Influence of Dirichlet boundary conditions on dissipative solitons in the cubic-quintic complex Ginzburg-Landau equation

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We investigate the influence of Dirichlet boundary conditions on various types of localized solutions of the cubic-quintic complex Ginzburg-Landau equation as it arises as an envelope equation near the weakly inverted onset of traveling waves. We find that various types of nonmoving pulses and holes can accommodate Dirichlet boundary conditions by having, for holes, two halves of a π hole at each end of the box. Moving pulses of fixed shape as they arise for periodic boundary conditions are replaced by a nonmoving asymmetric pulse, which has half a π hole at the end of the box in the original moving direction to guarantee that Dirichlet boundary conditions are met. Moving breathing pulses as they arise for periodic boundary conditions propagate toward one end of the container and stop moving while the breathing persists indefinitely. Finally breathing and moving holes are replaced by two (nonbreathing) half π holes at each end of the container and one hump in the bulk.

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I. INTRODUCTION AND MOTIVATION

Particle and holelike solutions have been observed experimentally for a number of pattern-forming systems with dissipation and dispersion [1–11]. These include various systems in nonlinear optics [10], binary fluid convection [2,3] and the Faraday instability in various complex fluids [11], as well as various chemical reactions [9], in particular on surfaces [1]. Such dissipative localized solutions have also been obtained from macroscopic equations (including hydrodynamic equations [12], model equations relevant in nonlinear optics [13–16] and reaction-diffusion systems [17–27]) as well as for order parameter equations [28–31] and phase equations [32–35], the analog of hydrodynamic equations for large aspect ratio pattern forming systems [36].

The cubic-quintic complex Ginzburg-Landau (CGL) equation is a prototype equation applicable near the weakly hysteretic onset of an oscillatory instability to traveling or standing waves [37]. We emphasize that the analysis presented in the following is also relevant for systems described by macroscopic basic equations and order parameter equations. Macroscopic basic equations, such as hydrodynamic equations, for example, for simple fluids, binary fluid mixtures or nematic liquid crystals, as well as reaction-diffusion equations used to describe chemical reactions in spatially extended systems, can be reduced near the onset of an instability systematically to an envelope equation, like the cubic-quintic CGL equation, via a reduced perturbation expansion in the distance from instability onset as a small parameter [38–40]. Sometimes, as for example for convection in binary fluid mixtures, envelope equations can only be used for qualitative comparisons, since boundary layers arise. Order parameter equations or Swift-Hohenberg equations [40,41] are constructed to contain near the onset of an instability the appropriate envelope equations, while incorporating simultaneously the correct symmetry of the pattern-forming system to be described as well as the nonlinearities, which are thought to be most important.

For the cubic-quintic CGL equations many different types of stable pulse [42–65] and holelike [59–62,66,67] solutions have been found and analyzed starting with the fixed shape pulse described first by Thual and Fauve [42]. Depending on the localized solutions in question they stably exist over a larger or smaller parameter range. They always coexist with either zero amplitude or a finite amplitude spatially homogeneous solution, which are also at least locally stable. Starting in the field on nonlinear optics these various types of localized solutions have been called dissipative solitons [63] in order to express the fact that they arise for strongly driven dissipative systems in contrast to classical solitons for which driving and damping is typically taken into account only perturbatively [39]. In contrast to the cubic-quintic CGL equation, for the cubic CGL equation no stable dissipative solitons are known, but one can have spatiotemporal chaos [68] and hole solutions, which are, however, not generically stable [40,61].

In most experimentally accessible systems one has Neumann or Dirichlet boundary conditions or a mixture thereof. To realize periodic boundary conditions (PBCs), as they are frequently used for numerical calculations, one can construct experimentally in one spatial dimension annular geometries [2,3]. In two dimensions PBCs are not realizable experimentally, because PBCs in two dimensions correspond to a torus, the surface of which is rather impractical to do, for example, well controlled convection experiments on.

