# Abel solution to a bremsstrahlung inverse problem

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Bremsstrahlung is correlated with high-energy electrons in laser-heated plasma [K. Brueckner, Phys. Rev. Lett. **36**, 677 (1976)]. Since the result is important to the National Ignition Campaign (NIC) we reconsider the derivation, and the energy dependence of the Gaunt factor(s). We find an expression for bremsstrahlung we can Abel invert, and we demonstrate the accuracy of the transform with a simple numeric exercise.

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Radiation at high-energy (>10 keV) is a signature for energetic electrons (20–500 keV) and laser-plasma interactions (LPI). Typically, the correlation is made with (a), a transform by K. Brueckner [1,2], and (b), a model by W. Kruer [3]. The result pertains to inertial-confinement-fusion (ICF) because fast electrons (>150 keV) penetrate low-$Z$ ablator (Be or CH) and preheat deuterium-tritium (DT) fuel. Since [1] is inaccurate [4,5], [3] is specific (the author assumes a Maxwell-boltzmann distribution for hot electrons), and the National Ignition Campaign (NIC) will assess preheat using x-ray diagnostics, we re-examine [1] and present new formula to reduce x-ray data.

The transform derived by Brueckner [1] is presented first. The bremsstrahlung emitted by a fast electron per unit path, per unit energy [6] is

$$\frac{d^2\epsilon_{\rm rad}}{dxd(h\nu)} = \frac{16}{3}\frac{e^2}{\hbar c}\frac{1}{mc^2}\langle Z^2\rangle\frac{Ne^4}{mc^2\beta^2}\ln\Lambda_r. \tag{1}$$

The change in energy per fast electron, per unit path [6] is

$$\frac{d\epsilon}{dx} = -4\pi\langle Z^1\rangle\frac{Ne^4}{mc^2\beta^2}\ln\Lambda_c. \tag{2}$$

The relevant Gaunt factor(s) are

$$\ln\Lambda_r = \begin{cases}\ln\left[\dfrac{\epsilon^{1/2}+(\epsilon-h\nu)^{1/2}}{\epsilon^{1/2}-(\epsilon-h\nu)^{1/2}}\right] & \text{for } h\nu\le\epsilon,\\ 0 & \text{for } h\nu\ge\epsilon,\end{cases}$$

and

$$\ln\Lambda_c = \ln(2\epsilon/\hbar\omega).$$

Combining Eqs. (1) and (2), the radiation emitted by a fast electron with initial energy $\epsilon_o$ is

$$\frac{d^1\epsilon_{\rm rad}}{d(h\nu)^1} = \frac{4}{3\pi}\frac{e^2}{\hbar c}\frac{1}{mc^2}\frac{\langle Z^2\rangle}{\langle Z^1\rangle}\int_{h\nu}^{\epsilon_o}\frac{\ln\Lambda_r}{\ln\Lambda_c}d\epsilon, \tag{3}$$

and the energy radiated by a distribution, $n_e$, is

$$\frac{d^1E_{\rm rad}}{d(h\nu)^1} = \int_{h\nu}^{\infty} n_e\frac{d^1\epsilon_{\rm rad}}{d(h\nu)^1}d\epsilon_o. \tag{4}$$

Since the inverse to Eq. (4) is unknown, we differentiate it in $h\nu$ (twice). If the integrand is well behaved [4] the first derivative with respect to $h\nu$ is

$$\frac{d^2E_{\rm rad}}{d(h\nu)^2} = -n_e\frac{d^1\epsilon_{\rm rad}}{d(h\nu)^1}\bigg|_{\epsilon_o=h\nu} + \int_{h\nu}^{\infty} n_e\frac{d^2\epsilon_{\rm rad}}{d(h\nu)^2}d\epsilon_o; \tag{5}$$

the second is

$$\frac{d^3E_{\rm rad}}{d(h\nu)^3} = -n_e\frac{d^2\epsilon_{\rm rad}}{d(h\nu)^2}\bigg|_{\epsilon_o=h\nu} + \int_{h\nu}^{\infty} n_e\frac{d^3\epsilon_{\rm rad}}{d(h\nu)^3}d\epsilon_o. \tag{6}$$

