Detection of weak and large electric fields through the transient dynamics of a Brownian particle in an electromagnetic field

J. I. Jiménez-Aquino*

Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, Apartado Postal 55-534, C.P. 09340, México, Distrito Federal, México

M. Romero-Bastida†

SEPI-ESIME Culhuacán, Instituto Politécnico Nacional, Av. Santa Ana No. 1000, Col. San Francisco Culhuacán, Delegación Coyoacán, Distrito Federal 04430, México

(Received 26 June 2009; revised manuscript received 3 February 2010; published 29 March 2010)

In this work we present a mechanism to detect the presence of an external electric field of either weak or large amplitude by means of the decay process from an unstable state, described by a bistable potential, of an electrically charged Brownian particle embedded in a uniform electromagnetic field. Since the detection process takes place around the initial unstable state of the bistable potential, our theoretical description is given in the linear approximation of the aforementioned potential. The decay process is characterized through the statistics of the passage time distribution calculated by means of two theoretical approaches relying on the overdamped Langevin equation: one is the quasideterministic approach valid for large times and used for the detection of weak signals, whereas the other one is the rotational approach, valid for intermediate times and adequate for the detection of large electric-field amplitudes.

DOI: 10.1103/PhysRevE.81.031128 PACS number(s): 05.70.Ln, 05.40.Jc

I. INTRODUCTION

The first passage time (FPT) characterization remains a topic of great interest in the study far from equilibrium systems where the stochastic fluctuations play an important role, as can be corroborated by the growing number of recent theoretical research on its application in fields as diverse as physics, chemistry, and biology [1–9]. Many of these studies have been formulated within the context of either the Langevin or Fokker-Planck-type equations for Markovian [1–17] and non-Markovian [18–21] processes. In 1989, it was shown by Vemuri and Roy [13] that weak optical signals can be detected via the transient dynamics of a laser much in the same way as the super-regenerative detection in radar receivers. The physical idea behind this proposal is that weak signals are greatly amplified when used to trigger the decay of an unstable state. This fact was immediately supported theoretically by means of the time characterization of such a transient dynamics [14,15] and experimentally corroborated by Littler *et al.* [16]. Also, in the context of the PT distribution, the detection of large optical signals in lasers was studied in Ref. [17]. At the beginning of this century, the proposal of Refs. [14,17] was extended to characterize the decay process of the rotating unstable systems for Gaussian white noise [22] and Gaussian colored noise [23]. Inspired by the works done by Vemuri and Roy [13], Balle *et al.* [14], and Dellunde *et al.* [17] we present in this work an alternative mechanism to detect weak and large amplitudes of an external electric field through the decay process from the unstable state of a two-dimensional bistable potential of a charged Brownian particle in a uniform electromagnetic field. The decay process is used in the same way as in a super-regenerative receiver and takes place in the following way: in the presence of just a magnetic field, pointing along the z axis, the decay process starts when the charged Brownian particle, initially located around the unstable state of a two-dimensional bistable potential $V(x,y)$, leaves this initial state by effect of thermal fluctuations (internal noise). Then it evolves downhill in the potential up to a prescribed reference value representing the potential absorbing barrier, which is taken as a quantity proportional to the steady-state value of the bistable potential. The force responsible of the rotational evolution of the particle is the magnetic force which lies on the x-y plane. If, at the initial unstable state, the particle is additionally subject to the action of an external constant electric field, then the decay process will be accelerated by the corresponding electric force. In the case where the amplitude of the electric field is less or of the same order than the intensity of the internal noise, the decay process is dominated by thermal fluctuations and the dynamical characterization is given through the quasideterministic (QD) approach. In the opposite case, if the amplitude of the electric field dominates over the noise intensity, the decay process is driven by this external field, and the dynamical relaxation is practically deterministic, being then described by another approach here named to as *rotational approach* (RA). Due to the initial presence of thermal fluctuations, the relaxation process is described in the overdamped approximation (diffusive regime) of the Langevin equation. Our theoretical description will be given in a transformed space of coordinates (x',y'), obtained by means of a time-dependent rotation matrix, where both approaches, QD and RA, are better formulated. We will show that, notwithstanding the differences between the charged Brownian particle and the laser system of Refs. [14,17], their dynamical behaviors during the decay process are completely similar. This might in turn motivate

*ines@xanum.uam.mx

†rbm@xanum.uam.mx

experimentalists to execute novel experiments to corroborate our theoretical results.

There exists another physical mechanism (phenomenon) known as stochastic resonance (SR) [24], capable of detecting and transmitting efficiently weak signals information in nonlinear systems due to the presence of random noise. The main characteristic of SR is the stochastic relaxation in modulated bistable systems. The phenomenon shows the role played by noise as it contributes to enhance the response of a bistable system to weak signals. The SR effect requires three basic ingredients: (i) a source of background noise, (ii) a weak coherent input (such as a periodic signal), and (iii) an energetic activation barrier providing a threshold that the system typically has to overcome in order to perform its useful task. This phenomenon was first introduced by Benzi *et al.* [25] and has widely been studied from the experimental and theoretical point of views in a variety of systems ranging from physics [26], chemistry [27], biology, and medicine [28–35]. An important application of SR in biological systems is its ability to enhance the detection of weak signals which improves the biological information processing. There are many biological systems in which SR has been applied. For instance, the mechanoreceptive system in crayfish [30], human tactile sensation [31], human brain system [32], neuron systems [33], ion channels [5,33], and so on. An amount of studies on SR has been done analyzing a paradigmatic system: a one-dimensional bistable potential (double-well), in the context of Langevin and Fokker-Planck equations. However, recent studies show the presence of SR in some more complex biological systems which have been described by a coupled set of Langevin equations [35].

