been devoted to the case of optically active scatterers randomly dispersed in an isotropic medium [10,11]. These studies belong to the general framework of the multiple scattering waves in the presence of random scatterers [2]. However, despite a number of studies, many points are not completely understood [12].

We focus in the present work on a somewhat different problem: the medium is optically active and anisotropic, with defects randomly placed in it. It can be the case for an anisotropic crystal with a small amount of structural defects [13], or a liquid crystal with colloidal particles dispersed in it [14], a dusty plasma [15], etc.

We will discuss the following points:

In Sec. II, we formulate the evolution of the Stokes parameters due to the possible changes in optical properties of the medium when external electromagnetic fields are applied. The equations are written under the formalism of a three-dimensional classical pseudospin (the Stokes vector) in a pseudomagnetic field. The value of the pseudomagnetic field is given explicitly in terms of the applied fields. In Sec. II D, the possible dissipation term is discussed as well as its relation with relaxation.

In Sec. III, we consider polarization stochasticity when the dielectric tensor experiences random components. For example, this can be due to impurities located randomly in a homogeneous optically active medium. Thus, the equation of motion for the Stokes parameters turns out to be a Langevin-type equation, from which the Fokker-Planck (FP) equation results. We address the question of the possible equilibrium distribution of the polarization states in Sec. III C. Moreover, an explicit solution of the Fokker-Planck equation is obtained by use of the functional integral approach in the case of the homogeneous Kerr and Faraday effects.

In the final Sec. V, we consider the Bloch-type relaxation equations, as they can be simply derived from the dissipation processes, or introduced in a phenomenological way similarly to relaxation of the spin systems. Within this framework, we study the case of the homogeneous Faraday effect plus a sinusoidal-modulated Kerr effect. Then, resonance of the polarization state can occur. We emphasize various ways which could be experimentally used to validate the Bloch equations in the optical case and to extract the actual values of the relaxation constants.

II. EVOLUTION OF THE STOKES VECTOR IN A PSEUDOMAGNETIC FIELD

We consider a monochromatic electromagnetic plane wave, of frequency ω and wavelength λ, propagating with wave number k along the direction z through a transparent medium of constant magnetic permeability, μ. The state of the wave is defined by its intensity, S_0, and the three-dimensional Stokes vector, $\mathbf{S}=(S_1,S_2,S_3)$ with S_0,S_1,S_2,S_3 the four Stokes parameters:

$$S_0 = |E_x|^2 + |E_y|^2,$$

$$S_1 = |E_x|^2 - |E_y|^2,$$

$$S_2 = E_x E_y^\star + E_x^\star E_y,$$

$$S_3 = i(E_x E_y^\star - E_x^\star E_y),$$

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written here in terms of the transverse wave electric field, $\mathbf{E}$. The indices x and y refer to two orthogonal directions perpendicular to the z direction.

A. Envelope approximation

If the direction z is the anisotropy axis, the electric permittivity $\hat{\varepsilon}$ can be written as the 3×3 matrix:

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}. \tag{1}$$

For later use, we write $\hat{\varepsilon}_{\perp}$ the projection of matrix (1) onto the plane x-y:

$$\hat{\varepsilon}_{\perp} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix}.$$

The Maxwell equations in the medium give the equations for the electric field, $\mathbf{E}$, of the wave:

$$\nabla \times (\nabla \times \mathbf{E}) + \mu \frac{\partial^2}{\partial t^2}(\hat{\varepsilon}\mathbf{E}) = 0, \tag{2}$$

$$\nabla \cdot (\hat{\varepsilon}\mathbf{E}) = 0. \tag{3}$$

The electric field is then transverse because of Eqs. (1) and (3), and the only projection of $\mathbf{E}$ onto the plane x,y will be considered.

If $\hat{\varepsilon}$ varies slowly in regard to the typical time and space variations of the electromagnetic field, the envelope approximation holds [16,17]:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \sqrt{S_0} e^{i(kz-\omega t)} \mathbf{F}(z), \tag{4}$$

with S_0 as the initial wave intensity, and the two-dimensional normalized Jones vector $\mathbf{F}$ changes smoothly with z and t, that is, $\partial/\partial z \ll k$, $\partial/\partial t \ll \omega$.

Using Eq. (4) in Eq. (2), and the envelope approximations, one finds the differential equation [5]

$$\frac{2i}{k}\frac{d\mathbf{F}}{dz} = \mathcal{H}\mathbf{F}, \tag{5}$$

with the 2×2 matrix $\mathcal{H}$:

$$\mathcal{H} = \mathcal{I}_2 - v_{\phi}^2 \mu \hat{\varepsilon}_{\perp}, \tag{6}$$

and $\mathcal{I}_2$ the 2×2 identity matrix. The phase velocity is $v_{\phi} = \omega/k$.

For Eq. (5) to be consistent with the envelope approximation, one has to fix the value of k such that matrix (6) is small with regard to k. It is realized when

$$k^2 = \frac{\omega^2 \mu}{2} \mathrm{Tr}\{\hat{\varepsilon}_{\perp}\},$$

which makes $\mathcal{H}$ traceless.

Moreover, the matrix $\mathcal{H}$ must be Hermitian for the medium to be transparent. Hence, we can write

$$\mathcal{H} = b\hat{\sigma}_1 + c\hat{\sigma}_2 + a\hat{\sigma}_3,$$

with the traceless Pauli matrices: $\{\sigma_i\}_{i=1,2,3}$. The dimensionless coefficients a,b,c are real and depend on the electromagnetic field.