Here, K. Brueckner substitutes $\ln\Lambda_c$ with its 'average'; the same is done for $\ln\Lambda_r$. If we use the notation in [1] and define

$$\frac{\lambda_Z}{2mc^2} = \frac{4}{3\pi}\frac{e^2}{\hbar c}\frac{1}{mc^2}\frac{\langle Z^2\rangle}{\langle Z^1\rangle}\frac{1}{\langle\ln\Lambda_c\rangle} \tag{7}$$

and

$$\langle\ln\Lambda_r\rangle = 2, \tag{8}$$

we find

$$\frac{d^1\epsilon_{\rm rad}}{d(h\nu)^1}\frac{2mc^2}{\lambda_Z} = +2(\epsilon_o-h\nu), \tag{9}$$

$$\frac{d^2\epsilon_{\rm rad}}{d(h\nu)^2}\frac{2mc^2}{\lambda_Z} = -2, \tag{10}$$

and

$$\frac{d^3\epsilon_{\rm rad}}{d(h\nu)^3}\frac{2mc^2}{\lambda_Z} = +0. \tag{11}$$

Now, if Eqs. (10) and (11) are substituted in Eq. (6), we find the Brueckner estimate to $n_e=n_e(E_{\rm rad})$,

$$\frac{d^3E_{\rm rad}}{d(h\nu)^3} = \frac{\lambda_Z}{mc^2}n_e(h\nu). \tag{12}$$

Unfortunately (as noted in [4,5]) we should *not* substitute $\ln\Lambda_r$ with a constant.

Instead, we seek a formula with the energy dependence of $\ln\Lambda_r$. To start, we substitute $\ln\Lambda_c$ with a series,

$$\frac{1}{\ln\Lambda_c} = \sum_{n=0}^{\infty}\left(\frac{d^n}{d\epsilon^n}\frac{1}{\ln\Lambda_c}\right)\bigg|_{\epsilon=\epsilon_o}(\epsilon-\epsilon_o)^n/n!, \tag{13}$$

and study its convergence in $n$. With little effect to $d^1\epsilon_{\rm rad}/d(h\nu)^1$, $d^2\epsilon_{\rm rad}/d(h\nu)^2$, and $d^3\epsilon_{\rm rad}/d(h\nu)^3$, we truncate the series at $n=0$ [7]. Equation (3) is simplified to

 

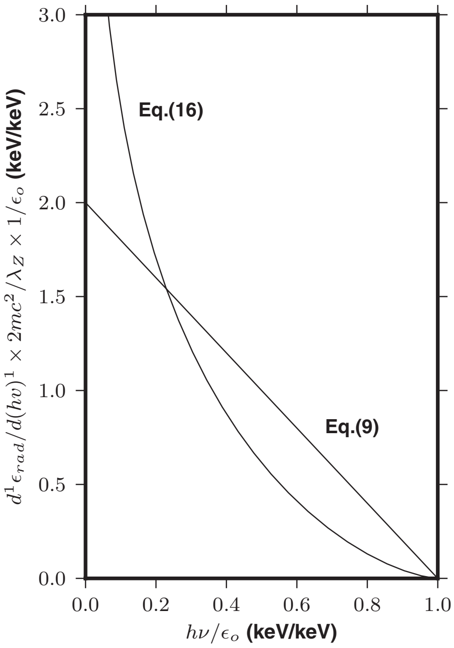


FIG. 1. The x-ray spectra for a single electron.

$$\frac{d^1\epsilon_{\mathrm{rad}}}{d(h\nu)^1} = \frac{\lambda_Z(\epsilon_o)}{2mc^2}\int_{h\nu}^{\epsilon_o} \ln \Lambda_r d\epsilon \tag{14}$$

and

$$\frac{\lambda_Z(x)}{2mc^2} = \frac{4}{3\pi}\frac{e^2}{\hbar c}\frac{1}{mc^2}\frac{\langle Z^2\rangle}{\langle Z^1\rangle}\frac{1}{\ln(2x/\hbar\omega)}. \tag{15}$$