Neumann boundary conditions correspond physically to zero flux boundary conditions, for example for the concentration in mixtures. Dirichlet boundary conditions correspond for an envelope equation to a vanishing value of the envelope, A, at the end of a quasi-one-dimensional box, as they will be discussed in the following: $A \equiv 0$. This is, for ex-

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ample, in many cases the appropriate boundary condition for the velocity of a fluid, simple or complex, at the end of the container.

Previous work on dissipative solitons in the cubic-quintic CGL equation has concentrated mostly on PBCs and, to some degree, on Neumann boundary conditions [66,69,70]. Significant changes have been found for the properties of dissipative solitons when going from PBCs to Neumann boundary conditions, in particular for propagating pulse und holelike solutions.

For binary fluid convection there has been a considerable amount of work by Cross for a combination of Neumann and Dirichlet boundary conditions to capture the reflection of waves at the end of the container for binary fluid mixtures thus allowing for the stabilization of localized patches of convection in the vicinity of either boundary [71].

The paper is organized as follows. In the next section we describe the model and the numerical technique used. In Sec. III, we describe our results followed in Sec. IV by a discussion and conclusions.

II. MODEL

We investigate the complex subcritical cubic-quintic Ginzburg-Landau equation in one spatial dimension,

$$\partial_t A = \mu A + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A + (D_r + iD_i)\partial_{xx} A, \quad (1)$$

where $A(x,t)$ is a complex field. In writing down this equation we have already transformed into the moving frame. To guarantee saturation to quintic order, γ_r is taken to be negative, while $\beta_r > 0$ to have a weakly inverted bifurcation. The diffusion coefficient, D_r is assumed to be positive. In the spirit of an envelope equation, the fast spatial and temporal variations have already been split off when writing down the envelope equation. To compare with measurable quantities such as, for example, temperature variations in fluid dynamics, these rapid variations must be taken into account [37,38,40,72,73].

While there have been quite a few investigations of the cubic-quintic CGL equation for periodic boundary conditions and some for Neumann boundary conditions including a recent one by the authors for Neumann boundary conditions [70], we are not aware of any work using Dirichlet boundary conditions. This type of boundary conditions can be expected to be relevant experimentally, because it corresponds to the case of vanishing modulus of the amplitude, A.

We have carried out most of our numerical studies (except for the ones for breathing pulses, compare further below) for the following values of the parameters [58], which we kept fixed for the present purposes, $\beta_r=1$, $\beta_i=0.2$, $\gamma_r=-1$, $\gamma_i=0.15$, $D_r=1$, and $D_i=-0.1$. We note that these parameter values have also been used in previous studies [58,59,62]. Thus, the only parameter value that is varied in most of the results described in the following is the distance from linear onset, μ. As a numerical method we used fourth order Runge-Kutta finite differencing. Typically we used $N=600$ points and a grid spacing $dx=0.4$. This corresponds to a box size $L=Ndx=240$. The typical time step was $dt=0.1$ for nonbreathing pulses and holes and $dt=0.05$ for breathing pulses and holes. For Dirichlet boundary conditions we have $A(0)=0$ and $A(L)=0$. We will compare our results obtained for Dirichlet boundary conditions with the ones found previously for periodic and Neumann boundary conditions.

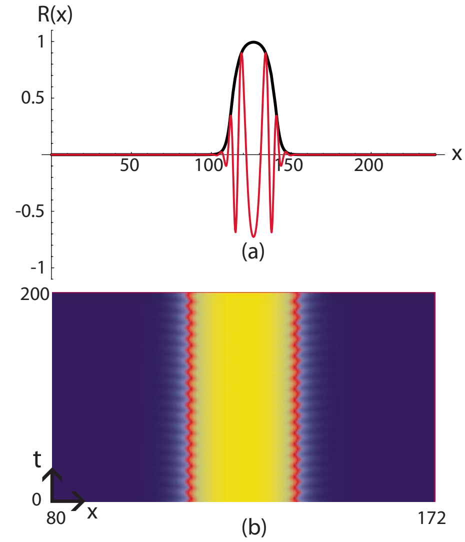

FIG. 1. (Color online) The figures show a breathing pulse in the asymptotic regime starting with in phase localized initial conditions. (a) snapshot and (b) $x-t$ plot for 4000 iterations. The parameter values used are given in the main text.