The detection of weak signals, in the decay process of unstable states, is something similar (not equal) to SR process. Both are basically related to stochastic relaxation in nonlinear systems in which the cooperative effect between weak signals and surrounding noise plays a fundamental role. To begin with this line of thought, in this work we study the simple case of a constant electromagnetic field in the detection of weak electric field. However, a SR-like effect can arise when we consider either a weak periodic electric field with fluctuating phase and constant magnetic field or a weak oscillating electromagnetic field. A detailed analysis of these cases will be the purpose of study in future works. Clearly, the model of a Brownian particle in a two-dimensional bistable potential in the presence of weak periodic electromagnetic field admits the description in terms of SR in a similar way as the one given in the one-dimensional case. Certainly, SR has been extended to a large number of applications. However, our proposal which is an alternative SR-like mechanism, opens new roads of investigation. It may serve to explore the detection process of weak signals in some other systems different of lasers as, for example, the stochastic relaxation in a bistable magnetic system [26,36], or that in a single bistable neuron [34], ion channels [5], in which the decay of the unstable state in the presence of weak-electromagnetic field is of practical interest.

The plan of the paper is as follows. In Sec. II, we give a brief description of the studied system as well as of the employed transformation that renders the equations employed in the rest of the manuscript. In Sec. III we study the QD approach and the criteria for the detection of weak intensities of the electric fields. The RA and the corresponding criteria for the detection of strong fields are studied in Sec. IV. Our concluding remarks are given in Sec. V.

II. LANGEVIN EQUATION

The Langevin equation for an ordinary Brownian particle of mass m embedded in a thermal bath with a friction coefficient α in the presence of an arbitrary two-dimensional potential $V(x,y)$ is

$$m\frac{d\mathbf{u}}{dt} = -\alpha\mathbf{u} - \nabla V + \xi(t), \quad (1)$$

where $\mathbf{u}\equiv d\mathbf{r}/dt=(u_x,u_y)$ is the two-dimensional velocity, $\mathbf{r}=(x,y)$ is the position vector, ∇ is the two-dimensional gradient operator, and $\xi(t)$ is the fluctuating force vector $\xi(t)=(\xi_x,\xi_y)$ with zero mean value $\langle\xi_i(t)\rangle=0$ and correlation function $\langle\xi_i(t)\xi_j(t')\rangle=2\lambda\delta_{ij}\delta(t-t')$. λ is the noise intensity which, according to the fluctuation-dissipation relation in the absence of time-dependent external forces, satisfies $\lambda=\alpha k_B T$, with k_B as the Boltzmann constant and T as the temperature of the surrounding medium (the bath). For the particular two-dimensional bistable potential $V(x,y)=-(a/2)(x^2+y^2)+(b/4)(x^2+y^2)^2$ with a, $b>0$, the associated force is $F=-\nabla V=a\mathbf{r}-br^2\mathbf{r}$, where $r^2=x^2+y^2$ is the square modulus of the vector $\mathbf{r}$. Furthermore, if the Brownian particle is electrically charged with charge q and is also under the action of a uniform electromagnetic field, a Lorentz force $F_L=(q/c)\mathbf{u}\times\mathbf{B}+q\mathbf{E}$ will be present. For the case $\mathbf{B}=(0,0,B)$ and $\mathbf{E}=(E_x,E_y)$ the two-dimensional Langevin equation reads as

$$\frac{d\mathbf{u}}{dt} = -\frac{\alpha}{m}\mathbf{u} + W\mathbf{u} + \frac{a}{m}\mathbf{r} - \frac{b}{m}r^2\mathbf{r} + \frac{q}{m}\mathbf{E} + \frac{1}{m}\xi(t), \quad (2)$$

with W a real 2×2 antisymmetric matrix given by

$$W = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \quad (3)$$

and $\Omega=qB/mc$ being the Larmor frequency. In the overdamped approximation the inertial term $m\dot{\mathbf{u}}$ can be neglected and the above Langevin equation reduces to

$$\frac{d\mathbf{r}}{dt} = \tilde{a}\mathbf{r} + \tilde{W}\mathbf{r} - br^2\Lambda\mathbf{r} + q\Lambda\mathbf{E} + \Lambda\xi(t), \quad (4)$$

where $\tilde{W}$ and Λ are matrices given by

$$\tilde{W} = \begin{pmatrix} 0 & \tilde{\Omega} \\ -\tilde{\Omega} & 0 \end{pmatrix}, \quad \Lambda = \frac{1}{\alpha_e}\begin{pmatrix} 1 & C \\ -C & 1 \end{pmatrix}, \quad (5)$$

with $\tilde{a}=a/\alpha_e$, $\tilde{\Omega}=\tilde{a}C$ such that $C=qB/c\alpha$ is a dimensionless parameter, and $\alpha_e=\alpha(1+C^2)$, which is an effective, magnetic-field-dependent friction coefficient. Our scheme will be formulated in the transformed space of coordinates $\mathbf{r}'=e^{-\tilde{W}t}\mathbf{r}$, wherewith Langevin Eq. (2) transforms into