B. Two-states Schrödinger-like equation for the Stokes parameters

It is convenient to write the equations above on the basis of the two circular-polarization states. The state vector of the wave is

$$\mathbf{\Psi} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \sqrt{S_0}\hat{A}\mathbf{F},$$

using the unitary matrix:

$$\hat{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

In this framework, the evolution of Ψ is governed by the homogeneous Schrödinger-like equation:

$$\frac{2i}{k}\frac{d\mathbf{\Psi}}{dz} = \hat{\mathcal{H}}\mathbf{\Psi}, \tag{7}$$

with the traceless Hermitian operator, $\hat{\mathcal{H}} = \hat{A}\mathcal{H}\hat{A}^{\dagger}$, which, using the Pauli matrices, writes

$$\hat{\mathcal{H}} = a\hat{\sigma}_1 + b\hat{\sigma}_2 + c\hat{\sigma}_3. \tag{8}$$

The four optical Stokes parameters, $\{S_j\}_{j=0,\ldots,3}$, can then be simply expressed as average values of the four Pauli matrices, namely,

$$S_j = \mathbf{\Psi}^{\dagger}\sigma_j\mathbf{\Psi}.$$

From the Schrödinger-like Eq. (7), the Ehrenfest theorem applied to the Pauli matrices states

$$\frac{2i}{k}\frac{d}{dz}(\mathbf{\Psi}^{\dagger}\hat{\sigma}_j\mathbf{\Psi}) = \mathbf{\Psi}^{\dagger}[\hat{\sigma}_j, \hat{\mathcal{H}}]\mathbf{\Psi},$$

for the Hermitian operator $\hat{\mathcal{H}}$. Using Eq. (8) and the commutation relation $[\hat{\sigma}_1, \hat{\sigma}_2] = 2i\hat{\sigma}_3$ (and cyclic permutations), one finds the evolution equation for the three-dimensional Stokes vector $\mathbf{S} = (S_1, S_2, S_3)$ as

$$\frac{d\mathbf{S}}{dz} = \gamma\mathbf{S} \times \mathbf{G}, \tag{9}$$

with the constant "gyrospin" factor: $\gamma = k$, and $\mathbf{G}$ the *pseudomagnetic field*:

$$\mathbf{G} = (-a, -b, -c). \tag{10}$$

In terms of the classical Hamiltonian, H, the pseudomagnetic field is

$$\mathbf{G} = -\nabla_{\mathbf{s}}H, \tag{11}$$

with $\nabla_{\mathbf{s}}$ as the gradient in the $\mathbf{S}$ space.

C. Realization of the pseudomagnetic field in the anisotropic transparent medium

Let us suppose that the transparent medium is isotropic, with the Hermitian electric permittivity tensor $\hat{\varepsilon}_o$:

$$\hat{\varepsilon}_o = \varepsilon_o \mathcal{I}_3,$$

where $\mathcal{I}_3$ is the 3×3 identity matrix.

The medium is submitted to an external electric field $\mathbf{E}^{ext}$ in the plane perpendicular to the z axis and to an external magnetic field $\mathbf{H}^{ext}$ along the z axis.

1. Faraday components

The magnetic field $\mathbf{H}^{ext}=(0,0,h_z)$ being applied to the medium, the additional 2×2 dielectric tensor writes

$$\Delta\hat{\varepsilon} = \begin{pmatrix} 0 & -ifh_z \\ ifh_z & 0 \end{pmatrix}, \tag{12}$$

with the constitutive parameter f depending on the medium and on the wavelength of the electromagnetic wave.

2. Kerr components

The electric field $\mathbf{E}^{ext}=(e_x,e_y,0)$ changes the dielectric tensor of the medium according to the formula

$$\Delta\hat{\varepsilon} = \kappa \begin{pmatrix} e_x^2 & e_x e_y \\ e_y e_x & e_y^2 \end{pmatrix}, \tag{13}$$

with the constitutive parameter κ depending on the medium and on the wavelength of the electromagnetic wave.

All together, the pseudomagnetic field $\mathbf{G}$ can readily be calculated using Eqs. (12) and (13) leading to

$$\mathbf{G} = \frac{1}{n\varepsilon_o}\left(\kappa\frac{e_x^2-e_y^2}{2}, \kappa e_x e_y, f h_z\right), \tag{14}$$

with

$$n^2 = 1 + \frac{\kappa}{\varepsilon_o}\frac{e_x^2+e_y^2}{2}.$$

The classical Hamiltonian H, as defined in Eq. (11), is

$$H = aS_1 + bS_2 + cS_3 = -\mathbf{G}\cdot\mathbf{S}. \tag{15}$$

D. Dissipative term and relaxation term

1. Dissipative term

When considering the second order in the wave intensity S_0, the linear evolution Eq. (9) for the vector $\mathbf{S}$ of constant magnitude must be modified in the Landau-Lifshitz-Gilbert form [20]

$$\frac{d\mathbf{S}}{dz} = \gamma\mathbf{S}\times\mathbf{G} + \eta\mathbf{S}\times(\mathbf{S}\times\mathbf{G}). \tag{16}$$

This phenomenological form is well known in the context of real spin dynamics. The positive coefficient η refers to the dissipation parameter in the (pseudo) spin dynamics. Even with the additional Landau-Lifshitz-Gilbert term, the intensity of the wave remains constant, $\mathbf{S}^2=S_0^2$.