III. RESULTS

A. Pulses

First we investigated how various types of pulselike solutions are affected by Dirichlet boundary conditions when compared to periodic or Neumann boundary conditions. It turns out that classical fixed shape pulses, which are well documented to exist over a large parameter interval, are hardly affected at all by the type of boundary conditions as long as their width is small compared to the box size chosen.

Next we investigated the case of breathing pulses as they have been studied before for periodic boundary conditions [49] and for Neumann boundary conditions [70]. In order to obtain breathing pulses the linear dispersion ($\sim D_i$) has to be large enough in magnitude compared to linear diffusion ($\sim D_r$). Specifically we chose for our investigations of breathing pulses the same parameter values as for the case of Neumann boundary conditions: $\beta_r=3$, $\beta_i=1$, $\gamma_r=-2.75$, $\gamma_i=1$, $D_r=0.9$, $D_i=-1.1$, $\mu=-0.19$, and $L=240$, $dx=0.4$, ($N=600$), and the time step $dt=0.05$. Using in phase boundary conditions (ICP) in the spirit of [58], we obtain asymptotically in time a breathing pulse shown as a snapshot in Fig. 1(a) and as an $x-t$ plot in Fig. 1(b) for $T=200$ (corresponding to $4 \cdot 10^3$ iterations). We note that in the case of ICP both sides of the breathing pulse are oscillating in phase. Using antiphase boundary conditions (ICA) in the spirit of [58], we obtain asymptotically in time a breathing pulse shown as a snapshot in Fig. 2(a) and as an $x-t$ plot in Fig. 2(b) for

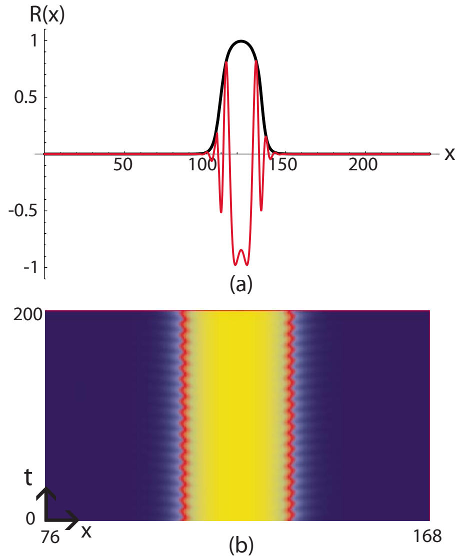

FIG. 2. (Color online) The figures show a breathing pulse in the asymptotic regime starting with antiphase localized initial conditions. (a) snapshot and (b) $x-t$ plot for 4000 iterations. The parameter values used are given in the main text.

$T=200$ (corresponding to $4\cdot10^3$ iterations). We note that in the case of ICA both sides of the breathing pulse are oscillating in antiphase. The solid thin (red) lines shown in the snapshots for the modulus, $R(x)$, in Figs. 1 and 2 as well as in all the following figures correspond to the real part of the complex field, A.

To investigate the influence of Dirichlet boundary conditions on moving pulses of fixed shape it is most convenient to start with antiphase initial conditions. Using the standard parameter values discussed in the last section and $\mu=-0.113\,55$ we obtain first moving fixed shape pulses of the type shown in Fig. 3, where we present the modulus along with the local wave vector.

Once the fixed shape pulse reaches the boundary (in the case shown the left boundary), a static asymmetric pulse located near the left boundary results (Fig. 4). Both, the modulus and the wave vector, are static. We emphasize that a pulse moving to the right generates a static pulse located near the right boundary thus underscoring the preservation of the symmetry of the deterministic equation.

