Excitation regeneration in delay-coupled oscillators

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A network of phase-coupled oscillators constitutes a system in which excitable 2π phase slips can be induced by noise. We show that such an excitation in one of the oscillators can be regenerated by subsequent oscillators and become self-sustained for certain topologies. We focus on the simplest such topology: two mutually coupled oscillators. Our analysis is bolstered by an experimental confirmation of the phenomenon via a pair of mutually delay coupled quantum-dot lasers. Both the intensities and phases of the laser outputs were measured confirming the interpretation.

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The synchronization of two or more oscillators is a topic of considerable interest in many areas of science with nonlinear dynamics such as multistability, excitability and chaos readily observed. Networks of coupled oscillators are of much interest to researchers in different fields. Examples include models of neural networks (see [1] and references therein), Josephson junction arrays [2], chemical oscillators [3], cardiac cells [4], and Landau damping in plasmas [5]. (For a detailed review of synchronization in complex networks, see [6].) An important feature that is fundamental to many systems arising in nature is time delay, due, for example, to the finite propagation speed of signals or the finite time of chemical reactions. Coupled models incorporating delay have been used to explain phenomena such as the passage of information through coupled neurons [7] and the synchronization of crickets chirping [8] and fireflies flashing [9]. The introduction of delay enriches the set of possible dynamics for most systems. For example, amplitude death typically requires many oscillators and various assumptions regarding the distribution of frequencies but occurs readily with the introduction of delay even for low numbers of oscillators [10]. Time delay is also critical to some systems such as coupled microwave oscillators and lasers [11] even for relatively short delays due to the high frequencies involved.

The Kuramoto model [12] in which oscillators are coupled via their phases is of fundamental significance in many of these cases. Yeung and Strogatz were the first to introduce delay in this model [13] and they showed that various phenomena arise (such as bistability) which do not occur in its absence. We consider such a delay model in this work and describe how an excitable response can be regenerated in a system because of the combination of delay and mutual coupling. To experimentally probe these dynamics, we use another system which involves such delayed coupling and for which a delayed Kuramoto model can be derived: namely, a set of mutually coupled semiconductor lasers. Specifically, we analyze the mutual delay coupling of two quantum dot (QD) semiconductor lasers. These devices have previously been shown to be very stable in mutually coupled configurations [14] even for very long delay times. For other semiconductor lasers, stable synchronization can only be found for very short delay times with chaos a typical feature even for short delays [15–17]. As well as increased stability, QD lasers have been shown to display very different dynamics compared to their quantum well and bulk counterparts in master-slave configurations [18,19] and when subjected to external optical feedback [20]. Of interest here is the phenomenon of excitable phase slipping observed in master-slave systems [18,21]. (The excitable pulses observed with QD lasers appear to result from the Adler mechanism [22]. In [23], excitable pulses not resulting from the Adler mechanism but from phase slips in so-called homoclinic teeth were predicted.) Of course, it is not only in master-slave systems of semiconductor lasers that excitability is observed and for a review of excitability in laser systems see [24]. The introduction of delayed mutual-coupling suggests that dynamical events might be "passed on" from one oscillator to another and that such a phenomenon can occur in systems of mutually coupled lasers has indeed been demonstrated for the low-frequency fluctuations (LFF) observed in semiconductor laser systems involving feedback [16,25]. Outside of laser physics, Roxin *et al.* showed [26] that a small world network of leaky integrate and fire neurons could generate self-sustained firing if there was a reinjection of the pulse to a neuron after its refractory time. Our coupled system also has analogies with a ring of Josephson junctions and a chain of highly damped pendula coupled via torsional springs, being a specific example of the more general formal analogy between systems with delay and spatially extended systems [27]. Let's consider the mechanical analog of a chain of coupled pendula. Suppose first we have an open chain of (an odd number of) pendula. If the pendulum at one end is excited into a 2π rotation this motion can be propagated along the chain (given appropriate torsion) and will be annihilated at the end of the chain. Similarly, if the pendula at both ends are excited simultaneously then the two excitations will meet at the center pendulum and will annihilate each other there. If the chain is now considered to be infinite, then the excitation can persist indefinitely and it is this case which is of most relevance for this work where we show that a network of delay phase-coupled oscillators can display an analogous behavior. The configuration considered in this work provides one the most simple topologies in which an excitation is reinjected after each round trip time, namely, two mutually coupled oscillators in a ring topology. A long delay between the oscillators is required to ensure a return to stable action before reinjection. In fact, multiple excitations per round trip