$$\frac{d\mathbf{r}'}{dt} = \tilde{a}\mathbf{r}' - br'^2\Lambda\mathbf{r}' + q\Lambda\mathcal{R}^{-1}(t)\mathbf{E} + \Lambda\mathcal{R}^{-1}(t)\xi(t), \quad (6)$$

where $\mathcal{R}(t)=e^{\tilde{W}t}$ is an orthogonal rotation matrix such that its transpose is also its inverse, i.e., $\mathcal{R}^T(t)=\mathcal{R}^{-1}(t)$, and $\mathcal{R}^{-1}(t)=e^{-\tilde{W}t}$ such that

$$\mathcal{R}(t) = \begin{pmatrix} \cos\tilde{\Omega}t & \sin\tilde{\Omega}t \\ -\sin\tilde{\Omega}t & \cos\tilde{\Omega}t \end{pmatrix}. \quad (7)$$

The third and fourth terms of Eq. (4) represent the time-dependent rotating electric and fluctuating forces respectively. Also $r'^2=x'^2+y'^2$ with $r'^2=r^2$, which means that the modulus of vector $\mathbf{r}$ remains invariant under the transformation $\mathcal{R}^{-1}(t)$.

III. QD APPROACH

We are interested in calculating the probability distribution of the passage time required by the charged particle to reach a prescribed reference value R' in its decay process from the initial unstable state of the bistable potential caused by both the internal fluctuations and the external electric field. In a similar way as done in the laser system of Refs. [14,16,17] the detection process must take place around the initial unstable state of the bistable potential; thus the decay process of the charged particle must stop at a fixed value, representing the absorbing potential's barrier, taken as $R^2=C_0r_{st}^2$, where $0<C_0<1$ and $r_{st}^2=a/b$. The value of C_0 must be determined by the experiment (in the laser system this value is 2% of the intensity steady-sate value). In our case, the steady-state value can be calculated from the deterministic evolution of Eq. (4) without the electric field in terms of the variable $\mathfrak{r}\equiv r^2$, and it satisfies $d\mathfrak{r}/dt=(2\tilde{a}/\mathfrak{r}_{st})\mathfrak{r}(\mathfrak{r}_{st}-\mathfrak{r})$, where $\mathfrak{r}_{st}=a/b$ is the stationary-state value. In a similar way, in the transformed space of coordinates the variable $\mathfrak{r}'\equiv r'^2$ satisfies the deterministic equation $d\mathfrak{r}'/dt=(2\tilde{a}/\mathfrak{r}'_{st})\mathfrak{r}'(\mathfrak{r}'_{st}-\mathfrak{r}')$, and also $\mathfrak{r}'_{st}=a/b$. To calculate the mean first passage time (MFPT) when the strength of the electric field is less or of the same order than the intensity of the noise, we use the formalism of the QD approach [14,22] which relies on the linear approximation of Eq. (6) that reads as

$$\frac{d\mathbf{r}'}{dt} = \tilde{a}\mathbf{r}' + q\Lambda\mathcal{R}^{-1}(t)\mathbf{E} + \Lambda\mathcal{R}^{-1}(t)\xi(t). \quad (8)$$

The solution of this last equation the initial conditions $\mathbf{x}'_0=(x'_0,y'_0)=(0,0)$ is given as

$$\mathbf{x}' = \mathbf{h}'(t)e^{\tilde{a}t}, \quad (9)$$

where

$$\mathbf{h}'(t) = \int_0^t e^{-\tilde{a}s}\Lambda\mathcal{R}^{-1}(s)[q\mathbf{E} + \xi(s)]ds. \quad (10)$$

In terms of its components we have

$$h'_i(t) = \int_0^t e^{-\tilde{a}s}\Lambda_{ij}(\mathcal{R}^{-1})_{jk}(s)[qE_k + \xi_k(s)]ds. \quad (11)$$

We recall that the purpose of the QD approach is to show that stochastic process (9) becomes quasideterministic [12,14,22] in the large time limit such that, for $t\gg 1/2\tilde{a}$, the stochastic process $h'_i(t)$ plays the role of an effective initial condition; i.e., $h'_i(\infty)$ behaves as a Gaussian random variable. That this is indeed the case can be corroborated by noting that, for small noise intensity and weak electric fields, it can be guaranteed that

$$\lim_{t\to\infty}\frac{dh'_i(t)}{dt} = \lim_{t\to\infty} e^{-\tilde{a}t}\Lambda_{ij}(\mathcal{R}^{-1})_{jk}(t)[qE_k + \xi_k(t)] \to 0, \quad (12)$$

and therefore $h'_i(\infty)$ becomes a Gaussian random variable, i.e., $h'_i(\infty)=h'_i$, and therefore in this limit case the stochastic process described by Eq. (9) becomes a quasideterministic one. In terms of the modulus r'^2 it reads as

$$r'^2(t) = h'^2e^{2\tilde{a}t}, \quad (13)$$

where $h'^2=h_1'^2+h_2'^2$. Therefore, the random first passage time required for the charged particle to reach a prescribed reference value R' is

