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Localized cyclotron mode driven by fast α particles under a nonuniform magnetic field

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Resonance requires precise synchronization. Surprisingly, relativistic cyclotron instability can survive under a magnetic field with its nonuniformity larger than the requirement of synchronism. Localized eigenmode observed in a hybrid simulation is found to be consistent with that predicted by an analytical theory including both profile and eigenvalue. Half of the spatial area of the wave profile is located where the frequency mismatch is negative as against to the positive requirement generally believed. The consequence on the α dynamics is also demonstrated.

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Resonance is a fundamental issue in science with wide applications and requires precise synchronization [1–5]. For a harmonic oscillation with damping, the width of the frequency mismatch between the driving force and the oscillator is required to be smaller than the damping rate in order to remain in resonance [1]. For cyclotron resonance, the wave frequency has to be very close to the harmonic frequency of charge particle's gyromotion under magnetic field [2–7]. Cyclotron instability through the resonance involving relativistic mass variation effect has been studied for half a century [2–16]. The cyclotron maser instability [2–5] is the most important mechanism responsible for generating state-of-the-art high power microwave source with critical applications in space, defense, and industry. In order to drive the instability, it is well known [2–5] that the precise synchronism has to be stably maintained and the frequency mismatch between the wave and the harmonic cyclotron frequency, ω_c, of the charge particles is required to be positive [2–5]. While the charge particles are losing their kinetic energy to the wave, their Lorentz factor decreases, their harmonic cyclotron frequency increases, the frequency mismatch decreases, and thus the resonance is enhanced to sustain the wave growth and the instability [2–4].

Fusion produced fast ions are the only direct source to reheat the plasmas to maintain the burning [17–19]. The fast α particles produced by the fusion reaction of deuterium and tritium have an energy of 3.5 MeV corresponding to a Lorentz factor of γ=1.000 94. It is interesting to find that the small factor of relativistic mass variation is capable to drive harmonic cyclotron instabilities [11–16]. The relativistic cyclotron instabilities can significantly affect fast ions' dynamics [20–24]. Moreover, the instability provides an explanation [25] for experimentally observed ion cyclotron emission in JET(Joint European Torus) [26–31] and causes the selective gyrobroadening [20–24] of α energy spectrum measured in Princeton's TFTR (Tokamak Fusion Test Reactor) [32,33]. However, the magnetic field in tokamak device is not uniform. The nonuniformity is usually much larger than the rate of wave growth in homogenous plasmas that shall be larger than the rate of damping. Thus, the relativistic effect is believed [16] to be minor in tokamak as echoed by the belief [10] that resonance synchronism can be easily destroyed by inhomogeneity.

In this paper, we will show the simulation and analytical results that impact our understanding of resonance and the requirement of frequency mismatch. In contrast to conventional wisdom [16], localized eigenmodes of cyclotron waves [34] are excited at the minimum of the magnetic field which is taken to be sinusoidally nonuniform. One of the possibilities is that resonance and the resultant structure of wave mode is a consequence of the need to drive instability for dissipating free energy and increasing the entropy. We even find that the wave mode can exist at where the wave eigenfrequency is lower than the local harmonic cyclotron frequency. An eigenmode theory derived is found to be consistent with the simulation results.

A one-spatial-and-three-momentum-dimension particle-in-cell code was developed to study the dynamics of the α particles and the resultant relativistic electrostatic ion cyclotron instability. The system is in the x direction with its periodical length of L=4096 Δx while the external magnetic field is in the z direction as $B=B_0[1+\delta\sin(2\pi x/L)]$, where the cell size Δx=0.043 cm, B_0=5 T, δ=0.01, and the minimum B is at the cell of 3072. To reduce the numerical noise, the quiet-start technique [35,36] was used for isotropic monoenergetic particle α with an initial maximum gyroradius of ρ_α=125 Δx at B_0 (where the subscript α is for the α particle), and for 5 keV particle deuterons that may also be described by fluid scheme; while the electrons are treated as a dielectric neutralizing background. The α density is 2×10^9 cm^{-3} and the deuteron density is 10^{13} cm^{-3}. For the particle deuterons case, the number of α particles is 5 038 848 and that of deu-

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terons is 11 390 625. On the other hand, for the fluid deuterium case, the number of α particles is 10 077 696. Besides, the unit time is $t_0=\omega_{c\alpha}^{-1}$ (where $\omega_{c\alpha}$ is at B_0) and the time step is 0.025. The wave number is $k=2\pi m/L$, where m is the mode number.