To close the section on pulses, we study the influence of Dirichlet boundary conditions on breathing moving pulses as they have been investigated quite recently [65]. They are generated from ICA as initial conditions and using the same parameter values as above for breathing nonmoving pulses. Using $\mu=-0.0872$ we obtain a moving breathing pulse as shown in Fig. 5.

Asymptotically in time a breathing pulse near the right boundary is obtained when the moving breathing pulse is originally moving to the right as in Fig. 6.

B. Holes

As it was shown in [69,70], Neumann boundary conditions lead to rather drastic changes for hole-type solutions, in particular for propagating holes, when compared to periodic boundary conditions. To investigate the case of the influence of Dirichlet boundary conditions, which might appear to be even more restrictive to be satisfied, we have started with two types of localized initial conditions, namely, (ICP–ICP) and (ICA–ICP) as plotted in Fig. 7. We note the combination (ICA–ICA) leads to the same results as (ICA–ICP).

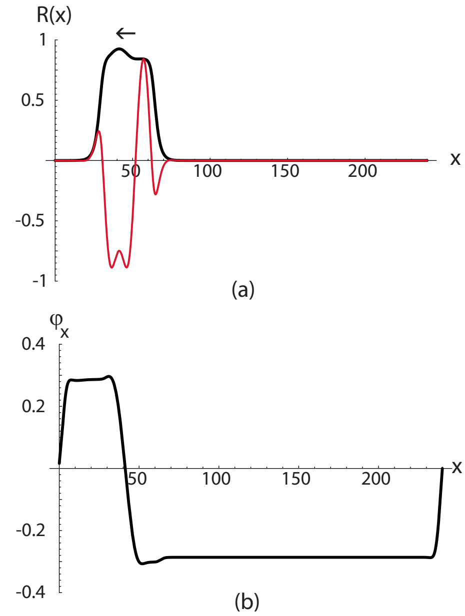

FIG. 3. (Color online) The figures show a moving fixed shape pulse starting with antiphase localized initial conditions. (a) snapshot of the modulus and (b) the local wave vector. The parameter values used are given in the main text.

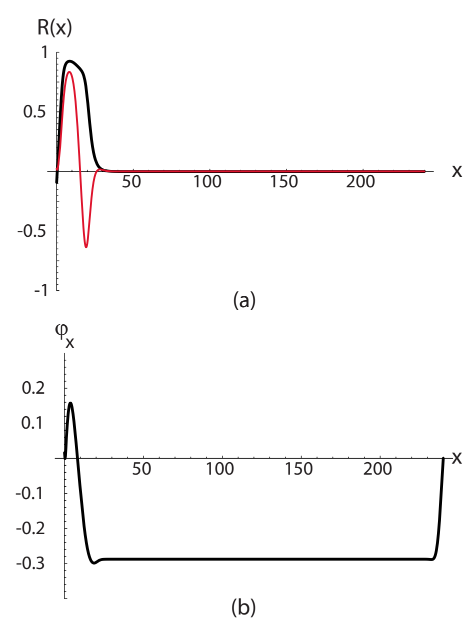

FIG. 4. (Color online) The figures show the final result after a moving fixed shape pulse reaches the boundary (in the case shown the left boundary) giving rise to a static modulus and a static distribution of the wave vector. (a) snapshot of the modulus and (b) the local wave vector.

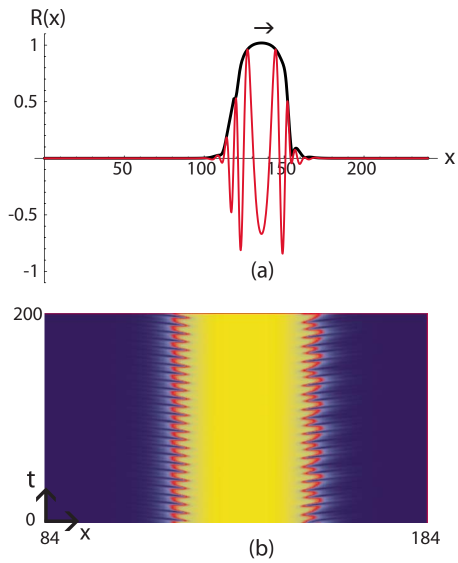

FIG. 5. (Color online) The figures show a breathing moving pulse prepared from ICA initial conditions. (a) snapshot and (b) $x-t$ plot for 4000 iterations. We note that the $x-t$ plot shows strong subharmonic contents, in particular $f/3$ with f the fundamental frequency. The parameter values used are given in the main text.