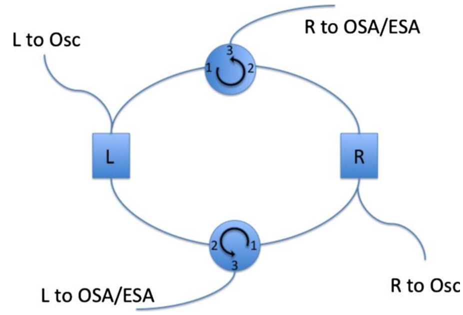

FIG. 1. (Color online) The experimental setup. The two lasers are labeled L and R. The circulators work as follows: light input at port 1 is output at port 2, light input at port 2 is output at port 3, and light input at port 3 is output at port 1. OSA, ESA, and Osc. are optical spectrum analyzer, electrical spectrum analyzer and oscilloscope, respectively.

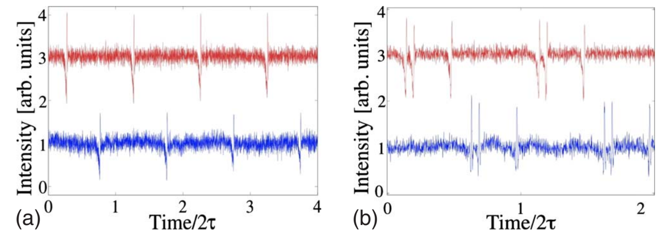

FIG. 2. (Color online) The intensity of the two lasers is shown in pulsating regimes. (The two signals are separated in average value to allow ease of view.) In the left, figure pulses in each laser occur with round-trip time separation 2τ while from one laser to the other are separated by the delay time τ. On the right, the intensity of the two lasers is shown in a regime where there is more than one pulse per round trip. The pattern is reproduced in each laser with round-trip time periodicity.

can coexist in the system for a long enough delay time. An experimental analysis of this behavior with semiconductor lasers has been heretofore restricted because of the need for short delays. The use of QD lasers however has changed this situation and such a system may now be realized in the laboratory.

In the first part of this work, we describe the experimental setup and results. This includes the first demonstration to the best of our knowledge of Adler-like excitability in mutually coupled semiconductor lasers and a measurement of the phase of the electric fields of the lasers when undergoing regenerative phase slipping. In the second part, we consider a simple model for two coupled oscillators, which explains these results and is of physical relevance being a limit of the more complicated laser rate equations. In the final part, we describe some of the more complex features associated with the system with asymmetric coupling and display the multistable nature of the system both theoretically and experimentally.

The experimental set up was as follows. Two single mode (distributed feedback) quantum dot lasers were selected so as to have as similar properties as possible (namely, operating wavelength, threshold current and slope efficiency). The devices were mounted on temperature controlled stages and light was collected from both facets by lensed fibers. The devices were mutually coupled in a ring topology as shown in the schematic in Fig. 1. Optical circulators ensured a unidirectional coupling and prevented unwanted effects arising from feedback. The system was analyzed using an optical spectrum analyzer, an electrical spectrum analyzer and a 14 GHz real-time digital oscilloscope. Polarization controllers were used to ensure TE polarized light was injected into both lasers. When free running, the lasers operated at approximately 1290 nm and the threshold currents were about 9 mA. The temperatures of the lasers were stabilized with resolution better then 0.01 K. The coupling delay between the lasers was 18.5 ns. The lasers were pumped up to 2.5 times their threshold currents and they were tuned by temperature. The coupling strength was estimated to be less than one percent.

Regions of stable frequency locking were observed at all coupling levels examined with dynamics appearing near the unlocking boundaries. For the coupling levels here, locking bands of approximately 2 GHz were obtained and the system was multistable for certain values of the detuning. Most importantly for this work, intensity pulses such as those in Fig. 2 were observed in the dynamical regions near the unlocking boundaries. Typically, an intensity pulse in one laser was followed by a pulse in the other after the delay-time τ and yet another in the first after a round-trip time 2τ and so on. It was possible to have more than one pulse per round trip and both the creation and death of pulses were observed. Superficially, it would appear that a leader-laggard mechanism is at play [16] but we believe that the truth may be more complicated as it can be phase perturbations which result in the initial symmetry breaking and these are not always readily apparent in the associated intensity time-series. In fact, simulations using instantaneous kicks rather than noise suggest that the initial perturbation may not even be in the laser showing the initial slip. A kick in one laser may not initially result in a full phase slip in the same laser but can cause a full slip in the second after one or more delay times and this can then be passed to the original laser after another delay time.