If we integrate Eq. (14) [8] we find

$$\frac{d^1\epsilon_{\mathrm{rad}}}{d(h\nu)^1}\frac{2mc^2}{\lambda_Z(\epsilon_o)} = \frac{\epsilon_o}{(h\nu)^0}[-\Delta + (1+\Delta^2)\mathrm{atanh}\,\Delta], \tag{16}$$

$$\frac{d^2\epsilon_{\mathrm{rad}}}{d(h\nu)^2}\frac{2mc^2}{\lambda_Z(\epsilon_o)} = \frac{\epsilon_o}{(h\nu)^1}[-\Delta - (1-\Delta^2)\mathrm{atanh}\,\Delta], \tag{17}$$

and

$$\frac{d^3\epsilon_{\mathrm{rad}}}{d(h\nu)^3}\frac{2mc^2}{\lambda_Z(\epsilon_o)} = \frac{\epsilon_o}{(h\nu)^2}(+\Delta^{-1}), \tag{18}$$

with $\Delta=(1-h\nu/\epsilon_o)^{1/2}$ [9]. Looking at Eq. (16) [and Eq. (9)] we find the error in Eq. (8) (see Fig. 1). Bremsstrahlung is overpredicted at $h\nu \sim \epsilon_o$; it is underpredicted at $h\nu \ll \epsilon_o$. Now, we find a new transform if we note $d^2\epsilon_{\mathrm{rad}}/d(h\nu)^2=0$ at $h\nu=\epsilon_o$. If we write Eq. (6) as

$$(h\nu)^2\frac{d^3E_{\mathrm{rad}}}{d(h\nu)^3} = \frac{1}{\pi}\int_{h\nu}^{\infty}\frac{n_e(\epsilon_o)}{c_e(\epsilon_o)}\frac{1}{\sqrt{\epsilon_o - h\nu}}d\epsilon_o \tag{19}$$

and

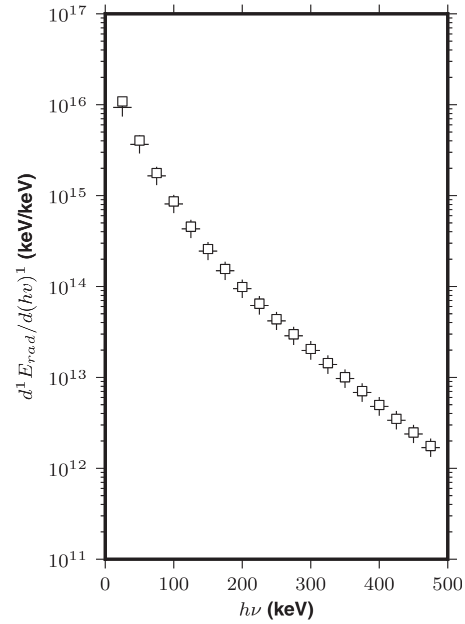


FIG. 2. The x-ray spectra given by Eqs. (1)–(4) (+) and Eq. (22) (□).

$$c_e(x) = \frac{1}{\pi}\frac{1}{x^{3/2}}\frac{2mc^2}{\lambda_Z(x)},$$

we can Abel invert to find

$$\frac{n_e(h\nu)}{c_e(h\nu)} = \left.\frac{t^2\dfrac{d^3E_{\mathrm{rad}}}{d(t)^3}}{\sqrt{t-h\nu}}\right|_{t=\infty} - \int_{h\nu}^{\infty}\frac{\left(t^2\dfrac{d^3E_{\mathrm{rad}}}{d(t)^3}\right)'}{\sqrt{t-h\nu}}dt. \tag{20}$$

For a fit we well resolve,

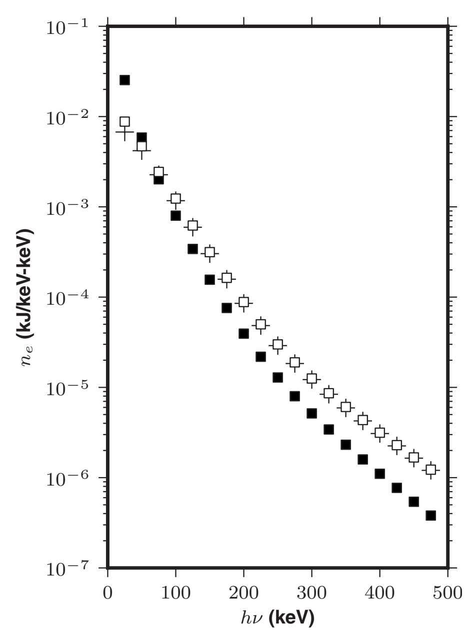


FIG. 3. The bi-Maxwellian (+), and the estimates given by Eqs. (12) (■) and (20) (□).