To investigate by what type of an object a nonbreathing π hole is replaced, we start with (ICP–ICP) initial conditions (for the range of μ for which π holes are generated for periodic boundary conditions and ICP) and with (ICA–ICP) initial conditions (for the range of μ for which π holes are generated for periodic boundary conditions and ICA). As a result we obtain a compound object with a π hole in the middle of the box, half of a π-hole at each end and, in between the π hole and each of the two half π-holes at either end, a hump. Figure 8 shows the modulus, the local wave vector, φ_x, and the phase, φ, for μ=−0.09. We note the singularity associated with the local wave vector for the compound object containing a π hole.

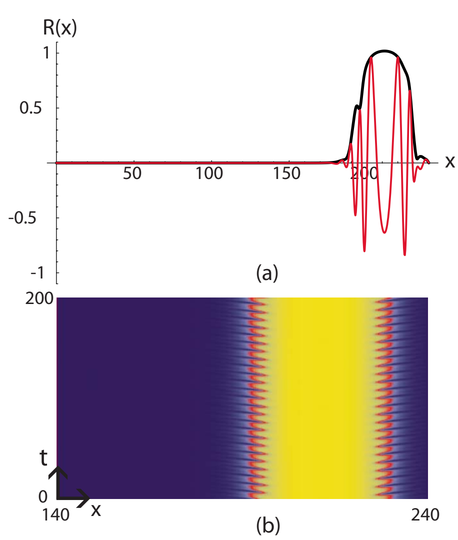

FIG. 6. (Color online) The figures show a breathing pulse near the right boundary of the box asymptotically in time prepared from ICA initial conditions and resulting from a moving breathing pulse. (a) snapshot and (b) $x-t$ plot for 4000 iterations. The parameter values used are given in the main text.

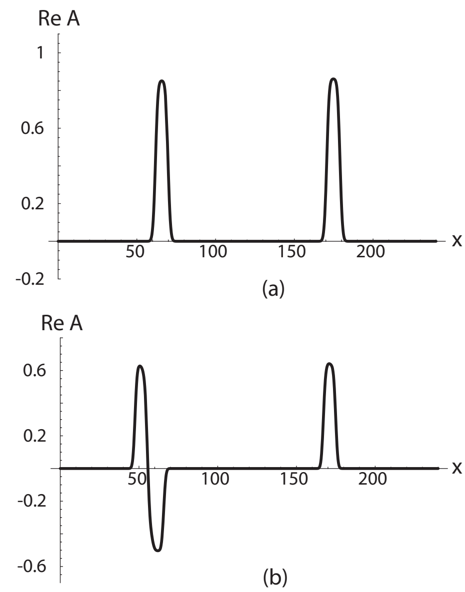

FIG. 7. The figures show the two initial conditions used to generate and to study hole-type solutions. (a) (ICP–ICP) initial conditions and (b) (ICA–ICP) initial conditions.

Thus we arrive at the conclusion that by adding half of a π hole at either end of the container Dirichlet boundary conditions are compatible with a compound object having a π hole in the center. This leads naturally to the question whether this is also possible for 2π holes. Starting with either (ICP-ICP) initial conditions (for the ICP range of 2π holes) or with (ICA-ICP) initial conditions (for the ICA range of 2π holes) we obtain indeed a compound object with a 2π hole in the center, half a π hole at either end and two humps in between, but close to either boundary. This compound object is shown in Fig. 9 for μ=−0.105 including the modulus and the local wave vector. Note the absence of a singularity associated with the local wave vector for the compound object containing a 2π hole.