The additional Gilbert part is indeed a dissipation term in the sense that the proper value of the Hamiltonian H, which corresponds to the "energy" of the polarization, is always decreasing, since

$$\frac{dH}{dz} = \eta[(\mathbf{G}\cdot\mathbf{S})^2 - \mathbf{G}^2\mathbf{S}^2] \leq 0.$$

Since $H=-\mathbf{GS}$, the interaction between the Stokes vector and the pseudomagnetic field tends to align the pseudospin in the antiparallel direction. The stable fixed point, $\mathbf{S}^\star$, of the Stokes vector is then of magnitude S_0, parallel to $\mathbf{G}$ and in the opposite direction, it writes then

$$\mathbf{S}^\star = -S_0\frac{\mathbf{G}}{|\mathbf{G}|}. \tag{17}$$

This is the only vector realizing $d\mathbf{S}/dz=0$, $dH/dz=0$, and $d^2H/dz^2<0$.

2. Bloch form

One can alternatively see the dissipative part of the equation as a relaxation process. A convenient way is to linearize the Landau-Lifshitz-Gilbert equation around the stable fixed point, $\mathbf{S}^\star$. Considering the auxiliary vector $\mathbf{A}\equiv\mathbf{S}-\mathbf{S}^\star$, of small magnitude with respect to S_0, one finds the approximate equation for $\mathbf{A}$:

$$\frac{d\mathbf{A}}{dz} = \gamma\mathbf{A}\times\mathbf{G} - \Lambda\mathbf{A}, \tag{18}$$

where the matrix Λ is

$$\Lambda = \frac{\eta S_0}{|\mathbf{G}|}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & c^2+a^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix},$$

and $|\mathbf{G}|^2=a^2+b^2+c^2$ after Eq. (10).

The three eigenvalues of the operator $\gamma\times\mathbf{G}-\Lambda$ are 0 and $-\eta S_0|\mathbf{G}|\pm i\gamma|\mathbf{G}|$. The real parts of the eigenvalues are all nonpositive, then Eq. (18) is essentially a relaxation process, with the relaxation length l such that $L^{-1}=\eta S_0|\mathbf{G}|$. The typical length $(\gamma|\mathbf{G}|)^{-1}$ appearing in the imaginary part of the eigenvalues corresponds to oscillations of the polarization along the z trajectory.

When the relaxation length, L, is much smaller than the overall path length, L_0, in the optical material (i.e., the intensity of the wave is large enough: $S_0 \gg 1/L_0\eta|\mathbf{G}|$), the relaxation is efficient and one can replace the nonlinear Landau-Lifshitz-Gilbert term by a simpler linear term in the Bloch form [22], namely,

$$\frac{d\mathbf{S}}{dz} = \gamma\mathbf{S}\times\mathbf{G} + \Lambda(\mathbf{S}^\star - \mathbf{S}),$$

where Λ is the relaxation matrix, with non-negative real eigenvalues. The fixed point $\mathbf{S}^\star$, as given in Eq. (17), is the limit polarization state of the wave for the infinite path length, whenever the Landau-Lifshitz-Gilbert equation holds for the optical system.

III. STOCHASTIC ASPECT

In this section, we consider explicitly randomness of the optical properties of the medium as a random effective force acting on the Stokes parameters.

A. Langevin-type equation for the Stokes parameters

In the presence of defects located randomly in the material and interacting with the wave, or if the material is slightly inhomogeneous, the polarization changes in a random way. To model this effect, we add a random force $\mathbf{R}(z)$ as an uncorrelated white noise to the evolution equation of $\mathbf{S}$:

$$\frac{d\mathbf{S}}{dz} = \gamma\mathbf{S} \times \mathbf{G} + \eta\mathbf{S} \times (\mathbf{S} \times \mathbf{G}) + \mathbf{R}(z). \tag{19}$$

The random field $\mathbf{R}$ is such that

$$\langle \mathbf{R}(z) \rangle = 0,$$

$$\langle \mathbf{R}(z) \cdot \mathbf{R}(z') \rangle = \mathcal{D}\delta(z' - z),$$

$$\mathbf{R}(z) \cdot \mathbf{S}(z) = 0. \tag{20}$$

The strength of the fluctuations, $\mathcal{D}$, is a positive constant proportional to the number of defects per unit volume and $\delta(u)$ is the Dirac distribution. We shall discuss below (Sec. III B 3) in more details about the origin of the random noise $\mathbf{R}$ and the physical meaning of relation (20).

B. Fokker-Planck equation in the S space

The FP equation corresponding to Eq. (19) can be derived using the functional integral approach [18]. We consider the Gaussian form for the probability distribution of the function $\mathbf{R}$, namely,

$$P[\mathbf{R}(z)] \propto \exp\left[-\frac{1}{2\mathcal{D}} \int_0^z \mathbf{R}^2(z')dz' \right].$$

Using this distribution, the propagator K between two $\mathbf{S}$ states at two different values of the coordinate, say 0 and z, is given by the functional integral [18]:

$$K[\mathbf{S}(z)|\mathbf{S}(0)] = \int \prod_\tau \delta\left[\frac{d\mathbf{S}}{dz} + \mathbf{F}(\mathbf{S}) - \mathbf{R}(z) \right] \times \exp\left[-\frac{1}{2\mathcal{D}} \int_0^z \mathbf{R}^2(z')dz' \right] \mathcal{D}(\mathbf{S})\mathcal{D}[\mathbf{R}(z)],$$

where we put

$$\mathbf{F}(\mathbf{S}) = -\gamma\mathbf{S} \times \mathbf{G} - \eta\mathbf{S} \times (\mathbf{S} \times \mathbf{G}).$$