$$t^\star = (1/2\tilde{a})\ln(R'^2/h'^2). \quad (14)$$

This random passage time requires of the marginal probability density $P(h')$ which can be calculated from the joint probability density given by the Gaussian distribution [37]

$$P(h'_1,h'_2) = N\exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}(\sigma^{-1})_{ij}(h'_i - \langle h'_i\rangle)(h'_j - \langle h'_j\rangle)\right], \quad (15)$$

where $N=1/2\pi(\det\sigma_{ij})^{1/2}$ is the normalization factor and $\sigma_{ij}=\langle h'_ih'_j\rangle-\langle h'_i\rangle\langle h'_j\rangle$ the correlation matrix. From Eq. (11) we have

$$\langle h'_i\rangle = q\int_0^\infty e^{-\tilde{a}s}\Lambda_{ik}(\mathcal{R}^{-1})_{kl}(s)E_lds. \quad (16)$$

$$\langle h'_ih'_j\rangle = \langle h'_i\rangle\langle h'_j\rangle + \int_0^\infty\int_0^\infty e^{-\tilde{a}(s+s')}\Lambda_{ik}\Lambda_{jl}(\mathcal{R}^{-1})_{km}(s) \times(\mathcal{R}^{-1})_{ln}(s')\langle\xi_m(s)\xi_n(s')\rangle dsds'. \quad (17)$$

After some algebra Eq. (17) reduces to $\langle h'_ih'_j\rangle=\langle h'_i\rangle\langle h'_j\rangle+(\lambda/\alpha a)\delta_{ij}$, which tells us that the variables h'_i are independent and that $\sigma_{ij}=(\lambda/\alpha a)\delta_{ij}$ is a diagonal matrix with elements $\sigma_{ii}\equiv\sigma^2=\lambda/\alpha a$. Therefore, joint probability density (15) now reduces to

$$P(h'_1,h'_2) = \frac{1}{2\pi\sigma^2}e^{-(1/2\sigma^2)[(h'_1-\langle h'_1\rangle)^2+(h'_2-\langle h'_2\rangle)^2]}. \quad (18)$$

The mean values $\langle h'_i\rangle$ can be calculated by assuming, without loss of generality, that $\mathbf{E}=(E,E)/\sqrt{2}$, with E being the modulus of this vector. In this case it is easy to show that $\langle h'_1\rangle$

$=\langle h_2'\rangle = qE/\sqrt{2}a$. The marginal probability density $P(h')$ can be calculated using the Jacobian transformation $dV=J(\mathbf{v})d\mathbf{v}$, where $\mathbf{v}=(h,\theta)$ is the new space of variables. In this case $dV=Ch'dh'$, with C as a constant. After some algebra we get [14,22]

$$P(h') = (h'/\sigma^2)I_0(p'h'/\sigma^2)e^{-(1/2\sigma^2)(h'^2+p'^2)}, \tag{19}$$

where $p'^2=\langle h_1'\rangle^2+\langle h_2'\rangle^2=(qE)^2/a^2$ and $I_0(x)$ is the modified Bessel's function of zeroth order [38]. The statistical properties of first passage time (FPT) distribution can be calculated through the moment generating function defined as $G(2\tilde{a}\lambda) \equiv \langle e^{-2\tilde{a}\lambda t}\rangle$, and thus $G(2\tilde{a}\lambda)=\langle (R'^2/h'^2)^{-\lambda}\rangle$. This generating function is calculated from the marginal probability $P(h')$ given by Eq. (19), giving as a result [22]

$$G(2\tilde{a}\lambda) = (R'^2/\sigma^2)^{-\lambda}e^{-\beta'^2}\sum_{m=0}^{\infty}\frac{\Gamma(m+\lambda+1)}{(m!)^2}\beta'^{2m} = G_0(2\tilde{a}\lambda)e^{-\beta'^2}M(\lambda+1,1,\beta'^2), \tag{20}$$

with $G_0(2\tilde{a}\lambda)=(R'^2/\sigma^2)^{-\lambda}\Gamma(\lambda+1)$. The moment generating function in the absence of the external electric field, $M(\lambda+1,1,\beta'^2)$, is the Kummer confluent hypergeometric function [38] with $\beta'^2=p'^2/2\sigma^2=\alpha(qE)^2/2a\lambda$. The MFPT is then calculated from $\langle 2\tilde{a}t^\star\rangle=[-dG(2\tilde{a}\lambda)/d\lambda]_{\lambda=0}$ and, after some algebra, leads to

$$\langle 2\tilde{a}t^\star\rangle = \langle 2\tilde{a}t^\star\rangle_0 + \sum_{m=1}^{\infty}\frac{(-1)^m\beta'^{2m}}{mm!}, \tag{21}$$

where

$$\langle 2\tilde{a}t^\star\rangle_0 = \ln(aR'^2/2\lambda) - \psi(1), \tag{22}$$

is the MFPT in the absence of the external electric field ($\beta'=0$) and $\psi(1)=-\gamma=-0.577$ the Euler constant. The variance of the passage time distribution, defined as $\langle(\Delta t^\star)^2\rangle = \langle t^{\star 2}\rangle - \langle t^\star\rangle^2$, can be calculated from $\langle(2\tilde{a}t^\star)^2\rangle = [-d^2G(2\tilde{a}\lambda)/d\lambda^2]_{\lambda=0}$, and, again after some algebra, we can show that [22]

$$\langle(2\tilde{a}\Delta t^\star)^2\rangle = \psi'(1) + 2\sum_{m=2}^{\infty}\left(\sum_{k=1}^{m-1}\frac{1}{k}\right)\frac{(-1)^m\beta'^{2m}}{mm!} - \left[\sum_{m=1}^{\infty}\frac{(-1)^m\beta'^{2m}}{mm!}\right]^2. \tag{23}$$