Breathing hole solutions come for periodic boundary conditions in two varieties, nonmoving and propagating. For the nonmoving breathing holes we find—in analogy to the two cases of nonbreathing π and 2π holes just discussed—the possibility of a compound object involving the breathing hole in the center, two humps near the ends of the container and a half π hole at either end. This is demonstrated in Fig. 10 for the modulus, the local wave vector and in an $x-t$ plot for μ=−0.0878, where we have used (ICP-ICP) initial conditions and where we have investigated the range, for which ICP gave nonmoving breathing holes for periodic BCs.

When the influence of Neumann boundary conditions was investigated for breathing moving holes as they exist for periodic BCs, we found that these were destroyed by Neumann boundary conditions and replaced by the spatially homogeneous solution for the modulus. Here, we present the corresponding results for the influence of Dirichlet boundary conditions. We start with initial conditions of the (ICP–ICP) type in the range, for which in phase boundary conditions give moving breathing holes for periodic BCs. Interestingly we

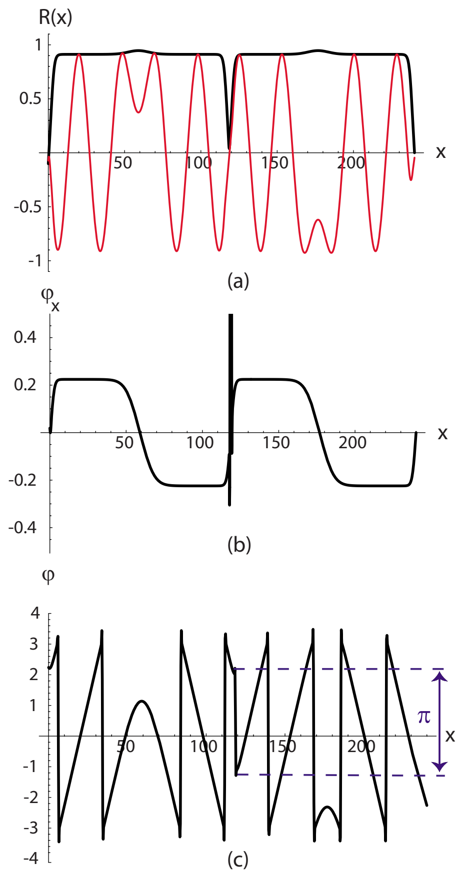

FIG. 8. (Color online) Compound object resulting for Dirichlet boundary conditions instead of a π hole for periodic boundary conditions (BCs). We note that both, the modulus and the local wave vector, are static. (a) Modulus $R(x)$, (b) local wave vector, $\varphi_x(x)$, and (c) phase, $\varphi(x)$.

obtain as a result of the influence of Dirichlet boundary conditions for $\mu-0.087\,97$ a compound state with a half π-hole at either end and a hump located near one end of the container (Fig. 11). The resulting state is static with respect to modulus and wave vector.

IV. DISCUSSION AND CONCLUSIONS

Here, we have presented the results of our studies on the influence of Dirichlet boundary conditions on various types of pulse and holelike solutions of the cubic-quintic complex Ginzburg-Landau equation as they are known to stably exist for periodic boundary conditions. In comparison to the results described before for Neumann boundary conditions (no flux), Dirichlet boundary conditions correspond to a pinning (fixed value zero) of the amplitude at the boundaries of the container. Depending on the type of solutions investigated we used four different types of initial conditions, in-phase or out of phase localized initial conditions for the various types of pulse solutions and either a pair of in-phase localized initial conditions or a combination of in-phase and out of phase initial conditions to obtain the various types of hole solutions and to study the influence of Dirichlet boundary conditions on those.