The integration measure $\mathcal{D}(\mathbf{S})$ includes the constraints: $\mathbf{S}(0)$ at $z=0$, $\mathbf{S}(z)$ at distance z, and $\mathbf{S}^2 = S_0^2$ anytime. Performing the integration over all the acceptable functions $\mathbf{R}(z)$, one finds

$$K[\mathbf{S}(z)|\mathbf{S}(0)] = \int e^{-\mathcal{S}/2\mathcal{D}} \mathcal{D}(\mathbf{S}), \tag{21}$$

with the action $\mathcal{S}$ defined as

$$\mathcal{S} = \int_0^z \left[\frac{d\mathbf{S}}{dz} + \mathbf{F}(\mathbf{S}) \right]^2 dz'. \tag{22}$$

Therefore, the probability distribution for the vector $\mathbf{S}$ taking the value $\mathbf{S}(0)$ at $z=0$, and the value $\mathbf{S}(z)$ at $z>0$, is given by the relation

$$P[\mathbf{S}(z)] = \int K[\mathbf{S}(z)|\mathbf{S}(0)]P[\mathbf{S}(0)]\mathcal{D}[\mathbf{S}(0)]. \tag{23}$$

The differential form derived from Eq. (23) leads to the FP equation

$$\frac{\partial P}{\partial z} = \mathcal{D}\nabla_{\mathbf{s}}^2 P + \boldsymbol{\nabla}_{\mathbf{s}} \cdot (\mathbf{F}P), \tag{24}$$

where $\nabla_{\mathbf{s}}^2$ is the Laplacian on the $\mathbf{S}$ sphere.

1. Equation of motion for the statistical average

From the Fokker-Planck equation above, one can verify directly the correctness of the evolution of the statistical average of the Stokes vector $\mathbf{S}$:

$$\langle \mathbf{S} \rangle = \int \mathbf{S}P(\mathbf{S})d\mathbf{S},$$

where integration takes place over the entire $\mathbf{S}$ space.

Introducing the current flow $\mathbf{J}$ as

$$\mathbf{J} = \mathcal{D}\boldsymbol{\nabla}_{\mathbf{s}}P + \mathbf{F}P,$$

such that the Fokker-Planck equation writes $\partial P/\partial z = \boldsymbol{\nabla}_{\mathbf{s}} \cdot \mathbf{J}$, one has

$$\frac{d\langle S_i \rangle}{dz} = \int S_i \frac{\partial P}{\partial z} d\mathbf{S} = \int S_i \boldsymbol{\nabla}_{\mathbf{s}} \cdot \mathbf{J} d\mathbf{S},$$

which becomes, using the identity $\boldsymbol{\nabla} \cdot (f\mathbf{A}) = \mathbf{A} \cdot \boldsymbol{\nabla} f + f\boldsymbol{\nabla} \cdot \mathbf{A}$,

$$\frac{d\langle S_i \rangle}{dz} = -\int F_i(\mathbf{S})P(\mathbf{S})d\mathbf{S},$$

resulting in the nonlinear equation

$$\frac{d\langle \mathbf{S} \rangle}{dz} = -\langle \mathbf{F}(\mathbf{S}) \rangle.$$

2. Fokker-Planck equation in the angular variables space

It may be convenient to express the Fokker-Planck Eq. (24) in terms of the angular variable. This can be obtained by using the spherical coordinates, $S_1/S_0 = \sin\theta\cos\phi$, $S_2/S_0 = \sin\theta\sin\phi$, and $S_3/S_0 = \cos\theta$, and noting

$$\sin\theta\nabla_{\mathbf{s}}^2 P = \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 P}{\partial\phi^2},$$

$$\boldsymbol{\nabla}_{\mathbf{s}}P \cdot (\mathbf{S} \times \boldsymbol{\nabla}_{\mathbf{s}}H) = \frac{S_0}{\sin\theta}\left[\frac{\partial}{\partial\phi}\left(\frac{\partial H}{\partial\theta}P \right) - \frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial\phi}P \right) \right],$$

and

$$\nabla_s \cdot (P\nabla_s H) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial H}{\partial\theta}P\right) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial H}{\partial\phi}P\right),$$

together with the identity

$$\nabla_s \cdot (\mathbf{S}\times\nabla_s H) \equiv 0.$$

After laborious calculations, Eq. (24), with the exact relation S_0^2=const, can be written in terms of the angular variables:

$$\sin\theta\frac{\partial P}{\partial z} = \frac{\partial}{\partial\theta}\left(\left[\eta\sin\theta\frac{\partial H}{\partial\theta} - \frac{\partial H}{\partial\phi}\right]P + \mathcal{D}\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\left[\frac{\partial H}{\partial\theta} + \frac{\eta}{\sin\theta}\frac{\partial H}{\partial\phi}\right]P + \frac{\mathcal{D}}{\sin\theta}\frac{\partial P}{\partial\phi}\right) \quad (25)$$

with the time variable $\tau = z/(1+\eta^2)$.