For practical purposes, if the parameter satisfies $\beta'^2 \le 1$, which implies that the strength of the electric field is less or of the same order than the intensity of the noise, the MFPT and its variance can be approximated by

$$\langle 2\tilde{a}t^\star\rangle = \langle 2\tilde{a}t\rangle_0 - \beta'^2 + (\beta'^4/4), \tag{24}$$

$$\langle(2\tilde{a}\Delta t^\star)^2\rangle = \psi'(1) - (\beta'^4/2), \tag{25}$$

Detection of weak electric fields

As we can see, results (24) and (25) are very similar to Eqs. (22) and (26) of Ref. [14], if $\beta'=\beta/2$. For the detection

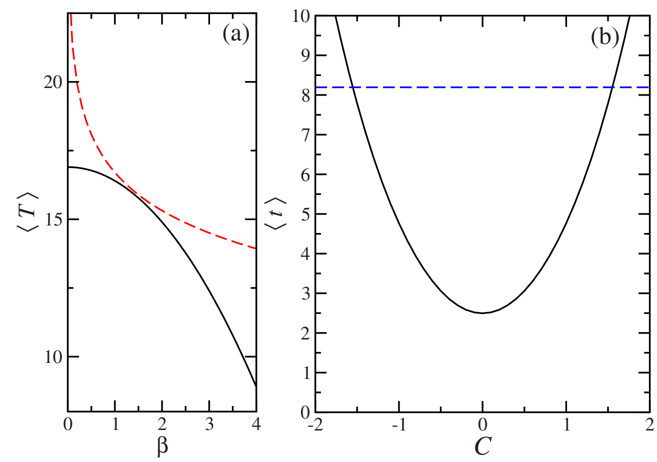

FIG. 1. (Color online) (a) Results for the mean passage time $\langle T\rangle=\langle 2\tilde{a}t^\star\rangle$ as a function of β obtained from Eq. (24) [solid line] and from the deterministic solution [Eq. (27)] [dashed line] for system parameters such that $2a\alpha R'^2/\lambda=1.8\times 10^7$. (b) Determination of the bandwidth detection for large amplitude of the electric field. The solid line is the third iteration of Eq. (38), with $a=300$, $\alpha=270$. The dashed corresponds to Eq. (22) for $C=0$. The range of intersection is $C\approx 3.0$.

of weak amplitudes of the electric field we can use the same criteria used in Ref. [14], which establishes that the difference between the time scales in the presence and in the absence of the electric field is greater or equal than the maximum variance; that is,

$$[\langle t^\star\rangle_{\beta_c'} - \langle t^\star\rangle_{\beta'=0}]^2 \ge \langle(\Delta t^\star)^2\rangle_{\beta'=0}. \tag{26}$$

Using Eqs. (24) and (25) we obtain the value of $\beta_c' \ge [\psi'(1)]^{1/4}\approx 1.13$, which is the critical value for which the weak electric field can be detected. However, for a rescaled parameter $\beta'=\beta/2$ and thus employing again the above equations, we get $\beta_c \ge [4\psi'(1)]^{1/4}\approx 1.6$, which is the same value calculated in Ref. [14]. Below this value the electric field cannot be detected. On the other hand, if $\beta'>\beta_c'$, the amplitude of the electric field can be efficiently detected because it dominates over the noise intensity and the dynamics is dominated by the deterministic evolution. In this case it can be shown that the MFPT and its variance are approximated by [14,22]

$$\langle 2\tilde{a}t^\star\rangle \approx \ln(aR'/qE)^2 \tag{27}$$

$$\langle(\Delta t^\star)^2\rangle \approx 1/2\tilde{a}^2\beta'^2 \tag{28}$$

In Fig. 1(a) we show that the critical value of $\beta_c\approx 1.6$ corresponds to the match between the two approximations [Eqs. (24) and (27)]. Here it is important to comment that Eqs. (24), (25), (27), and (28) have been derived from the QD approach. However, they are not actually appropriate to describe the rotational effects appearing in a natural way in the dynamical evolution of the charged particle in a magnetic field [see Eq. (4)] along its decay process to reach the value R'. This process is better understood in the transformed space of coordinates $\mathbf{r}'=(x',y')$, where the trajectory of the charged particle can be seen as rotational or not, depending

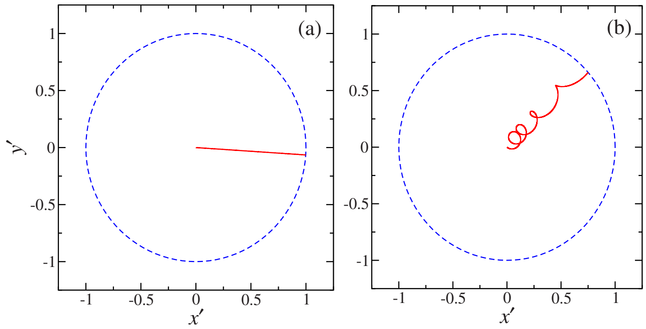

FIG. 2. (Color online) Dynamical evolution of a single trajectory of the system given by linear approximation of Eq. (8) in the (x',y') space for values $a=300$, $\alpha=270$, $C=10$, $R'=1.0$, (a) $qE=10^{-3}$ and (b) $qE=20.0$.

on which force, electric or fluctuating, is greater. In fact, if $\beta' \leq 1$, it is shown in Fig. 2(a) that the trajectory is practically a straight line, with no rotational effect associated whatsoever. This is the reason why the QD approach is better understood in the transformed space of coordinates.