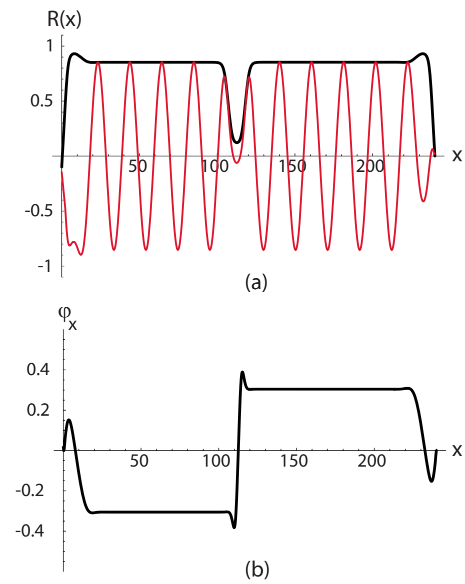

FIG. 9. (Color online) Compound object resulting for Dirichlet boundary conditions instead of a 2π hole for periodic BCs. We note that both, the modulus and the local wave vector, are static. (a) Modulus $R(x)$ and (b) local wave vector $\varphi_x(x)$

Several general observations emerge as an essence from our results immediately. First of all, all propagating pulses and holes for periodic BCs are replaced by nonpropagating, asymmetric pulses and holelike solutions for Dirichlet boundary conditions. A rather similar conclusion could already be drawn for the Neumann case. A second, and rather novel, feature is associated with all hole-type solutions investigated as well as with the final state of a moving, nonbreathing pulse. In all five cases one finds either one (in the case of a moving, nonbreathing pulse being replaced by an asymmetric pulse located at the end of the box in the original direction of propagation) or two half π holes (for the case of the nonbreathing analogs of π and 2π holes as well as for the hole solutions, which are breathing and nonmoving or moving before approaching the end of the container) at the end of the container to accommodate Dirichlet boundary conditions. Clearly the prediction of half π-holes at the end of container is suitable for experimental tests, which one could perform, for example. in an annulus to prepare the state, which has, in addition, an outlet followed by a straight section similar in spirit to accelarators in particle physics. Here, the analog of the particles to be investigated are dissipative solitons.

While there are dissipative solitons, which can adjust to all three types of boundary conditions (periodic, Neumann and Dirichlet) rather easily including nonmoving pulses that are either of fixed shape or breathing, other types of dissipative solitons change qualitatively when the boundary conditions are changed. One example is breathing moving pulses as they exist for periodic boundary conditions. Here we have demonstrated that they are replaced by breathing pulses located near the end of the container for Dirichlet BCs. Another outstanding example for qualitative changes is moving breathing holes (periodic BCs). When changing to Neumann conditions, they are wiped out [70] and replaced by the spa-

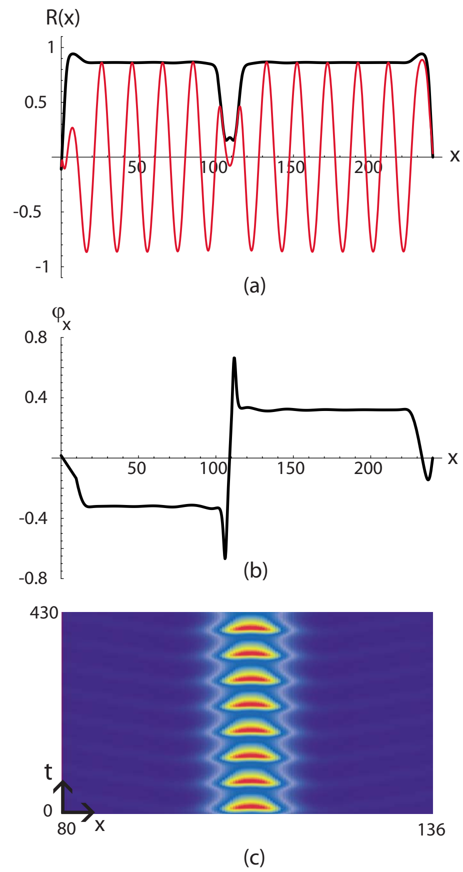

FIG. 10. (Color online) Compound object resulting for Dirichlet boundary conditions instead of a nonmoving breathing hole for periodic BCs. We note that both, the modulus and the local wave vector, are breathing. This is clearly brought out in the $x-t$ plot shown in (c) for 8600 iterations. (a) Modulus $R(x)$, (b) local wave vector $\varphi_x(x)$ and (c) $x-t$ plot.