The integral of the probability density P over the whole (θ,ϕ) space results from Eq. (25) (with the condition $\partial H/\partial\phi = 0$ when $\theta\to 0$ or π):

$$\frac{1}{4\pi}\int P(\theta,\phi)\sin\theta d\theta d\phi = 1.$$

The same equation has been derived using the standard Brownian motion theory in the context of the motion of a magnetic spin in a random magnetic field [20,21].

3. Origin of the random field R

The third condition (20) on the random noise insures that the relation $\mathbf{S}^2 = S_0^2$ is fulfilled anytime. This condition can be realized in several ways. In [20], the form $\mathbf{R}(z) = \mathbf{S}\times(\gamma\mathbf{h} + \eta\mathbf{S}\times\mathbf{h})$ was used, in which $\mathbf{h}$ is unconstrained uncorrelated Gaussian vectorial random field. It leads to the Fokker-Planck Eq. (25). In the present paper, we consider the random noise $\mathbf{R}(z)$ realizing generally condition (20), without any precise assumption about the analytical form of the vectorial function $\mathbf{R}$. This leads to the Fokker-Planck Eq. (24). We have seen that both choices lead to *the same* Fokker-Planck equation, a result from which one concludes that the correlations resulting from the analytical form $\mathbf{R}(z) = \mathbf{S}\times(\gamma\mathbf{h} + \eta\mathbf{S}\times\mathbf{h})$ are not relevant for the definite distribution of the Stokes parameters.

We can then be more precise about the physical origin of the random noise $\mathbf{R}$. In the case of the form $\mathbf{R}(z) = \mathbf{S}\times(\gamma\mathbf{h} + \eta\mathbf{S}\times\mathbf{h})$, the random noise $\mathbf{h}$ is a random component added to the vectorial field $\mathbf{G}$. It can be the case for the anisotropic material in which the optical parameters fluctuate randomly from place to place around an average value. In the case that we have considered in Sec. III B, the random field is no more related to the field $\mathbf{G}$. It can be the case for the homogeneous anisotropic material in which randomly oriented optically active small domains are included. The equivalence between Eqs. (24) and (25) shows that both problems can be investigated under the same framework.

C. Steady state

An important particular case relates to the possible steady state

$$\frac{\partial P}{\partial z} = 0. \quad (26)$$

Considering the auxiliary real function A such that $A = \eta H + \mathcal{D}\ln P$, Eq. (26) writes in terms of the angular variables:

$$\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial A}{\partial\theta}P\right] + \frac{\partial}{\partial\phi}\left[\frac{1}{\sin\theta}\frac{\partial A}{\partial\phi}P\right] + \frac{P}{\mathcal{D}}\left[\frac{\partial H}{\partial\theta}\frac{\partial A}{\partial\phi} - \frac{\partial H}{\partial\phi}\frac{\partial A}{\partial\theta}\right] = 0,$$

for which a trivial solution is $A\equiv 0$. This leads to the Boltzmann-like solution for the probability distribution in the equilibrium state, namely,

$$P = e^{-\eta H/\mathcal{D}}. \quad (27)$$

In a statistical mechanic sense, the equilibrium state is then characterized by an *effective temperature*, whose inverse is such that

$$\beta = \frac{\eta}{\mathcal{D}}.$$

This is the fluctuation-dissipation relation for the problem of the Stokes parameters evolution in a disordered transparent medium.

An interesting consequence of the Boltzmann-like solution of the polarization state is that the wave remains essentially circularly polarized because of the dissipation and of the diffusion in such a disordered medium embedded in an electromagnetic field. Indeed, equilibrium solution (27), with the general Hamiltonian (15), leads to the nonvanishing average value of the circular polarization degree. More precisely, the third component of the Stokes vector is the Langevin function:

$$\langle S_3\rangle = \frac{1}{\beta c} - \coth\beta c, \quad (28)$$

which is essentially proportional to the longitudinal magnetic field h_z for the small values of the field: $\langle S_3\rangle \sim -\eta f h_z/3\mathcal{D}n\varepsilon_o$, when we consider $\beta c \ll 1$ in Eq. (28).

A similar result was previously found in the context of the magneto-optical effects on multiple elastic scattering of light within the framework of transport theory [11]: for the longitudinal magnetic field, a nonzero circular Stokes parameter S_3 was found to persist in diffuse transmission, with a degree proportional to the magnetic-field strength.

IV. APPROXIMATE SOLUTION FOR THE S-DISTRIBUTION FUNCTION: THE LINEAR BIREFRINGENCE CASE

In some cases, the **S**-distribution function can be calculated within the approach detailed above. The problem goes simple when the dissipation is vanishing. Keeping constant the value of the effective temperature β, the fluctuation-dissipation relation leads to $\mathcal{D}\to 0$ which is similar to the semiclassical approximation of the usual path integral when the Planck constant is assumed to be small.

A. Constant electromagnetic field

Let us illustrate this idea with the homogeneous linear Kerr/Faraday case, namely,

$$\mathbf{G} = (-a,0,-c),$$

with constant parameters a and c. For simplicity in the equations below, we suppose that the initial wave is not linearly polarized, that is, $\mathbf{S}(0)=(0,\sin\theta_0,\cos\theta_0)$ for $z=0$, or equivalently, the initial spherical angles are $(\theta,\phi)=(\theta_0,\pi/2)$.