If $\beta' \gg 1$, the strength of the electric field is greater than the intensity of the noise and the dynamical evolution of the charged Brownian particle is rotational, as can be seen in Fig. 2(b). In this case, for both the dynamical characterization of this decay process and the efficient detection of the strong electric fields, we follow the RA studied in Refs. [17,22].

IV. ROTATIONAL APPROACH

To characterize the decay process from the unstable state, the statistics of the FPT distribution will be formulated in the limit of intermediate times. We also start from Eq. (9), but now, in the herein considered case, the random passage time $t^\star$ required to reach the R' value is

$$t^\star = (1/2\tilde{a})\ln(R'^2/h'^2(t^\star)) \qquad (29)$$

where $h'_i(t)$ is the same as Eq. (10). As we can see from Eq. (29), the random passage time is not easy to calculate because $h'(t)$ is also a function of time t. Therefore, for intermediate times, we take advantage of the statistical properties of the process $\mathbf{h}'(t)$, in such a way that

$$\langle h'_i(t)\rangle = q\int_0^t e^{-\tilde{a}s}\Lambda_{ik}(\mathcal{R}^{-1})_{kl}(s)E_l ds, \qquad (30)$$

$$\langle h'_i(t)h'_j(t)\rangle = \langle h'_i(t)\rangle\langle h'_j(t)\rangle + \frac{\lambda}{\alpha\tilde{a}}(1-e^{-\tilde{a}t})\delta_{ij}. \qquad (31)$$

To solve the problem we propose an alternative expression for $h'_i(t)$ compatible with Eqs. (30) and (31). This expression is [17,22]

$$h'_i(t) = \langle h'_i(t)\rangle + g(t)\eta_i, \qquad (32)$$

where $g^2(t)=(1-e^{-2\tilde{a}t})$ and η_i is a Gaussian random variable with zero mean value and correlation $\langle\eta_i\eta_j\rangle=(\lambda/\alpha a)\delta_{ij}$. Now, if the strength of the electric field dominates over the noise intensity, the first term of the right-hand side of Eq. (32) is the dominant one. Therefore, we can make a first-order series expansion in powers of η_i in Eq. (29) such that the passage time can be approximated by

$$t^\star = t_P - \frac{g(t_P)}{\tilde{a}}(t_1\eta_1 + t_2\eta_2) + \mathcal{O}(\eta_1^2,\eta_2^2), \qquad (33)$$

where t_P is the zeroth order passage time given by $t_P=(1/2\tilde{a})\ln(R'^2/|\langle\mathbf{h}'(t_P)\rangle|^2)$, with $t_i=\langle h'_i(t_P)\rangle/|\langle\mathbf{h}'(t_P)\rangle|^2$ and $|\langle\mathbf{h}'(t_P)\rangle|^2=\langle h'_1(t_P)\rangle^2+\langle h'_2(t_P)\rangle^2$. In this case the variance is also approximated by

$$\langle(\Delta t^\star)^2\rangle = \frac{\lambda}{\alpha a}\frac{g^2(t_P)}{\tilde{a}^2}(t_1^2+t_2^2). \qquad (34)$$

From Eq. (33), the mean passage time reads as

$$\langle t^\star\rangle = t_P = (1/2\tilde{a})\ln(R'^2/|\langle\mathbf{h}'(t_P)\rangle|^2). \qquad (35)$$

We again consider the electric field as $\mathbf{E}=(E,E)/\sqrt{2}$ and thus, after some algebra, we can show from Eq. (30) that $\langle h'_1(t)\rangle=A-iB$ and $\langle h'_2(t)\rangle=A+iB$, where $A=z(t)+z^*(t)$, $B=z(t)-z^*(t)$ and

$$z(t) = \frac{qE}{2\sqrt{2}a}(1-e^{-\tilde{\lambda}_1 t}), \quad z^*(t) = \frac{qE}{2\sqrt{2}a}(1-e^{-\tilde{\lambda}_2 t}), \qquad (36)$$

such that $\tilde{\lambda}_1=\tilde{a}+i\tilde{\Omega}$ and $\tilde{\lambda}_2=\tilde{a}-i\tilde{\Omega}$. Under these conditions it can be shown that

$$|\langle\mathbf{h}'(t)\rangle|^2 = 8z(t)z^*(t) = \frac{(qE)^2}{a^2}[1+\phi(t)], \qquad (37)$$