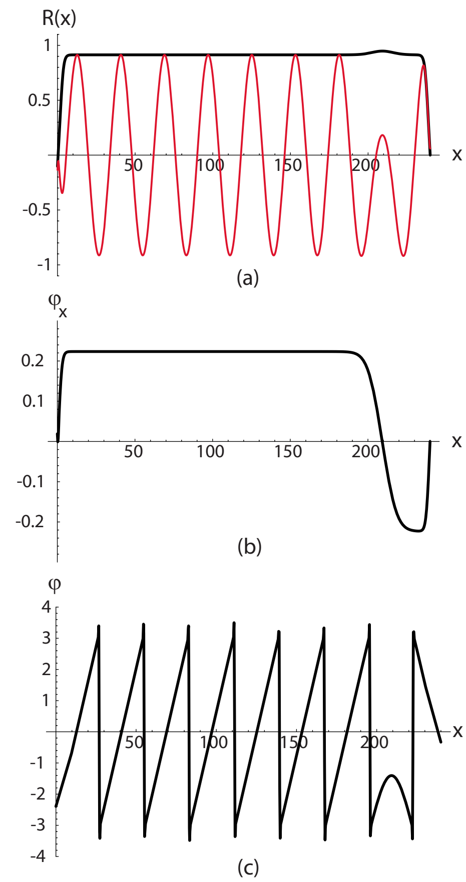

FIG. 11. (Color online) Compound object resulting for Dirichlet boundary conditions instead of a moving breathing hole for periodic BCs. We note that both, the modulus and the local wave vector, are static. (a) Modulus $R(x)$, (b) local wave vector, $\varphi_x(x)$, and (c) local phase, $\varphi(x)$.

tially homogeneous solution, while for Dirichlet boundary conditions a compound state of two half π-holes at the end of the container and a hump near the end of the container in the original direction of propagation arises. Clearly it is highly desirable to see experimental tests of our predictions. Possible systems include autocatalytic chemical reactions, binary fluid convection and systems from nonlinear optics for which the cubic-quintic complex Ginzburg-Landau equation applies.

In [74], it was discussed for static and standing wave solutions of several types of partial differential equations how one can find solutions for Neumann and Dirichlet boundary conditions by dividing by two the interval of certain classes of solutions satisfying periodic boundary conditions. As an additional important ingredient the authors of ref. [74] assume smooth periodic solutions of a certain periodicity as they can arise, for example, for reaction-diffusion systems, for Taylor vortex flow and for Rayleigh-Bénard convection. In the present paper we have discussed stable localized solutions, which are all of traveling wave nature (for the real and imaginary part of the envelope) and thus do not satisfy the requirements of the solutions studied in [74]. Furthermore, all the stable solutions discussed in the present paper involving a π-hole or half π holes are not smooth. In particular in ref. [74] the treatment of Dirichlet boundary conditions requires the symmetry $u(-x)=-u(x)$. The types of solutions we consider in the article do not satisfy this symmetry. We would like to emphasize, that in several cases the structure and the symmetry of the stable localized solutions found for periodic versus Dirichlet boundary conditions are qualitatively different. For example, a moving fixed shape solution for the modulus (for periodic boundary conditions) is replaced for Dirichlet boundary conditions by a nonmoving pulse of a different shape and with a static modulus and a static distribution of the local wave vector near one end of the container as it is shown in Figs. 3 and 4. In addition, many of the stable localized solutions presented here for Dirichlet boundary conditions have—to the best of our knowledge—not been anticipated even qualitatively in previous work. This includes in particular all solutions having one or two half π holes at the end of the box. Therefore, it will be most interesting to see, to what extent it will be possible to generalize the mathematical methods presented in [74] to the stable, localized solutions of traveling wave nature studied here for Dirichlet boundary conditions.

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