After Eq. (9), the classical trajectory on the $\mathcal{S}$ sphere is

$$\bar{S}_1(z) = \sin\bar{\theta}\cos\bar{\phi} = \frac{ac}{a^2+c^2}\cos\theta_0\left(1-\frac{\cos(\kappa z+\alpha)}{\cos\alpha}\right),$$

$$\bar{S}_2(z) = \sin\bar{\theta}\sin\bar{\phi} = \sin\theta_0\frac{\sin(\kappa z+\alpha)}{\sin\alpha},$$

$$\bar{S}_3(z) = \cos\bar{\theta} = \frac{c^2}{a^2+c^2}\cos\theta_0\left(1+\frac{a^2}{c^2}\frac{\cos(\kappa z+\alpha)}{\cos\alpha}\right),$$

where the bars over the variables refer to the classical trajectory. Moreover, the characteristic length κ^{-1} is such that $\kappa=\gamma\sqrt{a^2+c^2}$ and the angle α: $a\tan\alpha=-\sqrt{a^2+c^2}\tan\theta_0$.

We consider now all the trajectories of $\mathbf{S}$ close to the classical value $\bar{\mathbf{S}}$. In terms of the angular variables, it writes

$$\theta=\bar{\theta}+\vartheta, \quad \phi=\bar{\phi}+\varphi.$$

Action (22) writes here

$$\begin{aligned}\mathcal{S} &= \int_0^z\left(\frac{d\mathbf{S}}{dz}-\gamma\mathbf{S}\times\mathbf{G}\right)^2 dz' \\ &= \int_0^z\left(\frac{d\theta}{dz}+\gamma a\sin\phi\right)^2 dz' \\ &\quad + \int_0^z\left(\frac{d\phi}{dz}\sin\theta+\gamma a\cos\theta\cos\phi-\gamma c\sin\theta\right)^2 dz'.\end{aligned} \tag{29}$$

Expanding the integrand to the second order around the classical values, one finds in function of ϑ and φ:

$$\begin{aligned}\mathcal{S} = \int_0^z\Bigg[&\left(\frac{d\vartheta}{dz}+\gamma a\varphi\cos\bar{\phi}\right)^2 \\ &+\left(\frac{d\varphi}{dz}\sin\bar{\theta}-\gamma a\vartheta\frac{\cos\bar{\phi}}{\sin\bar{\theta}}-\gamma a\varphi\cos\bar{\theta}\sin\bar{\phi}\right)^2\Bigg]dz',\end{aligned}$$

where we used the relations equivalent to Eq. (9):

$$d\bar{\theta}/dz = -\gamma a\sin\bar{\phi}, \tag{30}$$

$$\sin\bar{\theta}d\bar{\phi}/dz = -\gamma a\cos\bar{\theta}\cos\bar{\phi}+\gamma c\sin\bar{\theta}. \tag{31}$$

B. Constant magnetic field

The other simple case is when the magnetic field is the only applied field (i.e., $a=0$). One has from Eqs. (30) and (31), $(\bar{\theta},\bar{\phi})=(\theta_0,\pi/2-\gamma cz)$, and the additional action $\mathcal{S}'$ writes, at the second order in the angular variables around their classical values $\vartheta=\theta-\theta_0$, $\varphi=\phi-\bar{\phi}$:

$$\mathcal{S}' = \int_0^z\left[\left(\frac{d\vartheta}{dz}\right)^2+\left(\frac{d\varphi}{dz}\right)^2\sin^2\theta_o\right]dz',$$

which leads to the two Hamilton equations

$$\frac{d^2\vartheta}{dz^2}=0, \quad \frac{d^2\varphi}{dz^2}=0.$$

Because of these equations, the first derivative of both functions θ and ϕ is constant (independent of z).

The acceptable functions realizing the definite values of (θ,ϕ) modulo 2π for $z'=z$ are then

$$\vartheta(z')=2\pi m\frac{z'}{z}, \quad \varphi(z')=2\pi n\frac{z'}{z},$$

with m and n as any integer numbers.

Complete solution in terms of a theta function

In this case we can write the exact form for propagator (21). Indeed, the propagator is written under the form

$$\begin{aligned}&K[\theta,\phi,z|\theta_0,\phi_0,0] \\ &= \int\exp\left[-\frac{1}{2\mathcal{D}}\int_0^z\left(\frac{d\phi}{dz}-\gamma c\right)^2 dz'\right] \\ &\quad\times\exp\left[-\frac{1}{2\mathcal{D}}\int_0^z\left(\frac{d\theta}{dz}\right)^2 dz'\right]\mathcal{D}(\theta(z),\phi(z)),\end{aligned}$$

where the integration occurs on all the allowed trajectories $\{\theta(z),\phi(z)\}$ realizing the constraints $\theta(0)=\theta_0$, $\phi(0)=\phi_0$, and the values θ,ϕ at the value $z'=z$ of the coordinate.

The possible orbits with the winding parameters m,n are such that

$$\frac{d\theta}{dz}=\frac{\theta-\theta_0+2m\pi}{z},$$

$$\frac{d\phi}{dz}=\frac{\phi-\phi_0+2n\pi}{z},$$

then, we have

$$S=\frac{(\theta-\theta_0+2\pi m)^2}{z}+\frac{(\phi-\phi_0-\gamma cz+2\pi n)^2}{z},$$

which, after summing over all the integer winding numbers, gives

$$\begin{aligned}&K[\theta,\phi,z|\theta_0,\phi_0,0] \\ &= \frac{4}{\pi^2}e^{-3[(\Delta\theta)^2+(\Delta\phi-c\gamma z)^2]/2\mathcal{D}z}\theta_3\left(\frac{\Delta\theta}{2},e^{-\mathcal{D}z/2}\right) \\ &\quad\times\theta_3\left(\frac{\Delta\phi}{2},e^{-\mathcal{D}z/2}\right),\end{aligned} \tag{32}$$

where the θ_3-Jacobi theta function is defined by

$$\theta_3(u,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inu},$$

and $\Delta\theta = \theta - \theta_0$, $\Delta\phi = \phi - \phi_0$.