Examples of the pulsing behavior and phase evolution are shown in Figs. 2 and 3. The phase was measured via the Hilbert phase method previously used in a master-slave configuration in [21]. Here, a low linewidth tunable laser is mixed with the output from one of the lasers. The Hilbert phase of this signal is given by $\omega_b t+\phi$ where ω_b is the frequency difference between the tunable laser and the laser

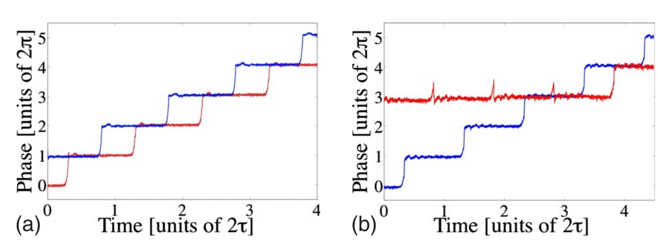

FIG. 3. (Color online) Experimental phase evolutions of the two outputs are shown in regions where the intensity was pulsating. In the figure on the left, both lasers are undergoing full phase slips. In that on the right, one laser makes $+2\pi$ rotations while the other sometimes follows but sometimes makes a small excursion from the steady state value and returns without performing a full rotation.

under test and ϕ is the phase of the laser. Thus, knowing the beat frequency one can find the phase evolution. The pulses are accompanied by two generic phase-trajectories. Either the phase changes by 2π (rotations of both $+2\pi$ and -2π were possible) and so returns to the same point physically, or it undergoes a short-lived perturbation from the original value. The intensity pulse shape is a result of the shape of the path taken by the electric field phasor. This path is not circular and so the intensity changes, much as it does in the case of master-slave configurations. However, it is the phase which is the important dynamical parameter.

Our system can be modeled as a pair of (Kuramoto) phase-coupled oscillators. It is of course possible to use the machinery of laser rate equations but the simple model captures much of the relevant physics and by restricting matters to a discussion of the phase only, the applicability to other systems should be more apparent. Further, the model has physical relevance being the limit of very weak coupling of the full model. (See, for example, [15,28].) The equations are

$$\dot{\phi}_1 = -\frac{\Delta}{2} - b_1 \sin[\phi_1(t) - \phi_2(t-\tau)] + \sqrt{2D_1}\xi_1(t),$$

$$\dot{\phi}_2 = \frac{\Delta}{2} - b_2 \sin[\phi_2(t) - \phi_1(t-\tau)] + \sqrt{2D_2}\xi_2(t).$$

Here, ϕ_i is the phase of oscillator i, Δ is the detuning in the reference frame centered on the average solitary frequency of the two oscillators, b_i describes the coupling strengths and τ is the delay time between the oscillators. In these equations, t is dimensionless due to a rescaling with respect to the photon lifetime (τ_{ph}) in the lasers. The equations are written as Langevin equations in order to excite slips from steady states: the D_i are the diffusion constants and the $\xi_i(t)$ are white Gaussian noise terms satisfying $\langle \xi_i(t)\xi_i(t')\rangle = \delta(t-t')$. We consider a relatively weak noise level only in this work. This model generalizes the Adler equation of master-slave coupling [22] to the case of mutual coupling and has been examined before to study the locking and unlocking properties of mutually coupled semiconductor lasers (see, for example, [29]) but not noise-induced phenomena, to the best of our knowledge. To begin with let us restrict ourselves to symmetric coupling, $b_1 = b_2 = b$. Depending on the sizes of b and τ, different numbers of frequency locked solutions may exist. For fixed τ and an appropriately low b, only one stable solution exists (born via a saddle-node bifurcation) and corresponds to fixed phases for both oscillators (in this particular frame). The central feature of Adler excitability then prevails for each individual oscillator: the existence of two fixed points one of which is stable and one unstable. The unstable point provides the threshold for the excitable trajectory. If the phase passes the unstable point, a full rotation results and it is the sign of the detuning which determines whether the phase increases or decreases. A positively detuned laser can only excite a phase slip of -2π and vice versa for a negatively detuned laser. In the mutually coupled system, at each locking boundary, one oscillator is negatively detuned and the other positively (in a frame between the two natural frequencies) and so both possibilities can arise near each boundary: the laser which is positively detuned can undergo a -2π phase slip and vice versa. As the magnitude of the detuning is increased, the fixed points for each oscillator move closer together and so less noise is required to excite a phase slip.