$$\frac{d^1 E_{\text{rad}}}{d(h\nu)^1} = \sum_i I_i f_i(h\nu),$$

we apply Eq. (20) (the NIC will resolve $d^1E_{\text{rad}}/d(h\nu)^1$ with 10 x-ray measurements from 20–500 keV). For a fit to limited data this is *not* done, but we still require an estimate of the fast electron energy, the characteristic energy per fast electron, and the risk to the DT fuel. In this circumstance we estimate $n_e$, but only if we assume the shape of the fast electron distribution, as in [3]. If we substitute Eq. (14) in Eq. (4) and assume a Maxwell-boltzmann distribution for hot electrons,

$$n_e(\epsilon_o,E_f,T_f) = \frac{4/3}{\pi^{1/2}} E_f T_f^{-2}\left(\frac{\epsilon_o}{T_f}\right)^{1/2} \exp\left(-\frac{\epsilon_o}{T_f}\right), \qquad (21)$$

we find

$$\frac{d^1 E_{\text{rad}}}{d(h\nu)^1}\frac{2mc^2}{\lambda_Z(h\nu)}\frac{1}{E_f} \cong \frac{3-\theta}{3}E_1(\theta) + \frac{5}{3}\exp(-\theta), \qquad (22)$$

where $T_f$ is the fast electron temperature, $\theta = h\nu/T_f$, and $E_f$ is the fast electron energy (the product of the fast electron count, $N$, and the average energy per fast electron, $3T_f/2$). With this approach we estimate the fast electron distribution, but only if the data can be fit to Eq. (22) [10].

Now, we test the relative merit(s) of Eqs. (12)–(22) against the x-ray spectra from Eqs. (1)–(4) for the bi-Maxwellian,

$$b_e(\epsilon_o) = n_e(\epsilon_o,E_a,T_a) + n_e(\epsilon_o,E_b,T_b), \qquad (23)$$

with $Z=79$, $\hbar\omega=100$ eV, $E_a=20$ kJ, $T_a=30$ keV, $E_b=2$ kJ, and $T_b=75$ keV. The spectra from Eqs. (1)–(4) is shown in Fig. 2, and the estimates given by Eqs. (12) and (20) appear in Fig. 3. Here, we see Eq. (12) is a reasonable estimate to Eq. (23), but the bi-Maxwellian is underpredicted at $h\nu \gg T$, and overpredicted at $h\nu \to 0$. Since Eq. (9) is a poor estimate to $d^1\epsilon_{\text{rad}}/d(h\nu)^1$ (see Fig. 1) this is expected; Eq. (20) is a much better fit to $b_e$. Still, Eq. (20) is less than exact, because $\ln \Lambda_c$ is evaluated at $\epsilon_o$ ($\ln \Lambda_c$ is fixed). This simplification introduces small errors at $\epsilon \sim \epsilon_o$, but larger errors at $\epsilon \ll \epsilon_o$ (or, at $h\nu \ll \epsilon_o$). Generally, this results in the overprediction of a distribution (and the bi-Maxwellian) at small $h\nu$ (see Fig. 3).

Given these result(s) we consider the uncertainty in Eq. (20). If hydrodynamic effects are negligible [1] we focus on the plasma background, the x-ray spectra, and the inverse transform. $Z$ and $\hbar\omega$ are set by the background. Since the uncertainty in $\lambda_Z/2mc^2$ is 10%–20%—and the uncertainty in $d^1E_{\text{rad}}/d(h\nu)^1$ is similar—the error we introduce with Eq. (18) should be insignificant (it is usually $\ll 10\%$). In this case the uncertainty in the fast electron distribution is estimated from the background (and/or the x-ray spectra) independent of the simplification to $\ln \Lambda_c$.

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[7] The error we introduce is $<20\%$ for $\epsilon_o = 10-1000$ keV, $h\nu = 10-1000$ keV, and $\hbar\omega = 10-100$ eV.
[8] Aided by a change-of-variables (with $1-h\nu/\epsilon$ replaced by $\chi^2$) and integration by parts.
[9] References [4,5] derive an expression equivalent to Eq. (16). Equations (17) and (18) are incorrect in [4].
[10] Reference [3] is consistent with Eq. (22), but the model in [3] has fewer dependencies.