According to Eq. (23), expression (32) is also the entire probability distribution $P[\mathbf{S}(z)]$, where $\mathbf{S}(z) = [\sin\theta(z)\cos\phi(z), \sin\theta(z)\sin\phi(z), \cos\phi(z)]$. A similar derivation can be done along the same lines for the pure electric case (i.e., $c=0$).

V. BLOCH-LIKE EQUATIONS FOR RELAXATION OF THE POLARIZATION

We have seen previously how the dissipative term appearing in the Landau-Lifshitz-Gilbert Eq. (16) leads to the Bloch-type Eq. (18) as the result of linearization. Such a linear relaxation equation can be generally written as

$$\frac{d\mathbf{S}}{dz} = \gamma\mathbf{S} \times \mathbf{G} + \Lambda_r(\mathbf{S}^\star - \mathbf{S}), \tag{33}$$

with Λ_r as a real positive matrix, and $\mathbf{S}^\star$ a constant vector related to the intrinsic (field-free) anisotropy of the material and giving the asymptotic polarization state when $z\to\infty$.

Here we use Eq. (33) as a general phenomenological equation including relaxation [19]. The precise form for Λ_r can have various physical origins other than dissipation. Relaxation mostly results from the relevant random processes.

Here, we consider the case where the relaxations along the S_1 and S_2 directions are identical, while the relaxation involved in the S_3 direction is different. In this case, $\Lambda_r = \mathrm{diag}(1/L, 1/L, 1/L')$, with L and L' corresponding to two relaxation lengths. In principle, the fixed point $\mathbf{S}^\star$ can get any value.

(i) As a first example, we consider the case where the pseudomagnetic field $\mathbf{G}$ is due to the homogeneous Faraday effect, namely, $\mathbf{G}=(0,0,-c)$. We have

$$\frac{dS_1}{dz} = -\gamma c S_2 + \frac{S_1^\star - S_1}{L},$$

$$\frac{dS_2}{dz} = \gamma c S_1 + \frac{S_2^\star - S_2}{L},$$

$$\frac{dS_3}{dz} = \frac{S_3^\star - S_3}{L'}.$$

It is convenient to introduce the asymptotic ($z\to\infty$) polarization state:

$$\mathbf{S}(\infty) = \begin{pmatrix} (S_1^\star - \alpha S_2^\star)/(1+\alpha^2) \\ (S_2^\star + \alpha S_1^\star)/(1+\alpha^2) \\ S_3^\star \end{pmatrix}, \tag{34}$$

with $\alpha = \gamma c L$. Then, the deviations to the limit polarization become simple. Using $\mathbf{S}' = \mathbf{S} - \mathbf{S}(\infty)$, one has

$$S_1'(z) = e^{-z/L}[S_1'(0)\cos\gamma c z - S_2'(0)\sin\gamma c z],$$

$$S_2'(z) = e^{-z/L}[S_2'(0)\cos\gamma c z + S_1'(0)\sin\gamma c z],$$

$$S_3'(z) = e^{-z/L'} S_3'(0).$$

We see that the value of α—then the value of the relaxation distance L, if γ and c are known—can be evaluated from the values of the asymptotic state $\mathbf{S}(z\to\infty)$, with and without the magnetic field. Nevertheless, the value of the second relaxation distance L' (the parameter along the direction of the pseudomagnetic field) does not appear in the asymptotic values of the Stokes parameters.

When the system is governed by dissipation process (16), the constants L and L' are, respectively, $L = 1/\eta S_0|c|$ and $L' = \infty$, as explained in Sec. II D. In this case, the third component of the Stokes vector propagates without attenuation [$S_3(z) = S_3(0)$], while the other components converge to the limit value (34) for the values of z beyond the typical length L.

(ii) A more complicated example is the case where the linear birefringence is modulated at the spatial frequency Ω_0. The pseudomagnetic field is the sum of a homogeneous magnetic field and of a rotating electric field, namely,

$$\mathbf{G} = (-a\cos\Omega_0 z, -a\sin\Omega_0 z, -c).$$

Introducing the rotation operator $\mathcal{R}$, represented as the 3×3 matrix,

$$\mathcal{R} = \begin{pmatrix} \cos\Omega_0 z & \sin\Omega_0 z & 0 \\ -\sin\Omega_0 z & \cos\Omega_0 z & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we notice that the pseudomagnetic field $\mathbf{G} = \mathcal{R}^{-1}\mathbf{G}'$, with the constant vector

$$\mathbf{G}' = (-a, 0, -c),$$

in the rotating frame. Defining then the vector $\mathbf{S}' = \mathcal{R}\mathbf{S}$, one has readily

$$\begin{aligned} \frac{d\mathbf{S}'}{dz} &= \frac{d\mathcal{R}}{dz}\mathcal{R}^{-1}\mathbf{S}' + \gamma\mathcal{R}(\mathcal{R}^{-1}\mathbf{S}' \times \mathcal{R}^{-1}\mathbf{G}') + \Lambda(\mathcal{R}\mathbf{S}^\star - \mathbf{S}') \\ &= \mathcal{M}\mathbf{S}' + \Lambda\mathcal{R}\mathbf{S}^\star, \end{aligned} \tag{35}$$

with the constant matrix $\mathcal{M}$ such as

$$\mathcal{M} = \begin{pmatrix} -1/L & \Omega_0 - \gamma c & 0 \\ -(\Omega_0 - \gamma c) & -1/L & -\gamma a \\ 0 & \gamma a & -1/L' \end{pmatrix}.$$