$\phi(t)$ being $\phi(t)=[e^{-2\tilde{a}t}-2e^{-\tilde{a}t}\cos(\tilde{\Omega}t)]$. Therefore, the MFPT required by the charged particle to reach the potential's barrier (circle or radius R'), as shown in Fig. 2(b), is approximated by

$$\langle t^\star\rangle = t_P = t_0 - \frac{1}{2\tilde{a}}\ln[1+\phi(t_P)], \qquad (38)$$

where $t_0=(1/2\tilde{a})\ln(aR'/qE)^2$. For large amplitudes of the electric field, i.e., $\beta' \gg 1$, it is possible to obtain, for the variance, the following expression:

$$\langle(\Delta t^\star)^2\rangle = \frac{g(t_P)}{2\tilde{a}^2\beta'^2[1+\phi(t_P)]}\left[1+\frac{\phi'(t_P)}{\tilde{a}[1+\phi(t_P)]}\right]^{-2}. \qquad (39)$$

Equations (38) and (39) satisfy the necessary requirements in the infinite-time limit. In this case $\phi(t)\to 0$, $\langle t^\star\rangle\to t_P$, and the variance vanishes as $\langle(\Delta t^\star)^2\rangle\to 1/2(\tilde{a}^2\beta'^2)$, consistently with Eqs. (27) and (28) as expected. The MFPT and its variance can be calculated through the iterative procedure $t_P^{(0)}=t_P$ and $t_P^{n+1}=t_P-(1/2\tilde{a})\ln[1+\phi(t_P^n)]$. The plot of the MFPT [Eq. (38)] and its variance Eq. (39), as well as the results given by Eqs. (27) and (28) together with the values computed from numerical simulations, is presented in Fig. 3. As we can see, we obtain an excellent agreement.

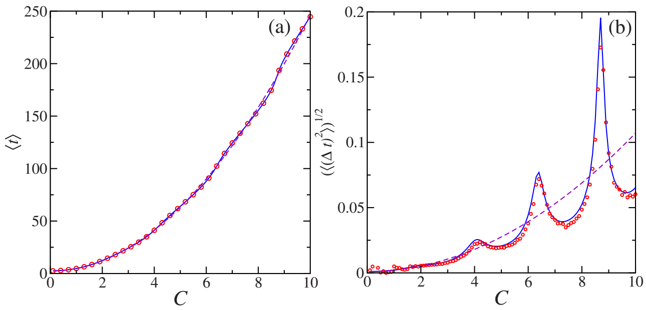

FIG. 3. (Color online) (a) MFPT and (b) variance as a function of dimensionless parameter C; same a, α, R, and qE values as in Fig. 2(b). The solid line corresponds to (a) the third iteration of Eq. (38) and to (b) the analytical result Eq. (39); circles are the simulation results. The dashed lines in (b) the variance given in Eq. (28).

Detection of large electric fields

The bandwidth detection can evaluated from Fig. 1(b), in a similar way as done in Refs. [17,22]. The solid line is the third iteration of Eq. (38), with $a=300$ s^{-2}, and $\alpha=270$ s^{-1}. The dashed line is the value of Eq. (24) in the absence of both the electric and magnetic fields ($C=0$). The intersection range is ≈ 3.0. Due to the number of involved parameters, their complicated interdependence, and the employed values, we obtain the bandwidth detection range in the order of Hz, instead of MHz as in the laser system [17]. One possible value for this bandwidth detection is obtained from $\tilde{\Omega} = aC/\alpha(1+C^2)$, which in this case we get $\tilde{\Omega}\approx 0.3$ Hz. We can see that, as the dashed line descends, the intersection range diminishes but $\tilde{\Omega}$ increases.

V. CONCLUSIONS

We have shown through the statistics of the FPT distribution that it is possible to detect weak and large amplitudes of an electric field by means of the decay process from the unstable state of a charged Brownian particle embedded in a uniform electromagnetic field. The decay process plays the role of a super-regenerative receiver that amplifies the external signal. For the detection of weak electric fields, we have found the same critical value $\beta_c \approx 1.6$ as that calculated in Ref. [14] for the laser system with $\beta' = \beta/2$. This is a surprising result due to the fact that the charged Brownian particle and the laser system are physically different systems. Equally surprising are the results obtained in the rotational characterization of the decay process of the Brownian particle, because they are very similar to those obtained in Ref. [17], although Eqs. (38) and (39) are slightly more complicated than those employed in the aforementioned reference due to the fact that the $\tilde{a}$ and $\tilde{\Omega}$ parameters are given in terms of the other parameters a, α, and $C=qB/c\alpha$. This is the reason why the bandwidth detection range is different in both systems.

Lastly, we would like to comment that the complete dynamical characterization, through the so-called nonlinear relaxation times, taking into account the nonlinear effects of the two-dimensional bistable potential is in progress.

ACKNOWLEDGMENTS

The authors wish to thanks R. M. Velasco for her valuable suggestions to improving our manuscript. We also thanks J. M. Sancho and L. Ramírez de la Piscina for their comments about our work.