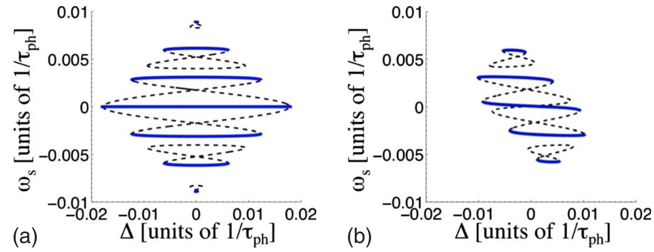

FIG. 4. (Color online) The figure on the left shows the stationary state frequency solutions ω_s for symmetric coupling with $b_1 = b_2 = 0.009$. The solid (blue) lines are the stable solutions while the dashed (black) lines are the unstable solutions. The figure on the right shows the steady state solutions for asymmetric coupling with $b_1 = 0.005$ and $b_2 = 0.009$. The delay time τ was taken to be 900 for these figures.

As the coupling strength is increased, other steady state solutions appear. The reduced model continues to be physically relevant to the experiment for relatively low coupling strengths. The steady states are created in further saddle-node bifurcations and Fig. 4 shows an example of the stable and unstable frequency solutions versus the detuning (a case with few steady states was chosen to keep the figure as simple as possible). The solid (blue) lines show the stable frequencies of the locked states while the dashed (black) lines show the unstable frequencies. Note that in all but one solution, the stable and unstable points have different frequencies. Just in one case (the $\omega = 0$ solution in the figure) are the frequencies the same and this corresponds to the situation described above where both solutions are fixed points in the same frame. Consider the mode with the lowest frequency, for example. In this mode, the unstable frequency is higher than the stable frequency and thus in the frame where the stable solution is a fixed point the unstable solution is a limit cycle of positive frequency. Phase slips generated by noise close to the boundaries of these modes cannot be sustained. Instead, the system moves to the next frequency solution after a limited number of slips—positive slips for frequencies below the central solution and vice versa. (In laser parlance, the system undergoes a mode hop.) Why this happens can be explained as follows. The average frequency of the field in a mode with frequency ω_s as it slips is $\omega_s \pm \pi/\tau$ (with the sign depending on the direction of the slips). However, the frequency difference between adjacent modes is approximately π/τ. Thus, the average frequency of the slipping mode is very close to that of the next mode and so one can intuit that rather than continuing to slip, the slips become smoothed out and the frequency of the next mode is attained. This seems to be borne out by simulation. The closer to the central mode, the more stable is the solution and so typically the system moves toward the center. Thus, one should not expect to observe sustained pulses except near the unlocking boundaries.

Consider now the situation where a slip is excited in one of the oscillators (both of which are in the central mode) and what occurs when this slip reaches the second oscillator. Numerically, it is found that three outcomes are possible: (i) the phase can change in the same manner; (ii) the phase can return to the original point without making a full rotation and (iii) the phase can change in the opposite manner where a change of $+2\pi$ in one triggers a change of -2π in the other and vice versa. For symmetric coupling, the most important factor in determining which does actually occur is the proximity to the unlocking boundary. For low detuning, the phases follow each other predominantly while for large detuning the phases change in the opposite direction predominantly. To explain this, suppose oscillator 2 is undergoing a phase rotation and let us approximate its phase as ωt over the course of the slip. The phase plots in Fig. 3 show that this is a reasonable approximation. The equation for ϕ_1 then becomes

$$\dot{\phi}_1 = \frac{\Delta}{2} - \mathrm{b}\,\sin(\phi_1 - \omega t)$$

(where we omit the noise for ease). This can be transformed to the usual Adler form by defining $\psi = \phi_1 - \omega t$. The resulting equation is

$$\dot{\psi} = \left(\omega + \frac{\Delta}{2}\right) - \mathrm{b}\,\sin(\psi),$$