A previous theory [37] also indicates the instability can survive a nonuniform magnetic field and shows a localized wave mode. The corresponding characteristics and physics of the instability and the localized wave mode are, however, different from the present results. The gyrokinetic theory derived an integral dispersion relation and solved it numerically in k space. The eigenfrequency is significantly lower than the local harmonic cyclotron frequency of the α particles at the magnetic minimum; i.e., the resonant condition cannot be satisfied for driving a relativistic instability [2]. Also, the localized wave mode has a different profile and may correspond to a different class of wave modes other than an absolute instability needed. Thus, an analytical theory based on the absolute instability condition is needed in order to obtain correct results consistent with the simulation and to explain the characteristics and the interesting resonance physics relevant to the observed localized eigenmode.

Consider the dispersion relation for a uniform magnetic field as $D\ (\omega,k)=0$ and for a nonuniform magnetic field as $D\ (\omega,k,x)\ \boldsymbol{\phi}(x)=0$. Let $\boldsymbol{\phi}(x)=\phi(x)\exp(ik_*x)$, where k_* represents the spatial fast-varying part of the wave function. By employing the two-scale-length expansions [38] with $\omega=\omega_*+\delta\omega$, $k=k_*-i\partial_x$, and $x=x_0+\delta x$, the corresponding eigenmode equation becomes

$$[D(\omega_*,k_*,x_0)+Q(\delta\omega)+\partial_k D(-i\partial_x)+1/2\partial_k^2 D(-i\partial_x)^2 + \partial_x D\delta x + 1/2\partial_x^2 D\delta x^2 + \partial_{\omega k}^2 D\delta\omega(-i\partial_x) + \partial_{kx}^2 D\delta x(-i\partial_x) + \partial_{\omega x}^2 D\delta\omega\delta x]\phi(x)=0, \quad (1)$$

where $Q(\delta\omega)=\partial_\omega D\delta\omega+\partial_\omega^2 D\delta\omega^2/2+\ldots$higher orders; all terms are at $\omega=\omega_*$, $k=k_*$, and $x=x_0$.

To investigate the localized modes, we make a parabolic approximation of the magnetic field at the minimum as $B(x)=B_0[1+\varepsilon_b(x-x_0)^2/2]$, where ε_b denotes the nonuniformity; for the simulation case of $\delta=0.01$, ε_b is 3.71×10^{-4}. Choose $x_0=0$ at the cell of 3072 for simplicity; all terms associated with the first order derivative to x_0 will vanish.

We further request k_* satisfying the absolute instability condition of $\partial_k D=0$ [38], which implies that the group velocity is zero. Also, in order to remove the term related to $\partial_{\omega k}^2 D$, let $\phi(x)=\Phi(x)\exp(ik_1x)$ and $k_1=-\partial_{\omega k}^2 D/\partial_k^2 D\delta\omega$. After some algebra, Eq. (1) can be transformed to a parabolic cylinder equation [39] as

$$\partial_t^2\Phi-(t^2/4-E)\Phi=0. \quad (2)$$

Then, we obtain the eigenvalue of $E=E_n=n+1/2=-Q(\delta\omega)/(\beta^2\partial_k^2 D)$ and the corresponding eigenfunction of $\Phi(t)=\Phi_n(t)=\mathrm{He}_n(t)\exp(-t^2/4)$, where $\beta^4\equiv\partial_x^2 D/\partial_k^2 D$, $t\equiv 2^{1/2}\beta x$, and the $\mathrm{He}_n(t)$ is the modified Hermite polynomial of order n, where the rank of the eigenvalue is $n=0,1,2,\ldots$. During the algebra, we had also neglected the term of $(\partial_{\omega k}^2 D)^2/\partial_k^2 D\delta\omega^2$ which should be much smaller than the term of $\partial_\omega^2 D\delta\omega^2$ in $Q(\delta\omega)$ when the charge particles strongly resonate with the wave. For localized solutions, Φ should vanish while away from x_0. Hence, only the solutions satisfying $\mathrm{Re}(t^2)>0$ are chosen to make sure the localization.

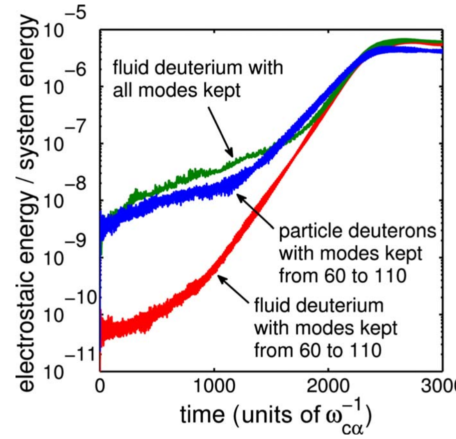

FIG. 1. (Color online) Histories of the total field energy from the simulation cases of particle deuterons with wave modes m kept from 60 to 110, fluid deuterium with all wave modes kept, and fluid deuterium with wave modes m kept from 60 to 110, respectively, where $m=kL/2\pi$.