[1] S. Condamin, O. Bénichou, V. Tejedor, R. Voituriez, and J. Klafter, Nature (London) **450**, 77 (2007).
[2] Vincent J. van Hijkoop, A. J. Dammers, K. Malek, and M. O. Coppens, J. Chem. Phys. **127**, 085101 (2007).
[3] C. L. Rhonald and A. Y. Grosberg, Phys. Rev. E **72**, 061918 (2005).
[4] G. Lakatos, T. Chou, B. Bergersen, and G. N. Patey, Phys. Biol. **2**, 166 (2005).
[5] I. Goychuk and P. Hänggi, Proc. Natl. Acad. Sci. U.S.A. **99**, 3552 (2002).
[6] H. Scher and E. Montroll, Phys. Rev. B **12**, 2455 (1975).
[7] M. Weiss, M. Elsner, F. Kartberg, and T. Nilsson, Biophys. J. **87**, 3518 (2004).
[8] I. Golding and E. C. Cox, Phys. Rev. Lett. **96**, 098102 (2006).
[9] H. Scher, G. Margolin, R. Metzler, J. Klafter, and B. Berkowitz, Geophys. Res. Lett. **29**, 1061 (2002).
[10] F. Moss and P. V. McClintock, *Noise in Nonlinear Dynamical Systems* (Cambridge University Press, Cambridge, England, 1989).
[11] F. Haake, Phys. Rev. Lett. **41**, 1685 (1978).
[12] F. de Pasquale, J. M. Sancho, M. San Miguel, and P. Tartaglia, Phys. Rev. Lett. **56**, 2473 (1986).
[13] G. Vemuri and R. Roy, Phys. Rev. A **39**, 2539 (1989).
[14] S. Balle, F. De Pasquale, and M. San Miguel, Phys. Rev. A **41**, 5012 (1990).
[15] J. I. Jiménez Aquino and J. M. Sancho, Phys. Rev. A **43**, 589 (1991).
[16] I. Littler, S. Balle, K. Bergmann, G. Vemuri, and R. Roy, Phys. Rev. A **41**, 4131 (1990).
[17] J. Dellunde, M. C. Torrent, and J. M. Sancho, Opt. Commun. **102**, 277 (1993).
[18] P. Hanggi and P. Talkner, Phys. Rev. A **32**, 1934 (1985).
[19] J. M. Sancho and M. San Miguel, Phys. Rev. A **39**, 2722 (1989).
[20] F. J. de la Rubia, E. Peacock-López, G. P. Tsironis, K. Lindenberg, L. Ramírez-Piscina, and J. M. Sancho, Phys. Rev. A **38**, 3827 (1988).
[21] J. Casademunt, J. I. Jiménez-Aquino, and J. M. Sancho, Phys. Rev. A **40**, 5905 (1989).
[22] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E **66**, 061101 (2002).
[23] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E **69**, 062101 (2004).
[24] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998); P. Jung and P. Hänggi, Phys. Rev. A **44**, 8032 (1991); B. McNamara and K. Wiesenfeld, *ibid.*

39, 4854 (1989).
[25] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981).
[26] A. N. Grigorenko, P. I. Nikitin, and G. V. Roshchepkin, Pis'ma Zh. Eksp. Teor. Fiz. **65**, 788 (1997); A. N. Grigorenko and P. I. Nikitin, IEEE Trans. Magn. **31**, 2491 (1995); A. Pérez-Madrid and J. M. Rubí, Phys. Rev. E **51**, 4159 (1995).
[27] L. F. Yang, Z. H. Hou, and H. W. Xin, J. Chem. Phys. **110**, 3591 (1999); K. Miyakawa, T. Tanaka, and H. Isikawa, Phys. Rev. E **67**, 066206 (2003).
[28] K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995); K. Wiesenfeld and F. Jaramillo, Chaos **8**, 539 (1998).
[29] S. M. Bezrukov and I. Vodyanoy, Nature (London) **378**, 362 (1995); I. Goychuk and P. Hänggi, Phys. Rev. E **61**, 4272 (2000).
[30] X. Pei, L. Wilkens, and F. Moss, Neurophysiology **76**, 3002 (1996); J. K. Douglass, L. Wilkens, E. Pantazelou, and F. Moss, Nature (London) **365**, 337 (1993).
[31] J. J. Collins, T. T. Imhoff, and P. Grigg, Nature (London) **383**, 770 (1996).
[32] K. Kitajo, D. Nozaki, L. M. Ward, and Y. Yamamoto, Phys. Rev. Lett. **90**, 218103 (2003).
[33] T. A. Engel, B. Helbig, D. F. Russell, L. Schimansky-Geier, and A. B. Neiman, Phys. Rev. E **80**, 021919 (2009); J. J. R. Duarte, M. V. D. Vermelho, and M. L. Lyra, Physica A **387**, 1446 (2008); D. Nozaki, D. J. Mar, P. Grigg, and J. J. Collins, Phys. Rev. Lett. **82**, 2402 (1999).
[34] D. Jianghai, S. Aiguo, W. Yiqing, and W. Xiulan, *IEEE International Conference on Neural Networks and Signal Processing*, Nanjing, China, December 14–17 (IEEE, Piscataway, NJ, 2003) Vol. 1, pp. 193–196.
[35] H. Li, J. Ma, Z. Hou, and H. Xin, Acta Physico-Chimica Sinica **24**, 2203 (2008); H. Li, Z. Hou, and H. Xin, Chem. Phys. Lett. **402**, 444 (2005).
[36] I. Klik and L. Gunther, J. Stat. Phys. **60**, 473 (1990).
[37] N. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (Elsevier, Amsterdam, 2007).
[38] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).