for which a locked solution exists only if $|\omega + \frac{\Delta}{2}| \le b$. That is, oscillator 1 can remain locked to oscillator 2 during the rotation only if this condition is satisfied. Thus, for low detuning ϕ_1 can rotate with ϕ_2 and undergo the same phase change. However, for large detuning, oscillator 1 cannot remain locked to oscillator 2 when the slip arrives. Suppose for clarity that ϕ_2 is increasing during a slip. Then, while unlocked, ϕ_1 will decrease. When ϕ_2 returns to its locked position, ϕ_1 will also return to its locked position but how it does so depends on its value when ϕ_2 settles. Once ϕ_2 has resettled, the two fixed points for ϕ_1 re-emerge and so depending on its value relative to these it can either return directly to the locked value or perform a slip of its own. Thus, both perturbations away from and back to the fixed point and full rotations can occur closer to the unlocking boundary. (Very close to the boundary one should observe each phase slipping but in opposite directions.) Of course, the rotations are always a multiple of 2π and so the end states are always physically equivalent. Naturally, a new slip can be excited by noise in the time between round trip separated pulses in the system and this can also propagate through the system. Likewise, noise can suppress a pulse and cause it to disappear. In general, a mixture of the different processes is observed. Two examples of numerical phase evolutions are shown in Fig. 5 in excellent qualitative agreement with the experimental cases.

In conclusion, we have described a mechanism by which an excitable response can be regenerated in a phase-coupled system due to the combination of mutual coupling and delay. A pair of mutually coupled quantum dot lasers was used to examine the system experimentally. The phenomenon described should be robust to the inclusion of additional oscillators; for networks containing closed loops one should expect it to be a generic feature. The results confirmed that with weak coupling the system is an excellent approximation to a pair of Kuramoto coupled oscillators. While in times less than a delay time, each laser in the system can be described as the slave in a master slave configuration, it is the subsequent regeneration of the phase slip that provides the difference over the master-slave system. Of course, a quantum well laser can also be expected to undergo such a phase slip and a mutually coupled configuration of quantum well lasers should be investigated. However, one might expect the resulting dynamics to become chaotic in that case due to both the requisite short delay times and more weakly damped relaxation oscillations. If the relaxation oscillation damping of a quantum well were to be higher than typical values, then one would imagine that the observed dynamics would be possible. The phase measurement works very well for the situation where the time between successive pulses is long. However, the technique runs into difficulty as more pulses appear as the correct beating frequency which must be subtracted is more difficult to ascertain. For the case of injection experiments, this can be overcome with the phasor technique described in [30]. Such an alternative would be most welcome for the case considered here but as yet none exists. The modeling also requires further work to incorporate both the electric field amplitude and carrier density. Qualitatively, one can expect similar results but some differences will occur. For example, in the phase model each oscillator is symmetric in relation to the detuning. This would not be the case in the full model where phase-amplitude coupling via the linewidth enhancement factor induces an inherent asymmetry to each laser. This work suggests that coupled quantum dot lasers could be of interest in the construction of Kuramoto networks and in particular network motifs. Future work could involve the prospect of using such ensembles to carry out logical processes and as artificial neural networks. One can easily envisage the coupling of a number of such devices in numerous topologies involving all three of feedback, master-slave and mutual couplings. Finally, a natural extension would be to introduce external forces to mimic the currents applied to neurons or to Josephson junctions.

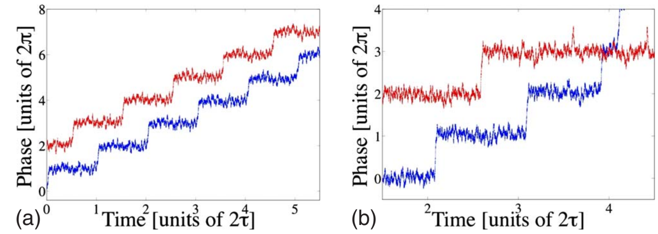

FIG. 5. (Color online) Numerical phase evolutions. For the figure on the left, the parameters were $b_1 = b_2 = 0.09$, $\Delta = -0.05$, $\tau = 2000$, $D_1 = 0.15$, and $D_2 = 0.15$. For the figure on the right, the parameters were the same but with $\Delta = -0.06$.

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