To write Eq. (35), we used the identity $\mathcal{R}(\mathcal{R}^{-1}\mathbf{S}' \times \mathcal{R}^{-1}\mathbf{G}' = \mathbf{S}' \times \mathbf{G}')$, valid for any rotation $\mathcal{R}$, and any vectors $\mathbf{S}'$ and $\mathbf{G}'$.

Equation (35) is easy to solve in the general case, but we only show hereafter the solution for the special case $\mathbf{S}^\star = (0,0,S_3^\star)$, which leads to simple expressions. Indeed, the asymptotic value for $\mathbf{S}'$ when $z\to\infty$, namely, $-\mathcal{M}^{-1}\Lambda\mathcal{R}\mathbf{S}^\star$, writes

$$\mathbf{S}'(\infty) = \frac{L' S_3^\star}{L(\Omega^2 + 1/L^2) + L' \gamma^2 a^2} \begin{pmatrix} -\gamma a \Omega \\ -\gamma a/L \\ \Omega^2 + 1/L^2 \end{pmatrix}, \quad (36)$$

where we put $\Omega = \Omega_0 - \gamma c$. The complete solution for $\mathbf{S}'$ is then

$$\mathbf{S}'(z) = \mathbf{S}'(\infty) + e^{\mathcal{M}z}[\mathbf{S}'(0) - \mathbf{S}'(\infty)]. \quad (37)$$

From Eq. (36), the maximum value of $\mathbf{S}'^2(\infty)$ is usually reached for $\Omega = 0$. The particular frequency $\omega = \gamma c$ can then be seen as a resonance frequency [23].

In the general case, the values of the relaxation lengths L and L' could in principle be determined through the shape of $|\mathbf{S}(\infty)|$ versus Ω_0. For example, the width of the $|\mathbf{S}(\infty)|$ distribution around the resonance case $\Omega_0 = \gamma c$ depends explicitly on L and L', as it is clear from Eq. (36). Alternatively, the actual values of the external electromagnetic fields, directly related to a and c through Eqs. (10) and (14), could be used as well for the determination of the relaxation distances L and L'.

One can give here the example of dissipation process (16) at the resonant frequency. In this case, the z evolution of $\mathbf{S}'$ writes as in Eq. (37), with the matrix

$$e^{\mathcal{M}z} = e^{-z/2L} \times \begin{pmatrix} e^{-z/2L} & 0 & 0 \\ 0 & \cos \rho z - \frac{2}{\rho L}\sin \rho z & -\frac{\gamma a}{\rho}\sin \rho z \\ 0 & \frac{\gamma a}{\rho}\sin \rho z & \cos \rho z + \frac{2}{\rho L}\sin \rho z \end{pmatrix},$$

where we put $\rho = \sqrt{\gamma^2 a^2 - 1/4L^2}$, and, as usual in this case, $L = (\eta S_0 |\mathbf{G}|)^{-1}$. One can note here, that, generally, the relaxation toward the limit value occurs essentially with the parameter L. However, when the applied electric field is small ($\gamma a L < 1/2$), oscillations are forbidden and replaced by a damping which makes here the second and third components of the Stokes vector to relax to the asymptotic value with a larger relaxation length L_a, namely, $L_a = L/(1 - \sqrt{1 - 4\gamma^2 a^2 L^2})$, as the result of the strong dissipation.

VI. SUMMARY

In the present work, we have developed a theory of stochastic wave polarization in a disordered medium submitted to electromagnetic fields. The starting point is a Langevin-type equation for the Stokes parameters of a monochromatic wave propagating in the forward direction. This stochastic approach is written assuming a random force, for example, caused by the optically active impurities.

In the case of the linear birefringent medium, we have been able to find the approximate solution of the Fokker-Planck equation for the distribution of the Stokes parameters of the wave using a semiclassical functional integral technique. This leads to the knowledge of the complete probability distribution of the polarization ratios in the case of the disordered optical material.

We also discussed the possible relaxation processes occurring in the polarization of a wave propagating in an optical medium. Such relaxation phenomena have been carried out by analyzing the Bloch-type equations, which are formally similar to the Bloch equations for a spin system. As a particular case, these Bloch-type equations are recovered after linearizing the Landau-Lifshitz-Gilbert equation close to the fixed point in the polarization space. We solved these Bloch-type equations in the case of an electromagnetic field made of the superposition of a sinusoidal space-modulated electric field and a homogeneous magnetic field. We could then discuss the appearance of polarization resonance, as well as the role of the relaxation in such an effect, in a way similar to the Rabi resonance for spins in a sinusoidal magnetic field. We also emphasize that the dissipative term is necessary for the system to reach an equilibrium state in the statistical sense. In this case a natural effective temperature can be defined for the electromagnetic wave according to its polarization-state distribution.

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