The waves can propagate in both positive and negative x directions. Therefore, with $k_0=k_*+k_1$, the wave function can be written as $\psi(x)=\Phi(x)[\exp(ik_0x)+\exp(-ik_0x)]/2=\Phi(x)\cos(k_0x)$, which is a symmetric and localized solution.

To study the behavior of instabilities under nonuniform magnetic field, we use the dispersion relation derived for a uniform B [11] and follow the procedures described above. From the dispersion relation $D(\omega,k)=0$, ω can be determined from k numerically with Muller's method. There are several solutions satisfying the $\partial_k D=0$ condition; the mode with a highest growth rate is of interest. The corresponding wave function is then calculated.

Figure 1 shows the histories of the field energy from the simulations. While the deuterons are treated as particles and all the wave modes are kept, the system is too noisy. By reducing the wave modes kept ($110\geq m\geq 60$; i.e., $21.09\geq k\geq 11.50$) or treating the deuterium as a fluid, the wave grows with some noises. With both, the wave grows exponentially from a low noise and then saturates at a similar level as other cases. The normalized growth rate is 3.3×10^{-3}.

A snapshot of the field profile in real space near the minimum B is shown in Fig. 2. A twin-wavelet structure is clearly formed; the width at each side is about four times the α's initial maximum gyroradius. Note that the variation in the normalized frequency mismatch across the wave profile is about 3.5×10^{-2} that is much larger than the normalized damping rate of 1.4×10^{-3}, estimated from the difference between the normalized growth rates in the homogenous plasma and in the nonuniform B, and even the normalized growth rate of 4.7×10^{-3} in the homogenous plasma. The relativistic effect cannot satisfy the requirement of synchronism [1,2,10] for oscillation resonance with damping (or growth). However, our results show that the relativistic instabilities can still survive and play an important role in this

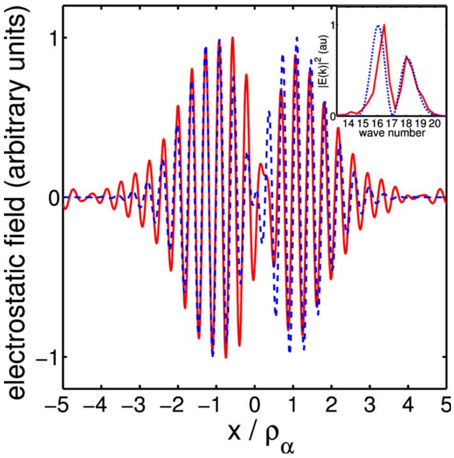

FIG. 2. (Color online) The electrostatic field at the simulation time of 1900 (red solid curve) during the linear wave growth for the simulation case of fluid deuterium and m from 60 to 110 as well as the wave function predicted by the analytical theory (blue dashed curve). The inserted figure shows the k space profiles of the electrostatic field snapshot from the simulation (red solid curve) and the analytical wave function (blue dashed curve).

system. The wave function of the rank $n=1$ obtained from the theory is also plotted in Fig. 2 and is in a good agreement with the simulation result in terms of the overall shape and its width. The theoretical normalized growth rate of 2.54×10^{-3} agrees with that of the simulation (e.g., 3.3×10^{-3}) as well. The figure inserted in Fig. 2 shows the theoretical and simulation profiles in the k space. As also agreeing, both show that there are two peaks located near 16 and 18, respectively, while the growing k modes of the spectrum are from 14 to 20; these k modes have been included in the simulation case of wave modes limited.

Figure 3 shows the power spectrum of the wave frequency and the amplitude vs the position. Within the area of the twin-wavelet structure, the wave frequency measured at

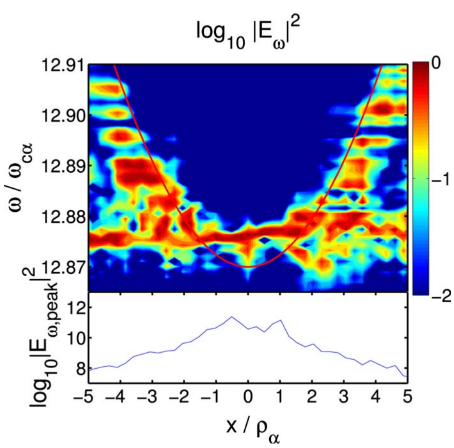

FIG. 3. (Color online) The power spectrum in log scale vs the position. The upper panel is for the normalized wave frequency; at each position, the power intensity is normalized to the corresponding peak value that is shown in the lower panel. The red curve is for the local relativistic 13-harmonic cyclotron frequency of the α particles at initial.

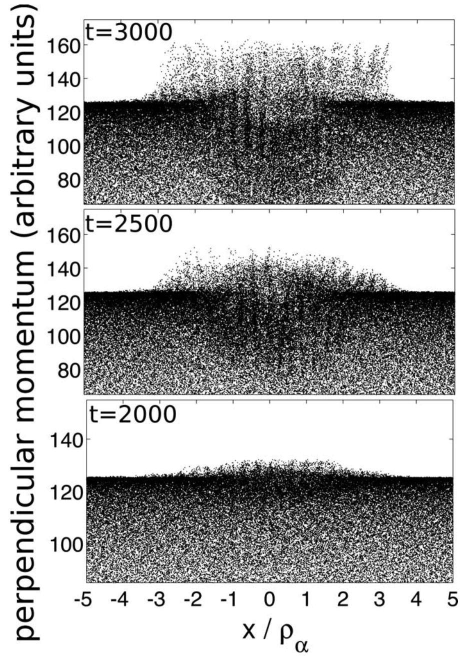

FIG. 4. Scatter plots of the α particles in the phase space of perpendicular momentum vs position at $t=2000$, 2500, and 3000, respectively.

different position is almost as a constant of 12.875 that is close to 12.8705 from the analytical theory. This evidences that the localized wave mode is an eigenmode predicted by the theory. The wave profile is symmetrical and its width at both sides is about four α Larmor radiuses. This may be related to the fact that the excursion of one magnetized charge particle is two Larmor radiuses. Moreover, while the frequency mismatch in the central region is positive as required [2–5] by the relativistic cyclotron instabilities, the mismatch at outer region from both the simulation and the theory is negative as against the well-known resonant requirement. The peaks of the wave mode are located at a small positive mismatch and at about one α Larmor radius away from the center as also indicated in Fig. 2. It is interesting to note that the area of the wave profile at the outer region and the number of α particles within are larger than those at the central region. The shape of the profile may be resulted from the need of the resonant interaction to drive the instability to dissipate the free energy associated with the energetic α particles.

The localized cyclotron wave mode has profound effects on the dynamics of the α particles. Figure 4 shows the snapshots of the α particles in the phase space of perpendicular momentum and real space. When the wave is still in its linear growth stage at $t=2000$, some of the α particles are being accelerated while some are decelerated; the acceleration is stronger at the central region. At the stage of the wave saturation at $t=2500$, the acceleration and deceleration of the α particles are significant. At the end of the simulation at $t=3000$, while some of the α particles have been accelerated almost to double its perpendicular energy, the overall accelerated phase space becomes a rectangle shape and be almost independent of the wave profile. At the edges of the wave profile, although taking a longer time, the much weaker wave field is still capable of affecting the α dynamics as that at the peak.

For simplicity, the nonuniformity studied analytically is parabolic at the minimum of the sinusoidal function employed in the simulation. The eigenmode driven is near the minimum, i.e., the zero at the derivation of B. As concerned for possible applications in a realistic device, this kind of nonuniformity may occur as a local approximation for the magnetic field along toroidal direction due to the sharp pressure gradients in high-β tokamak discharges or along poloridal direction especially for trapped α. While the relativistic cyclotron instability studied here is for energetic ions, the findings may still have important implications and thus applications for electrons such as the increasing of tunability in gyrotrons. The width of the wave eigenmode being eight α Larmor radiuses is not small in a device. Besides the direct consequences of the instability on the α dynamics, the details of the nonlinear plasma dynamics are not within the scope of this paper. While absolute instability is studied here, convective instability and its consequences driven by the energetic α particles remain an interesting topic to be investigated.

In summary, the relativistic ion cyclotron instability under nonuniform magnetic field has been studied with a hybrid simulation and an analytical theory. Although the nonuniformity is too large to satisfy the synchronism requirement of resonance, localized cyclotron mode is still observed in the simulation and is found to be consistent with that predicted by the theory questing for an absolute instability. Half of the spatial area of the eigenmode is located where the frequency mismatch is negative as against the positive requirement generally believed for driving relativistic cyclotron instabilities. The localized cyclotron wave demonstrates the profound effects on the dynamics of the α particles involved. We have also assessed the condition and possible applications in realistic devices such as high-β fusion tokamak plasmas.

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